PROJECTIVE MODULES OVER THE GERASIMOV–SAKHAEV COUNTEREXAMPLE

NIKOLAY DUBROVIN, PAVEL PŘÍHODA, AND GENA PUNINSKI

Abstract. We investigate the structure of the so–called Gerasimov–Sakhaev counterexample, which is a particular example of a universal localization, and classify (both finitely and infinitely generated) projective modules over it.

1. Introduction

The story began when Lazard [15, Prop. 5] proved that over a commutative ring every projective module finitely generated modulo its Jacobson radical is finitely generated. This result has been further extended to rings with polynomial identities [12], with the ascending chain condition on one-sided annihilators [23], and with one-sided Krull dimension [19]. The question whether this property holds true for an arbitrary ring is often referred to as Lazard's conjecture (see [11] and [12]).

Later Sakhaev [20] and independently Zöschinger [23] completely characterized rings R possessing a non-finitely generated projective module finitely generated modulo its Jacobson radical. This happens exactly when there are $n \times n$ matrices x, y over R such that $yx = 0$, the matrix $1 - x - y$ is in the Jacobson radical of the ring $M_n(R)$ of $n \times n$ matrices over R, and $y(x+y)^{-1}x \neq 0$. Here n stands for the number of generators of $P/\operatorname{Jac}(P)$, therefore, if $P/\operatorname{Jac}(P)$ is cyclic, then x and y are elements of R. It is not difficult to find a ring satisfying the first and the last condition: take the free algebra $k\langle x, y\rangle$ over a field k, impose the relation $yx = 0$, and adjoin the two-sided inverse to $x + y$ universally. But it is more difficult to satisfy the second condition.

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Gerasimov and Sakhaev [11] succeeded in finding an example as above by localizing $R = k\langle x, y\rangle/yx = 0$ with respect to the set Σ of all square matrices that become invertible under each of the following evaluations: $x \to 0, y \to 1$ and $x \to 1, y \to 0$ (since $x + y$ is sent to 1 by both maps, it is invertible in R_Σ). It follows from the general theory of universal localization that $R_\Sigma/\operatorname{Jac}(R_\Sigma) \cong k \oplus k$, in particular R_Σ is a semilocal ring with exactly two maximal (left, right or two-sided) ideals, and $\operatorname{Jac}(R_\Sigma)$ contains $1-x-y$. The main problem in [11] was to check that $y(x+y)^{-1}x \neq 0$ in R_Σ. This difficulty was overcome by a clever use of the property of Σ being independent, which led to a relatively simple criterion for when an element of R_Σ is equal to zero. From this criterion it was derived that the canonical map $R \to R_\Sigma$ is an embedding and $y(x + y)^{-1}x \neq 0$ in R_Σ, thereby settling Lazard's conjecture in the negative.

A concrete construction that comes from Sakhaev and Zöschinger then gives a non-finitely generated projective right R_Σ-module P such that $P/\operatorname{Jac}(P)$ is a cyclic module generated by $\bar{x}$, the image of x. Furthermore, the dimension of P (that is, the vector counting the multiplicities of the two simple R_Σ-modules as composition factors of $P/\operatorname{Jac}(P)$) is $(1, 0)$, hence P is indecomposable. Since R_Σ itself has dimension $(1, 1)$, fairly general arguments yield that every finitely generated projective R_Σ-module is free of a unique rank, that is, R_Σ is projective-free. In this paper we classify non-finitely generated projective (right) modules over R_Σ by showing that every such module is isomorphic to a direct sum of copies of P and R_Σ, and this decomposition is essentially unique except for the relation $P^{(\omega)} \oplus R_\Sigma^{(\omega)} \cong R_\Sigma^{(\omega)}$.

Despite being easy to formulate, the proof of this result is quite involved. First we investigate the structure of the universal localization R_Σ. As was noticed by Dicks and Sontag [5], if $R = k\langle x, y\rangle/yx = 0$, then the 3×3 matrix ring $M_3(R)$ is a coproduct over $k \oplus k \oplus k$ of $M_3(k)$ and a serial 5-dimensional k-algebra of global dimension 2. Using some general knowledge on coproducts (see [1, 2]) one can conclude that R is coherent of global dimension 2. Unfortunately no such transfer of properties is known for universal localizations (except when R is hereditary or R_Σ is flat over R), so we proceed with a careful elementwise analysis of R_Σ.

First we prove that $(1 - y)R_\Sigma$ and $R_\Sigma(1 - x)$ are the only maximal two-sided (and one-sided) ideals of R_Σ (both of codimension 1) and calculate the lattice of two-sided ideals of R_Σ above the ideal $\langle xy\rangle$ generated by xy. For instance, we show that $\operatorname{Jac}(R_\Sigma)$ is generated by $1 - x - y$ as a two-sided ideal, and the only idempotent two-sided ideals of R_Σ not contained in $\operatorname{Jac}(R_\Sigma)$ are $\langle x\rangle$ and $\langle y\rangle$. Note that $\langle x\rangle$ is the trace of the projective right R_Σ-module P and $\langle y\rangle$ is the trace of a projective left R_Σ-module Q.

From general arguments it follows that to complete our classification (of projective R_Σ-modules) it suffices to prove that $\langle y\rangle$ is not the trace of a projective right R_Σ-module. At this point we have not been able to find a direct proof of this result. We resolve the problem by using the following bypass.

A different example of a k-algebra S with a 'strange' projective module was found by Puninski [18]. In his example S is the endomorphism ring of a uniserial module, with the following properties. There are $f, g \in S$ such that f is mono not epi, g is epi not mono, $gf = 0$, and $1 - f - g \in \mathrm{Jac}(S)$ (the last condition means just that $1-f-g$ is neither epi nor mono). Using the extensive knowledge of the structure of S (see [19], [6]) we prove that the map α from R to S sending x to f and y to g makes every matrix in Σ invertible, hence is uniquely extendable to a morphism $\bar{\alpha} : R_\Sigma \to S$. Now, if R_Σ had a projective right R-module P' with trace $\langle y\rangle$, then the trace of the induced projective right S-module $P' \otimes_{R_\Sigma} S$ would be a subset of SgS, which is known to be impossible.

Note that the rings R_Σ and S exhibit similar features, but with a strange twist. For instance, Sf and gS are the only maximal (two-sided or one-sided) ideals of S, but the 'corresponding' ideals $(1-y)R_\Sigma$ and $R_\Sigma(1-x)$ are principal on the opposite side. It is known (see [6]) that S is coherent of global dimension 2, but we have not been able to verify these properties for R_Σ. Apart from the above similarity we have just one result to support that this may be true for R_Σ. Namely, we prove that for any $r \in R$, its right (left) annihilator in R_Σ is free as a right (left) R_Σ-module. A typical example is given by $y \in R_\Sigma$ whose right annihilator xR_Σ is free (since x is a right nonzero divisor in R_Σ). Furthermore, we will show that R_Σ is not flat as a right or left R-module.

Of course there is no reason to believe that a 'small' ring R_Σ and its 'huge' counterpart S are too close in their properties. However some properties of R_Σ are easily verified using S. For instance, the image of $y(x+y)^{-1}x$ in S is $g(f+g)^{-1}f = f(f+g)^{-1}g$ which is clearly nonzero (because g is epi and f is mono), therefore $y(x+y)^{-1}x \neq 0$ in R_Σ. On the other hand we should expect a great similarity between R_Σ and its image $S' = \bar{\alpha}(R_\Sigma) \subseteq S$ which is the rational closure in S of the subring generated by $1, f$ and g. From general theory it follows that S' is the division closure of the same set in S, hence every element of S' can be written as a 'rational function' in $1, f$ and g. Unfortunately calculating in S' is harder than in S. For instance, we do not know if $\bar{\alpha} : R_\Sigma \to S'$ is an isomorphism even for a very particular choice of f and g.

One source of our interest in R_Σ is that it represents a 'universal' semi-local ring with two maximal ideals and an infinitely generated projective module P of dimension $(1,0)$. An open problem in this area is to find a projective module Q of dimension $(0,\omega)$. By the results of this paper there is no such module Q over R_Σ, but it may exist over an appropriate factor of R_Σ. This module would exhibit a 'perfect' direct sum decomposition behavior: $Q \cong Q^{(\alpha)}$ for every $1 \le \alpha \le \omega$, and those are the only possible direct sum decompositions.

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2. Projective modules

A *ring* R in this paper will always mean an associative ring with unity, and all *modules* will be unital and mostly right modules over R. Thus we apply morphisms of right modules on the left: if f and g are morphisms, then in fg one should apply g first, and f after that. An R-module F is said to be *free*, if F is isomorphic to a direct sum of copies of R, that is, $F \cong R^{(I)}$ for some set I, and the cardinality of I will be called the *rank* of R. For instance, R considered as a right module over itself is free of rank 1. An R-module P is said to be *projective* if P is isomorphic to a direct summand of a free R-module. Clearly every free module is projective, but the converse is usually not true even for finitely generated modules. Below we will see some examples of non-finitely generated projective modules which are indecomposable, hence definitely non-free.

A classical theorem by Kaplansky (see [7, Cor. 2.48]) says that every projective module is a direct sum of countably generated (projective) modules. Furthermore, every countably generated projective can be constructed as a countable direct limit of free finite rank modules. This construction is due to Whitehead [22] and is similar to a Bass's classical representation of countably generated flat modules as direct limits of free finite rank modules.

If A is an $m \times n$ matrix over a ring R then the left multiplication by A defines a morphism of free right R-modules $f : R^n \to R^m$ of ranks n and m. Following [16] we say that a sequence of matrices $\{A\} = A_1, A_2, \dots$ is *multiplicative* if all consecutive products $A_{i+1}A_i$ are defined. Thus in the corresponding sequence $\{f\} = f_1, f_2, \dots$ of morphisms, we can compose f_{i+1} and f_i to get the following chain complex $F_1 \xrightarrow{f_1} F_2 \xrightarrow{f_2} \dots$, where F_i are free right R-modules of appropriate finite ranks. Let $P = P(f)$ denote the direct limit of this directed system. If $\overline{x}_i$ is a standard basis of F_i, then P is isomorphic to the module with generators $\overline{x}_1, \overline{x}_2, \dots$ and relations $\overline{x}_{i+1}A_i = \overline{x}_i$. Furthermore, being a direct limit of free modules, P is flat, and this is exactly the kind of module that was considered by Bass.

To make P projective, we should impose one extra condition on the above chain of maps. We say that a multiplicative sequence $\{f\} = f_1, f_2, \dots$ is *stable* if, for every $i \geq 1$, there exists a morphism $g_{i+1} : F_{i+2} \to F_{i+1}$ such that $g_{i+1} f_{i+1} f_i = f_i$. In terms of matrices this means that for every $i \geq 1$ there exists a matrix C_{i+1} such that $C_{i+1} A_{i+1} A_i = A_i$. We illustrate this with the following diagram:

$$\begin{tikzcd} F_1 \arrow[r, "f_1"'] & F_2 \arrow[r, "f_2"'] & F_3 \arrow[l, dashed, bend right, "g_2"'] \arrow[r, "f_3"'] & F_4 \arrow[l, dashed, bend right, "g_3"'] \arrow[r, "f_4"'] & \cdots \end{tikzcd}$$

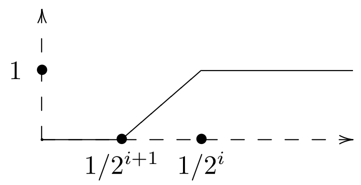
The following fact, which is essentially due to Whitehead, characterizes countably generated projective modules as direct limits of stable sequences (see [16, Sect. 2] for more explanations).

Fact 2.1. [22] If $\{f\} = f_1, f_2, \dots$ is a stable sequence of maps between free finite rank right R-modules, then the corresponding direct limit $P = P(f)$ is a countably generated projective R-module. Conversely, every countably generated projective right R-module is isomorphic to a module $P(f)$ for some stable sequence $\{f\}$.

Recall that the *trace*, $\mathrm{Tr}(P)$, of a projective module P is the sum of all images of maps from P to R_R. In general (see [14, Prop. 2.40]) $\mathrm{Tr}(P)$ is an idempotent ideal such that $P{\cdot}\mathrm{Tr}(P) = P$, and a less known fact is that $\mathrm{Tr}(P)$ is the least element among the ideals I of R such that $PI = P$. Recall also (see [13, Thm. 24.7]) that $\mathrm{Tr}(P)$ is never contained in the Jacobson radical of R. If P is constructed by the above sequence of matrices $\{A\}$, then the trace of P is generated by the entries of the A_i.

Some special cases of this construction are of a particular interest. Suppose first that there exists a sequence $\{r\} = r_1, r_2, \dots$ of elements of R such that $r_{i+1} r_i = r_i$ for every $i \geq 1$. Then $\{r\}$ is stable by a trivial reason (taking $c_{i+1} = 1$ for every i). Thus the corresponding sequence of morphisms between free rank one right R-modules has the projective module $P = P(r)$ as its direct limit. In this case we obtain an increasing chain $r_1 R \subseteq r_2 R \subseteq \dots$ of right ideals of R, and it is easily seen that P is isomorphic to the union $\cup_{i=1}^{\infty} r_i R$ of this chain. Furthermore P is a *pure* right submodule of R, which is equivalent to saying that R/P is a flat right R-module. Also the trace of P is a two-sided ideal generated by the r_i.

To give an example of this construction, let $R = C[0, 1]$ be the ring of continuous real valued functions on the interval $[0, 1]$ and let r_i, $i \geq 1$ be the following functions:



Then clearly $r_{i+1}r_i = r_i$, therefore we obtain a projective module $P = \cup_{i=1}^{\infty} r_i R$. It is readily checked that P consists of all continuous functions vanishing in some neighborhood of zero, hence P is Kaplansky's well-known example of a projective module (see [14, Exam. 2.12D]). One can show that P is indecomposable and not finitely generated. For more on the theory of projective $C[0,1]$-modules the reader could consult [16, Sect. 9]. Note that, if e is idempotent, then $r_i = e$ is a stable sequence, and the resulting projective module is eR. Furthermore, one can replace the elements $r_i \in R$ in the above construction by square R-matrices of fixed size. However, in this general framework it is difficult to say anything essential about the properties of P.

The following is a further refinement of the construction, a good account of which can be found in Facchini, Herbera and Sakhaev [10]. Suppose that there is an element $s \in R$ and a unit $u \in R$ such that $s^2 = us$, hence $u^{-1}s^2 = s$. Set $r_i = u^{-i-1}su^i$, in particular $r_0 = u^{-1}s$ (it is convenient to start from zero in this case) and $r_1 = u^{-2}su$. Straightforward calculations show that $r_{i+1}r_i = r_i$ for every $i \geq 0$. Thus we obtain a projective module $P = P(s)$ as a union of the ascending chain $r_0R \subseteq r_1R \subseteq \dots$ of right ideals of R, and the trace of P is equal to RsR. For P to be finitely generated, this ascending chain must eventually become stationary, and the following fact states exactly when it happens.

Fact 2.2. [10, Prop. 5.3] 1) there exists $i \geq 0$ such that r_i is an idempotent if and only if $su^{-1}s = s$, and then every r_i is an idempotent (because all the r_i are conjugates);

2) $r_iR = r_{i+1}R$ if and only if $su^{-2}s = u^{-1}s$, and then this equality holds for every i. In particular, the projective module $P = P(s)$ is finitely generated if and only if $su^{-2}s = u^{-1}s$.

Note that, multiplying $su^{-2}s = u^{-1}s$ by s on the right (and taking into account $u^{-1}s^2 = s$), we obtain $su^{-1}s = s$. Thus, when 2) takes place, so is 1). The converse is not true as we will show by a counterexample, but let us first get an idea why it may happen. Indeed, 1) says that the idempotents su^{-1} and $u^{-1}s$ generate isomorphic right (and left) ideals. On the other hand 2) written in the form $su^{-1} \cdot u^{-1}s = u^{-1}s$ claims that these right ideals coincide (because also $u^{-1}s \cdot su^{-1} = u^{-1}s^2u^{-1} = su^{-1}$).

Example 2.3. Suppose that M is a nonzero module such that $M \cong M \oplus M$. Then there are endomorphisms u, s of M that satisfy 1) but not 2) of Fact 2.2.

Proof. Let α be an isomorphism from M to $M \oplus M$. Extending this we obtain a decomposition $M = M \oplus M \oplus M$ and let π_1, π_2 denote the canonical projections onto the first and the second coordinates. Using α and the following diagram

$$\begin{tikzcd}
M \arrow[d, "\varphi"'] & = & M \arrow[d] \arrow[drr, "\alpha"] & \oplus & M \arrow[drr] & \oplus & M \arrow[d, "\alpha^{-1}"'] \\
M & = & M & \oplus & M & \oplus & M
\end{tikzcd}$$

we obtain an automorphism φ of M. Clearly $\varphi^{-1}(\pi_1 + \pi_2) = \pi_1\varphi^{-1}$, hence $\pi_1 + \pi_2 = \varphi\pi_1\varphi^{-1}$. Setting $u = \varphi^{-1}$ and $s = \pi_1\varphi^{-1}$ we obtain $su^{-1} = \pi_1\varphi^{-1}\varphi = \pi_1$ and $u^{-1}s = \varphi\pi_1\varphi^{-1} = \pi_1 + \pi_2$. Therefore $su^{-1} \cdot s = \pi_1 \cdot \pi_1\varphi^{-1} = \pi_1\varphi^{-1} = s$, but $su^{-2}s = su^{-1} \cdot u^{-1}s = \pi_1(\pi_1 + \pi_2) = \pi_1$ is not equal to $u^{-1}s = \pi_1 + \pi_2$. $\square$

Problem 2.4. *Describe the modules M whose endomorphisms rings contain elements u, s satisfying 1) but not 2) of Fact 2.2.*

The proof of Example 2.3 also works if $M \cong M \oplus N \cong M^2 \oplus N$. However, as was pointed out by Pere Ara to the third author, it does not suffice to require that M is directly infinite (that is, $M \cong M \oplus N$ for a nonzero module N). Namely, if R is the ring $k\langle xy\rangle/xy = 1$, then R_R is directly infinite (since $R_R \cong (1 - yx)R \oplus yxR$ and $R \cong yxR$), but $R = \operatorname{End}(R_R)$ contains no elements s, u such that u is a unit, $su^{-1}s = s$ and $su^{-2}s \neq su^{-1}$. The proof of this fact suggested by Pere Ara will lead us too far into K-theory, so we skip it.

Now we are arriving at the object of our main interest. To make the references easy, we gather all we need in the following lemma.

Lemma 2.5. *Let u, s be elements of a ring R such that $s^2 = us$ and $1 - u \in \operatorname{Jac}(R)$. Let $r_i = u^{-i-1}su^i$, $i \geq 0$ and let $P = \cup_{i=0}^{\infty} r_iR$ be the corresponding projective module. Then*

1) $P/\operatorname{Jac}(P)$ is a cyclic $R/\operatorname{Jac}(R)$-module generated by $\bar{s}$, the image of s;

2) P is finitely generated if and only if $su^{-1}s = s$.

Proof. 1) is a kind of folklore (see [19, Fact 3.1]). For instance, $r_0 s = u^{-1}s \cdot s = u^{-1}s^2 = s$, hence $s \in P$ and $s - r_0 = (1-u^{-1})s \in P \cap \operatorname{Jac}(R) = \operatorname{Jac}(P)$ (the last equality holds true because P is pure in R_R). By similar arguments $s - r_i \in \operatorname{Jac}(P)$ for every i.

2) Suppose that $su^{-1}s = s$ and we prove that P is finitely generated. By Fact 2.2 every r_i is an idempotent and it suffices to show that $su^{-2}s = u^{-1}s$. Otherwise by the same fact all the inclusions $r_iR \subset r_{i+1}R$ are proper, hence P is a countable direct sum of nonzero modules. But then the same is true for $P/\operatorname{Jac}(P)$, a contradiction to 1). □

Sometimes it is advantageous to make a change of variables in the above lemma. Namely, let us set $y = u - s$ and $x = s$ in this lemma. Then $yx = (u-s)s = us - s^2 = 0$, $1 - x - y = 1 - u \in \operatorname{Jac}(R)$ and it is easily checked that $su^{-1}s = s$ if and only if $y(x+y)^{-1}x = 0$, hence we have arrived at the conditions discovered by Sakhaev and Zöschinger. We single this out as a special corollary.

Corollary 2.6. *Suppose that x, y are elements of a ring R such that $yx = 0$ and $1 - x - y \in \operatorname{Jac}(R)$. Denote the unit $x + y$ by u, let $r_i = u^{-i-1}xu^i$, $i \geq 0$ and let $P = \cup_{i=0}^{\infty} r_iR$ be the corresponding projective module. Then $P/\operatorname{Jac}(P)$ is a cyclic module generated by $\bar{x}$; and P is finitely generated if and only if $y(x+y)^{-1}x = 0$. Furthermore, P is isomorphic to the direct limit of the following chain of morphisms: $R \xrightarrow{x\times -} R \xrightarrow{x\times -} R \xrightarrow{x\times -} \ldots$.*

Thus having at hands a pair of elements x, y as described in the corollary we can construct a projective module P whose factor by the Jacobson radical is cyclic. The following fact shows that all projective modules which are cyclic modulo their Jacobson radical are generated in this way. The proof of this fact is implicit in [10], but was given more attention in Příhoda [17].

Fact 2.7. Let P be a projective right R-module such that $P/\operatorname{Jac}(P)$ is a cyclic module. Then P is isomorphic to the module $P(s)$ as in Lemma 2.5 (or Corollary 2.6). This module is finitely generated if and only if $su^{-1}s = s$ (equivalently $y(x+y)^{-1}x = 0$).

By Morita equivalence, this fact clearly applies to the case when $P/\operatorname{Jac}(P)$ is a finitely generated R-module. Namely, if n is the number of generators for $P/\operatorname{Jac}(P)$ then, instead of elements u and s (or x and y), we obtain two $n \times n$ matrices over R with similar relations.

Note also that hypotheses of Corollary 2.6 are left-right symmetric, therefore the direct limit of the following chain of morphisms $R \xrightarrow{-\times y} R \xrightarrow{-\times y} R \xrightarrow{-\times y} \ldots$ is a projective left R module finitely generated modulo its Jacobson radical. Thus Fact 2.7 shows that projective modules finitely generated modulo Jacobson radical occur in pairs.

3. The ring $k\langle x, y\rangle/yx = 0$

We begin with a ring which is the basis of the Gerasimov–Sakhaev construction. Let k be any field and let $k\langle x, y\rangle$ be the free algebra in the noncommuting variables x and y. Let $R = k\langle x, y\rangle/yx = 0$ be the factor of $k\langle x, y\rangle$ by the (two-sided) ideal generated by yx. The elements of R can be considered as polynomials in x and y. It is easily seen that $1, x^i y^j$, $i + j \geq 1$ form a k-basis for R. It follows that x is a right non-zero divisor and y is a left non-zero divisor in R. For a reason that will be explained later, it is convenient to represent elements of R in the form $r = \alpha + xf(x) + yg(y) + \sum_{i,j\geq 1} \alpha_{ij} x^i y^j$, where $f(x)$ is a polynomial in x, $g(y)$ is a polynomial in y, and $\alpha, \alpha_{ij} \in k$. Then $r(0, y) = \alpha + yg(y)$, therefore $r(0, y) = 0$ implies $\alpha + yg(y) = 0$ yielding $r = xs$ for some $s \in R$. Similarly $r(x, 0) = \alpha + xf(x)$, therefore $r(x, 0) = 0$ yields $r = ty$ for some $t \in R$.

First we describe ring theoretic properties of R that can be derived from very general arguments. A ring R is said to be *projective-free* if every finitely generated projective R-module is free of a unique rank.

Fact 3.1. R is a projective-free coherent ring of global dimension 2. Furthermore, every projective R-module is free.

Proof. The first part follows from Dicks and Sontag [5, p. 264–265]. Namely, it was noticed there that the ring $M_3(R)$ of 3×3 matrices over R is isomorphic to the coproduct over $k \oplus k \oplus k$ of the matrix ring $M_3(k)$ and the ring S which is a factor of the upper triangular matrix ring $\left(\begin{smallmatrix} k & k & k \\ 0 & k & k \\ 0 & 0 & k \end{smallmatrix}\right)$ by the two-sided ideal $\left(\begin{smallmatrix} 0 & 0 & k \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$. Thus we are in the framework of Bergman's theory [1, 2] of coproducts over a semisimple artinian ring. It is easily seen that S has global dimension 2, hence R has global dimension 2 by [1, Cor. 2.5]; and Corollary 2.11 of the same paper yields that R is projective-free. Furthermore, by [1, Cor. 2.6], every projective R-module P is induced from projective modules over $M_3(k)$ and S, hence P is free. (Later we will see that over R_Σ the situation is less satisfactory: all finitely generated projective modules are induced from R, but there is a non-finitely generated projective which is not induced).

The ring R is coherent, as it was mentioned in [5] without proof, and below we will demostrate general arguments that the authors of [5] probably had in mind. □

For unexplained terminology on coproducts of rings the reader is referred to Bergman [1].

Lemma 3.2. *Suppose that R is a coproduct of faithful R_0-rings R_λ, $\lambda \in \Lambda$ over a semisimple artinian ring R_0. If every R_λ is right coherent, then R is right coherent.*

Proof. By definition of coherency (see [21, Sect. 1.13]) R is right coherent if and only if every finitely generated right ideal I of R is finitely presented. If I has n generators, then there is an epimorphism $f : R^n \to I$ and we have to prove that the kernel of f, $\ker(f)$, is finitely generated. Being a submodule of the standard module R, by [1, Thm. 2.2], I is standard, that is, isomorphic to a module $\bigoplus_\lambda M_\lambda \otimes_{R_\lambda} R$ for some R_λ-modules M_λ. Furthermore, since R^n is finitely generated and f is onto, by [1, Thm. 2.3], one can choose a standard representation $R^n = \bigoplus_\lambda N_\lambda \otimes_{R_\lambda} R$ such that $f(N_\lambda) \subseteq M_\lambda$ for every λ, and clearly each $f_\lambda = f|_{N_\lambda} : M_\lambda \to N_\lambda$ is onto. Since R_λ is right coherent and M_λ is a finitely generated R_λ-submodule of R (which is free as an R_λ-module), we conclude that M_λ is finitely presented. It follows that $\ker(f_\lambda)$ is a finitely generated right R_λ-module. Since R is flat as a left R_λ-module, it follows easily that $\ker(f) \cong \bigoplus_\lambda \ker(f_\lambda) \otimes_{R_\lambda} R$. Because only finitely many summands in this sum are nonzero, we conclude that $\ker(f)$ is a finitely generated R-module, as desired. □

The sum $\sum_{i=1}^{\infty} x^i yR$ of right ideals of R is direct, therefore R is neither right nor left noetherian. However, using [11, L. 1], it is not difficult to show that every right ideal I of R with $I(x, 0) \neq 0$ is finitely generated. Furthermore, [11, L. 2] can be used to give a direct proof of the fact that R is right and left coherent and projective-free.

Note also that $(y-1)R$ is a two-sided ideal of R of codimension 1 containing x (since $-(y-1)x = x$); and $R(x-1)$ is a two-sided ideal of codimension 1 containing y (since $-y(x-1) = y$). It readily follows (see also Section 5) that their intersection is a semiprime ideal of codimension 2 generated by $1 - x - y = (y-1)(x-1)$.

4. Universal localizations.

For a general theory of universal localization with respect to a set Σ of square matrices the reader is referred to Cohn's book [4]. We give only definitions and facts that are required for our particular construction, and in this we will stay close to Gerasimov's approach (see [4, Sect. 7.11]). Let $R = k\langle x, y\rangle / yx = 0$ be as in the previous section. Let α_1 be a morphism from R to k sending x to 0 and y to 1. Thus, if $r \in R$ is written as a polynomial in x and y, then $\alpha_1(r) = r(0, 1) \in k$. Let α_2 be a similar morphism from R to k sending x to 1 and y to 0, therefore $\alpha_2(r) = r(1, 0)$; and let $\alpha = (\alpha_1, \alpha_2)$ be the corresponding morphism from R to $k \oplus k$. Let Σ be the set of all $n \times n$ matrices over R whose α-image is invertible. Thus

$A \in \Sigma$ if both $\alpha_1(A)$ and $\alpha_2(A)$ are invertible k-matrices. In particular, $x + y \in \Sigma$, because $\alpha_1(x + y) = \alpha_2(x + y) = 1$. We define R_Σ to be the universal localization of R with respect to Σ. Since the image of every matrix in Σ is invertible in $k \oplus k$, by the universal property, α is uniquely extended to a morphism $\bar{\alpha} : R_\Sigma \to k \oplus k$ making the following diagram commutative (here λ denotes the canonical morphism $R \to R_\Sigma$):

$$\begin{tikzcd} R \arrow[r, "\lambda"] \arrow[d, "\alpha"'] & R_\Sigma \arrow[dl, "\bar{\alpha}"] \\ k \oplus k & \end{tikzcd}$$

Some properties of R_Σ could be extracted from a fairly general point of view. Recall that a ring T is said to be *semilocal* if the factor of T modulo its Jacobson radical is a semisimple artinian ring. For instance, a commutative ring is semilocal if and only if it has finitely many maximal ideals. The following is discussed in [8] in a more general framework.

Fact 4.1. (see [8, Thm. 3.3]) The map $\bar{\alpha}$ induces an isomorphism from $R_\Sigma / \operatorname{Jac}(R_\Sigma)$ onto $k \oplus k$. In particular R_Σ is a semilocal ring with exactly two (two-sided and one-sided) ideals; and, if $r \in R$, then $\lambda(r) \in \operatorname{Jac}(R_\Sigma)$ if and only if $\alpha(r) = 0$.

It follows from [11] that $\lambda : R \to R_\Sigma$ is an embedding, therefore we will identify elements of R with their images in R_Σ. In particular, $\alpha(1 - x - y) = 0$ implies that $1 - x - y \in \operatorname{Jac}(R_\Sigma)$. It also follows from [11] that $y(x + y)^{-1}x \neq 0$ in R_Σ, hence one can construct a 'strange' projective R_Σ-module P. Namely, as in Section 2, we set $u = 1 - x - y$ and $r_i = u^{-i-1}xu^i$. Then (see Corollary 2.6) $P = \cup_{i=0}^{\infty} r_i R_\Sigma$ is a non-finitely generated projective module whose factor $P/ \operatorname{Jac}(P)$ is cyclic generated by $\bar{x}$.

Despite R being non-noetherian, R_Σ is very similar to universal localizations in noetherian rings considered by Cohn [3]. Namely, it is easily checked that Σ consists of all matrices which are regular (hence invertible) modulo the semiprime ideal $(y - 1)R \cap R(x - 1)$ of R.

To move on we should get some insight into elements of R_Σ. According to Gerasimov (see [4] again), a typical element of R_Σ (or rather its equivalence class) is a blocked matrix

$$(*) \qquad t = \left(\begin{array}{c|c} p & r \\ \hline A & q \end{array} \right),$$

where $r \in R$, p is a row of elements of R, q is a column of elements of R, and $A \in \Sigma$. Note that t can be thought of as a 'usual' element $r - pA^{-1}q \in R_\Sigma$.

In particular, the image in R_Σ of $r \in R$ is the following matrix

$$\left(\begin{array}{c|c} & r \\ \hline & \end{array}\right)$$

with empty blocks; and the element $(x+y)^{-1}$ is represented by the matrix

$$\left(\begin{array}{c|c} -1 & 0 \\ \hline x+y & 1 \end{array}\right),$$

because $0 + 1 \cdot (x+y)^{-1} \cdot 1 = (x+y)^{-1}$.

To simplify notation we will skip the lines determining block partitions of elements of R_Σ. For a definition of operations in R_Σ see [4, Sect. 7.11]. We just explain how to multiply and add matrices of a very special kind. First of all to multiply an element $t \in R_\Sigma$ by $s \in R$ on the right is the same as to multiply by s on the right the last column of the matrix representing t:

$$ts = \begin{pmatrix} p & r \\ A & q \end{pmatrix} \cdot s = \begin{pmatrix} p & rs \\ A & qs \end{pmatrix}.$$

Similarly to multiply by s on the left one should multiply on the left the first row of the representing matrix:

$$st = s \cdot \begin{pmatrix} p & r \\ A & q \end{pmatrix} = \begin{pmatrix} sp & sr \\ A & q \end{pmatrix}.$$

The elements with the same last (blocked) row or the same last (blocked) column can be added using standard rules:

$$s + t = \begin{pmatrix} p & r \\ A & q \end{pmatrix} + \begin{pmatrix} p' & r' \\ A & q \end{pmatrix} = \begin{pmatrix} p+p' & r+r' \\ A & q \end{pmatrix}$$

and

$$\begin{pmatrix} p & r \\ A & q \end{pmatrix} + \begin{pmatrix} p & r' \\ A & q' \end{pmatrix} = \begin{pmatrix} p & r+r' \\ A & q+q' \end{pmatrix}.$$

Furthermore, to add $s \in R$ to an element of R_Σ is just to add this element to the corresponding block:

$$\begin{pmatrix} p & r \\ A & q \end{pmatrix} + s = \begin{pmatrix} p & r+s \\ A & q \end{pmatrix}.$$

The following is a useful criterion for when an element of R_Σ is equal to zero. It is an essential simplification of the general criterion which is due to the fact that Σ is an independent set of matrices (see [11, Prop. 1] and [4, Prop. 11.14]).

Fact 4.2. An element $(*)$ of R_Σ is equal to zero if and only if there is a row b over R, a column c, and matrices $B, C \in \Sigma$ such that the following holds:

$$\left(\begin{array}{c|c} p & r \\ \hline A & q \end{array}\right) = \left(\begin{array}{c|} b \\ \hline B \end{array}\right) \cdot \left(\begin{array}{c|c} \hline C & c \end{array}\right) .$$

Thus an element of R_Σ is equal to zero if and only if its representing matrix is 'non-full'. We also need a tool to transform matrix representations of elements of R_Σ without changing their equivalence class. The obvious candidates are elementary transformations (but not all of them). For instance, we can take any left linear combination of rows of A and add it to the first row:

$$\begin{pmatrix} p & r \\ A & q \end{pmatrix} = \begin{pmatrix} sA+p & sq+r \\ A & q \end{pmatrix},$$

for any row s over R of an appropriate length. Indeed, the last element of R_Σ can be written as $sq+r-(sA+p)A^{-1}q = r - pA^{-1}q$. Similar operations are possible with columns of A whose right linear combination can be added to the last column:

$$\begin{pmatrix} p & r \\ A & q \end{pmatrix} = \begin{pmatrix} p & r+pt \\ A & q+At \end{pmatrix},$$

for any column t over R of an appropriate height.

We need one general result that will simplify the following calculations greatly. Roughly speaking it says that the the operations of taking the universal localization and passing to a factor ring commute.

Proposition 4.3. *Suppose that I is an ideal of (an arbitrary) ring R and let Σ be a set of square R-matrices. Denote by $\overline{\Sigma}$ the image of Σ with respect to the canonical projection $\pi : R \to R/I$. Then there is a natural isomorphism $(R/I)_{\overline{\Sigma}} \cong R_\Sigma / R_\Sigma \lambda(I) R_\Sigma$, where λ is the canonical morphism $R \to R_\Sigma$.*

Proof. Because $\lambda : R \to R_\Sigma$ sends I to $R_\Sigma \lambda(I) R_\Sigma$, it induces a map $\lambda_I : R/I \to R_\Sigma / R_\Sigma I R_\Sigma$ such that the upper square in the following diagram is commutative:

$$\begin{tikzcd}
R \arrow[r, "\lambda"] \arrow[d, "\pi"'] & R_\Sigma \arrow[d, "\pi_\Sigma"'] \arrow[ddr, dashed, "h"] & \\
R/I \arrow[r, "\lambda_I"] \arrow[drr, bend right, "f"'] & R_\Sigma / R_\Sigma \lambda(I) R_\Sigma \arrow[dr, dashed, "g"] & \\
 & & T
\end{tikzcd}$$

It suffices to prove that $\lambda_I : R/I \to R_\Sigma/R_\Sigma\lambda(I)R_\Sigma$ is a universal $\overline{\Sigma}$-inverting ring. Thus we have to show that for any $\overline{\Sigma}$-inverting map f from R/I to a ring T there is a unique morphism $g : R_\Sigma/R_\Sigma\lambda(I)R_\Sigma \to T$ such that $f = g\lambda_I$. Clearly the composition $f\pi : R \to T$ is Σ-inverting, hence there is a unique map $h : R_\Sigma \to T$ such that $f\pi = h\lambda$. Then h sends $\lambda(I)$ to zero, hence factors uniquely through π_Σ: there exists $g : R_\Sigma/R_\Sigma\lambda(I)R_\Sigma \to T$ such that $h = g\pi_\Sigma$. Now it is easily checked that $f = g\lambda_I$ and g is unique (because h is unique). □

5. Some calculations in R_Σ

In this section we get some insight into the structure of R_Σ.

Lemma 5.1. *$(y-1)R_\Sigma$ is a maximal two-sided ideal of R_Σ containing x such that $R_\Sigma/(y-1)R_\Sigma \cong k$.*

Proof. To prove that $(y-1)R_\Sigma$ is a two-sided ideal we have to check that, for every $t \in R_\Sigma$, there exists $s \in R_\Sigma$ such that $t(y-1) = (y-1)s$. This is clearly true for every $r \in R$, because y commutes with $y-1$, and $x = -(y-1)x$ is already in $(y-1)R_\Sigma$. As the next step let us consider $t = (x+y)^{-1}$. We have

$$(x+y)^{-1}(y-1) = \begin{pmatrix} -1 & 0 \\ x+y & 1 \end{pmatrix} \cdot (y-1) = \begin{pmatrix} -1 & 0 \\ x+y & y-1 \end{pmatrix} \xrightarrow{R_1+R_2}$$

$$\begin{pmatrix} (y-1)+x & y-1 \\ x+y & y-1 \end{pmatrix} = \begin{pmatrix} (y-1)-(y-1)x & y-1 \\ x+y & y-1 \end{pmatrix} =$$

$$(y-1) \cdot \begin{pmatrix} 1-x & 1 \\ x+y & y-1 \end{pmatrix} = (y-1)[1+(x-1)(x+y)^{-1}(y-1)]\,,$$

and the resulting equality $(x+y)^{-1}(y-1) = (y-1)[1+(x-1)(x+y)^{-1}(y-1)]$ can also be verified directly.

The general strategy is similar: we take an arbitrary element t of R_Σ, multiply it by $y-1$ on the right and using elementary transformations will try to factor $y-1$ on the left. We have

$$t(y-1) = \begin{pmatrix} p & r \\ A & q \end{pmatrix} \cdot (y-1) = \begin{pmatrix} p & r(y-1) \\ A & q(y-1) \end{pmatrix} = t'\,.$$

From $A \in \Sigma$ it follows that $A(0,1)$ is an invertible matrix over k. Thus, using a left linear combination of rows of A we can clear the 'free term' $p(0,1)$ of p, hence we may assume that t' is of the form

$$\begin{pmatrix} p' & r'(y-1) \\ A & q(y-1) \end{pmatrix},$$

where $p'(0,1) = 0$. Writing each entry of p' as a polynomial in x and $y-1$ we conclude that $p' = xs+(y-1)u$ for some rows s, u over R. From $x \in (y-1)R$ it follows that $p' = (y-1)p''$ for some row p'' over R. Furthermore, as we have already noticed, $r'(y-1) = (y-1)r''$ for some $r'' \in R$, hence $y-1$ can be factored from t' on the left.

Thus $(y-1)R_\Sigma$ is a two-sided ideal of R_Σ. Now we apply Proposition 4.3 to calculate the factor $R_\Sigma/(y-1)R_\Sigma$. For this we first should factor R by the ideal $(y-1)R$, hence set $y = 1$ and $x = 0$ (since $x \in (y-1)R$) getting a one dimensional vector space over k, and then localize with respect to $\overline{\Sigma}$. By the definition of R_Σ, all matrices in $\overline{\Sigma}$ are invertible, hence there is nothing to localize. □

By symmetry we obtain the following.

Lemma 5.2. *$R_\Sigma(x-1)$ is a maximal two-sided ideal of R_Σ containing y such that $R_\Sigma/R_\Sigma(x-1) \cong k$.*

Now we describe the Jacobson radical of R_Σ.

Proposition 5.3. *The Jacobson radical of R_Σ is generated by $1-x-y = (1-y)(1-x)$ as a two-sided ideal. Furthermore, $\mathrm{Jac}(R_\Sigma) = (y-1)R_\Sigma \cap R_\Sigma(x-1) = (y-1)R_\Sigma \cdot R_\Sigma(x-1)$ (the ordering in the latter product is essential).*

Proof. Since $1-x-y = (1-y)(1-x) \in \mathrm{Jac}(R_\Sigma)$, it suffices to prove that the factor $R_\Sigma/R_\Sigma(1-x-y)R_\Sigma$ is isomorphic to $k \oplus k$. To calculate this factor, by Proposition 4.3, we should first factor R by $J = R(1-x-y)R$ getting $R' = R/J$, and then localize. This is the same as to impose the relation $1-x-y = 0$ on R, hence to set $\bar{y} = \bar{1} - \bar{x}$ in R'. Then $yx = 0$ yields $\bar{x} \cdot (1-\bar{x}) = 0$, hence $\bar{x}$ is an idempotent in R'. Therefore R' is isomorphic to $k[\bar{x}]/\bar{x}^2 = \bar{x}$, hence to $k \oplus k$, via $1 \to (1,1)$ and $\bar{x} \to (0,1)$. Since $\bar{y}$ goes to $(1,0)$ under this map, all matrices from $\overline{\Sigma}$ are invertible in $R' = k \oplus k$.

By Fact 4.1, R_Σ has exactly two maximal ideals. Furthermore, looking at the corresponding factor rings it is easily seen that the maximal ideals $(y-1)R_\Sigma$ and $R_\Sigma(1-x)$ are incomparable. Thus they are the only maximal (one-sided and two-sided) ideals of R_Σ, and $\mathrm{Jac}(R_\Sigma)$ is equal to their intersection.

From $1-x-y = (1-y)(1-x) \in (y-1)R_\Sigma \cdot R_\Sigma(x-1)$ it follows that $\mathrm{Jac}(R_\Sigma) \subseteq (y-1)R_\Sigma \cdot R_\Sigma(x-1)$. On the other hand, $(y-1)R_\Sigma \cdot R_\Sigma(x-1) \subseteq (y-1)R_\Sigma \cap R_\Sigma(x-1) = \mathrm{Jac}(R_\Sigma)$. □

Now we are ready for the big picture.

Proposition 5.4. *Figure 1 shows the lattice of two-sided ideals of R_Σ above the two-sided ideal generated by xy.*

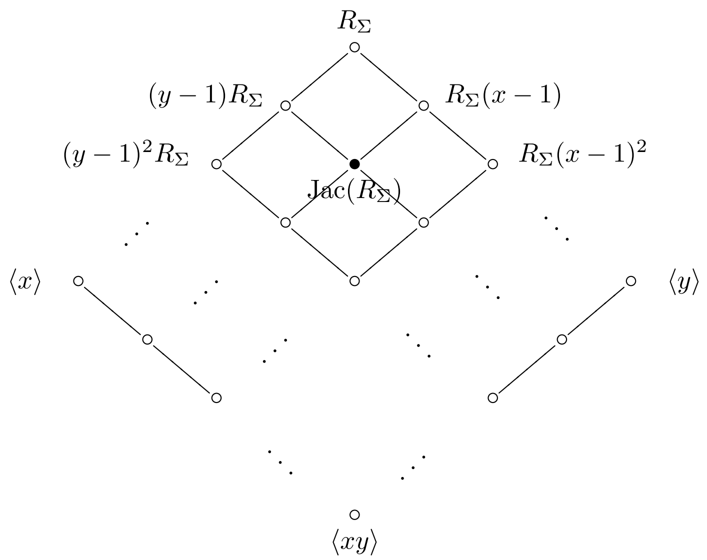
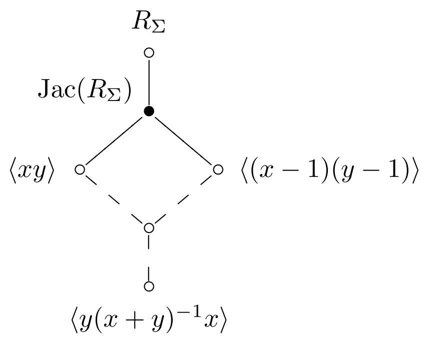


Figure 1

Proof. There is a one-to-one correspondence between two-sided ideals of R_Σ above $\langle xy\rangle$ and two-sided ideals of the factor ring $R_\Sigma/\langle xy\rangle$. By Proposition 4.3, we first factor R by $I = RxyR$, and then localize with respect to $\overline{\Sigma}$. Clearly $R' = R/I$ is isomorphic to the commutative ring $R' = k[x,y]/xy = 0$. Therefore to invert the matrices in $\overline{\Sigma}$ is the same as to invert their determinants. It readily follows that we should localize R' with respect to the complement of $(y-1)R' \cup (x-1)R'$, in particular $x+y$ is invertible in $R'_{\overline{\Sigma}}$. From $(x+y)^{-1}x^2 = x$ and $y^2(x+y)^{-1} = y$ it follows that $e = x(x+y)^{-1}$ and $f = y(x+y)^{-1}$ are orthogonal idempotents in $R'_{\overline{\Sigma}}$ such that $e+f = 1$, hence $R'_{\overline{\Sigma}}$ is a direct sum of two rings isomorphic to the localizations $k[y]_{(y-1)}$ and $k[x]_{(x-1)}$ (in fact, the former is isomorphic to $R_\Sigma/\langle x\rangle$, and the latter is isomorphic to $R_\Sigma/\langle y\rangle$). These rings are commutative valuation domains whose nonzero ideals are powers of a unique maximal ideal. □

But there are some two-sided ideals of R_Σ that do not fit into the above diagram. In the following lemma we calculate one of those ideals (or rather the corresponding factor of R_Σ). In particular it shows that the ordering of factors in the product $(y-1)R_\Sigma \cdot R_\Sigma(x-1)$ is essential.

Lemma 5.5. *Let* $J = \langle (x-1)(y-1)\rangle = R_\Sigma(x-1)\cdot(y-1)R_\Sigma$ *be the ideal of* R_Σ *generated by* $(x-1)(y-1)$*. Then* J *is a subspace of* $\mathrm{Jac}(R_\Sigma)$ *of codimension one such that the factor* R_Σ/J *is isomorphic to the ring of* 2×2 *upper triangular matrices over* k*. Furthermore* J *fits into the following diagram that complements Figure 1.*



(the dotted lines mean that we do not know the exact value of the intersection, it may be equal to $\langle y(x+y)^{-1}x\rangle$).

Proof. To calculate R_Σ/J, we first factor R by $I = R(x-1)(y-1)R$ and then localize with respect to $\overline{\Sigma}$. In R/I we have $1 = x+y-xy$ (we omit bars to simplify notations). Multiplying by x on the right we obtain $x = x^2$, hence x is an idempotent in R/I. Let $e_{11} = x$, $e_{12} = xy$ and $e_{22} = 1-x = y-xy$ (hence $y = e_{12}+e_{22}$). It is readily verified that the e_{ij} satisfy the usual identities for matrix units. For instance $e_{11}\cdot e_{12} = x\cdot xy = xy = e_{12}$ and $e_{12}\cdot e_{22} = xy(1-x) = xy = e_{12}$. It is also clear that R/I is a 3-dimensional vector space over k spanned by the e_{ij}, therefore it is isomorphic to $\left(\begin{smallmatrix} k & k \\ 0 & k\end{smallmatrix}\right)$. Since $xy \in \mathrm{Jac}(R/I)$, it is easily seen that all the matrices from $\overline{\Sigma}$ are invertible in R/I, hence $R_\Sigma/J \cong R/I$. Thus J has codimension 3 in R_Σ.

From $(x-1)(y-1) \in (y-1)R_\Sigma \cap R_\Sigma(x-1) = \mathrm{Jac}(R_\Sigma)$, we conclude that $\langle (x-1)(y-1)\rangle = R_\Sigma(x-1)\cdot(y-1)R_\Sigma$ is a subspace of $\mathrm{Jac}(R_\Sigma)$ of codimension $3-2=1$.

Since $R_\Sigma/\langle xy\rangle$ is a commutative ring and $yx = 0$ in R_Σ, it follows that $y(x+y)^{-1}x = 0$ in $R_\Sigma/\langle xy\rangle$, hence $\langle y(x+y)^{-1}x\rangle \subseteq \langle xy\rangle$. Furthermore, from $x = e_{11}$ and $y = e_{12}+e_{22}$ in $R_\Sigma/\langle (x-1)(y-1)\rangle$ we conclude that $(x+y)^{-1} = 1-e_{12}$, therefore $y(x+y)^{-1}x = (e_{12}+e_{22})(1-e_{12})e_{11} = 0$. Thus $y(x+y)^{-1}x \in \langle (x-1)(y-1)\rangle$ yields $\langle y(x+y)^{-1}x\rangle \subseteq \langle (x-1)(y-1)\rangle$.

Since $xy = e_{12} \neq 0$ in $R/\langle (x-1)(y-1)\rangle$ and $(x-1)(y-1) = 1-x-y \neq 0$ in $R/\langle xy\rangle$, the ideals $\langle (x-1)(y-1)\rangle$ and $\langle xy\rangle$ are incomparable, therefore both inclusions $\langle y(x+y)^{-1}x\rangle \subseteq \langle xy\rangle, \langle (x-1)(y-1)\rangle$ are proper. $\square$

Question 5.6. *Calculate* $\cap_{n=1}^{\infty} \operatorname{Jac}^n(R_\Sigma)$. *Is it equal to zero?*

Now we calculate annihilators in R_Σ of some elements of R. If $r \in R$ then $\operatorname{ann}(r)(R_\Sigma) = \{s \in R_\Sigma \mid rs = 0\}$ will denote the *right annihilator* of r in R_Σ, and similarly $\operatorname{ann}(R_\Sigma)(r) = \{t \in R_\Sigma \mid tr = 0\}$ is the *left annihilator* of r in R_Σ.

Lemma 5.7. *x is a right non-zero divisor in R_Σ and y is a left non-zero divisor in R_Σ.*

Proof. Suppose that $xt = 0$ for some $t \in R_\Sigma$. By Fact 4.2 we obtain

$$xt = x \cdot \begin{pmatrix} p & r \\ A & s \end{pmatrix} = \begin{pmatrix} xp & xr \\ A & s \end{pmatrix} = \begin{pmatrix} b \\ B \end{pmatrix} \cdot \begin{pmatrix} C & c \end{pmatrix}$$

for some $B, C \in \Sigma$. Plugging $x = 0$ into $xp = bC$ we obtain $0 = b(0, y)C(0, y)$. But $C \in \Sigma$ implies that $C(0, 1)$ is invertible over k, hence $C(0, y)$ is invertible in $k(y)$, the field of rational functions. It follows that $b(0, y) = 0$, hence (see Section 3 for explanations), $b = xb'$ for some row b' over R. Plugging this into $xp = bC$ and canceling by x (because x is a right non-zero divisor in R), we obtain $p = b'C$. Similarly $xr = bc$ yields $r = b'c$. Then

$$\begin{pmatrix} p & r \\ A & s \end{pmatrix} = \begin{pmatrix} b' \\ B \end{pmatrix} \cdot \begin{pmatrix} C & c \end{pmatrix}$$

shows that $t = 0$ in R_Σ (by Fact 4.2).

The second part of the statement follows by symmetry. □

Lemma 5.8. $\operatorname{ann}(y)(R_\Sigma) = xR_\Sigma$ *and* $\operatorname{ann}(R_\Sigma)(x) = R_\Sigma\, y$.

Proof. From $yx = 0$ it follows that $xR_\Sigma \subseteq \operatorname{ann}(y)(R_\Sigma)$. Suppose that $yt = 0$ for some $t \in R_\Sigma$, therefore

$$yt = y \cdot \begin{pmatrix} p & r \\ A & s \end{pmatrix} = \begin{pmatrix} yp & yr \\ A & s \end{pmatrix} = \begin{pmatrix} b \\ B \end{pmatrix} \cdot \begin{pmatrix} C & c \end{pmatrix}$$

for some $B, C \in \Sigma$. Plugging $y = 0$ into $yp = bC$ we obtain $0 = b(x, 0)C(x, 0)$. Since $C(x, 0)$ is invertible over $k(x)$ we conclude that $b(x, 0) = 0$, hence $b = b'y$ for some row b' over R (here we consider y as a diagonal matrix with all diagonal entries equal to y). Then $yp = bC$ can be rewritten as $yp(0, y) = b'yC(0, y) = b'C(0, y)y$. Since y is a left non-zero divisor in R_Σ, we can cancel by y getting $p(0, y) = b'C(0, y)$.

Similarly from the equality $yr = bc$ we deduce that $r(0, y) = b'C(0, y)$. Subtracting

$$\begin{pmatrix} b' \\ B \end{pmatrix} \cdot \begin{pmatrix} C & c \end{pmatrix} = \begin{pmatrix} b'C & b'c \\ A & s \end{pmatrix}$$

(that is, the zero of R_Σ) from t we obtain

$$t = \begin{pmatrix} p & r \\ A & s \end{pmatrix} = \begin{pmatrix} p & r \\ A & s \end{pmatrix} - \begin{pmatrix} b'C & b'c \\ A & s \end{pmatrix} = \begin{pmatrix} p - b'C & r - b'c \\ A & s \end{pmatrix}.$$

Writing $b' = b'(0,y)+xb''$ for some row b'' over R, we obtain $(p-b'C)(0,y) = p(0,y)-b'(0,y)C(0,y) = p(0,y)-b'C(0,y)+xb''C(0,y) = xb''C(0,y) \in xR$, hence $p - b'C \in xR$, and similarly $r - b'c \in xR$. Thus in the above matrix representation of t we can factor x out on the left, hence $t \in xR_\Sigma$.

The second statement follows by symmetry. □

It is not difficult to improve this lemma by calculating the annihilator of any $r \in R$. Namely, if $r = r'y^n$ and $r'(x,0) \neq 0$, then $\operatorname{ann}(r)(R_\Sigma) = x^n R_\Sigma$, if $n \geq 1$, and zero otherwise; and similarly for left annihilators.

Now we will make some guess on the global dimension of R_Σ.

Lemma 5.9. *yR_Σ is not flat as a right R_Σ-module and $R_\Sigma x$ is not flat as a left R_Σ-module. In particular, both global and weak dimension of R_Σ is at least* 2.

Proof. By Lemma 5.8, $yR_\Sigma \cong R_\Sigma/xR_\Sigma$ is a finitely presented right R_Σ-module. If yR_Σ were flat, then (see [14, Thm. 4.30]) it would be projective, hence xR_Σ would be generated by a nontrivial idempotent. But (see Corollary 6.4 below) R_Σ has no nontrivial idempotents. Thus yR_Σ is not flat as a right R_Σ-module, hence (since $xR_\Sigma \cong R_\Sigma$) both flat and projective dimensions R_Σ/yR_Σ are equal to 2. The rest of the statement follows by symmetry. □

Note that Lemmas 5.7, 5.8 provide some support for the following conjecture.

Conjecture 5.10. *R_Σ is a coherent ring of global dimension* 2.

Of course it would be easy to calculate the global dimension of R_Σ if it were flat as a right or left R-module. Unfortunately this is not the case.

Proposition 5.11. *R_Σ is not flat as a left or right R-module.*

Proof. It is well known that every universal localization $R \to R_\Sigma$ is an epimorphism in the category of rings. Let $t = y(x+y)^{-1}x = x(x+y)^{-1}y \in R_\Sigma$ and let $I = \{s \in R \mid ts \in R\}$ be the right ideal of R. If R_Σ were flat as a right R-module, then [21, Thm. 11.2.1] would imply that $IR_\Sigma = R_\Sigma$. We will show that $I = xR_\Sigma$ getting a contradiction.

Clearly $tx = x(x+y)^{-1}yx = 0$ implies $x \in I$. Suppose that $ts = x(x+y)^{-1}ys = r \in R$ for some $s \in R$. Since $ys = ys(0,y)$ and $s - s(0,y) \in$

$xR \subseteq I$, we may assume that $s = s(0, y)$ is a polynomial in y. We have

$$\begin{pmatrix} x & 0 \\ x+y & y \end{pmatrix} \cdot s + r = \begin{pmatrix} x & r \\ x+y & ys \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \cdot \begin{pmatrix} c & d \end{pmatrix}$$

for some $b, c \in \Sigma$ (see Fact 4.2). By plugging $x = 0$ into $x = ac$ we obtain $0 = a(0, y)c(0, y)$, hence $a(0, y) = 0$ (because $c \in \Sigma$). It follows that $a = xa'$ for some $a' \in R$, therefore $x = xa'c$ yields $1 = a'c$. This implies that $0 \neq c(x, 0) = \alpha \in k$. Taking $y = 0$ in $x+y = bc$ we obtain $x = b(x, 0)c(x, 0)$, hence $b(x, 0) = \alpha^{-1}x$.

If $s \neq 0$, then write $s = y^k s'$ for some $k \geq 0$ such that $s'(0) = \beta$ for some $0 \neq \beta \in k$ (recall that s is a polynomial in y). Plugging $y = 0$ into $ys = bd$ we obtain $0 = b(x, 0)d(x, 0)$. Since $b \in \Sigma$ it follows that $d(x, 0) = 0$, hence $d = d'y$ for some $d' \in R$; and then $ys = bd'y$ implies $s = bd'$. Continuing this way (if $k > 0$) we eventually obtain $s' = bd''$ for some $d'' \in R$. Then $0 \neq \beta = s'(0) = b(x, 0)d''(x, 0) = \alpha^{-1}x\, d''(x, 0)$, a contradiction.

The proof that R_Σ is not flat as a left R-module is similar. □

6. Projective modules over R_Σ

Recall from the previous sections that $yx = 0$, $1 - x - y \in \operatorname{Jac}(R_\Sigma)$, hence $u = x + y$ is a unit and $y(x + y)^{-1}x \neq 0$. If $r_i = u^{-i-1}xu^i$ then (see Lemma 2.6) $P = \cup_{i=0}^{\infty} r_i R_\Sigma$ is a non-finitely generated projective right R_Σ-module such that $P/\operatorname{Jac}(P)$ is a cyclic $R_\Sigma/\operatorname{Jac}(R_\Sigma)$-module generated by $\bar{x}$, and the trace of P is equal to $\langle x \rangle$. Furthermore P is isomorphic to the direct limit of the following chain of morphisms: $R_\Sigma \xrightarrow{x\times -} R_\Sigma \xrightarrow{x\times -} R_\Sigma \xrightarrow{x\times -} \dots$.

Suppose that M is an arbitrary right R_Σ-module. Define the dimension of M, $\dim(M)$, as a pair of cardinals $(\alpha(M), \beta(M))$, where $\alpha(M)$ is the dimension of $M/M \cdot R_\Sigma(x-1)$ as a vector space over $R_\Sigma/R_\Sigma(x-1) \cong k$ and $\beta(M)$ is the dimension of $M/M(y-1)R_\Sigma$ as a vector space over $R_\Sigma/(y-1)R_\Sigma \cong k$. We will be mostly interested in the dimensions of right projective R_Σ-modules, so let us give some examples.

Remark 6.1. $\dim(P) = (1, 0)$ *and* $\dim(R_\Sigma) = (1, 1)$.

Proof. From $\operatorname{Tr}(P) = \langle x \rangle \subseteq (y-1)R_\Sigma$ it follows that $P = P(y-1)R_\Sigma$, hence $\beta(P) = 0$. Furthermore, since $\operatorname{Jac}(R_\Sigma) = (y-1)R_\Sigma \cap R_\Sigma(x-1)$, we conclude that $\operatorname{Jac}(P) = P \cdot \operatorname{Jac}(R_\Sigma) = P(y-1)R_\Sigma \cap PR_\Sigma(x-1) = PR_\Sigma(x-1)$, hence $PR_\Sigma(x-1)$ is a proper submodule of P, and then $\alpha(P) \neq 0$. Because $P/\operatorname{Jac}(P)$ is a cyclic $R_\Sigma/\operatorname{Jac}(R_\Sigma)$-module, therefore $\alpha(P) = 1$.

It remains to notice that $R_\Sigma/\operatorname{Jac}(R_\Sigma) \cong k \oplus k$, hence $\dim(R_\Sigma) = (1, 1)$. □

The following fact is a consequence of a general result by Příhoda [17] saying that over semilocal rings projective modules are uniquely determined by their dimensions.

Fact 6.2. Suppose that P' and Q are projective right R_Σ-modules. Then $\dim(P') = \dim(Q)$ implies $P' \cong Q$ (and vice versa).

The following arguments are well known (see [19] or [17]), but because of their striking simplicity it is worthwhile to repeat them. As we will see, some non-finitely generated projective modules are strict supervisors of finitely generated ones!

Corollary 6.3. *Every finitely generated projective right R_Σ-module is free of a unique rank. Thus R_Σ is a projective-free ring.*

Proof. Suppose that Q is a finitely generated projective right R_Σ-module, hence $\dim(Q) = (m, n)$ for finite m and n. If $m = n = 0$, then P is a zero module (because both have the same dimension). Otherwise assume first that $m \geq n$, $m \neq 0$. From $\dim(R_\Sigma) = (1, 1)$, using projective covers (see [13, p. 350]), it follows that $Q \cong P' \oplus R_\Sigma^n$, hence $\dim(P') = (m - n, 0)$. If $m - n = 0$, then $P' = 0$, hence $Q \cong R_\Sigma^n$. Otherwise $\dim(P') = \dim(P^{m-n})$, therefore $P' \cong P^{n-m}$ by Fact 6.2. But P' is finitely generated, and P is not, a contradiction.

Now suppose that $m < n$. Comparing dimensions, we obtain $P^{n-m} \oplus Q \cong R_\Sigma^n$, therefore P is finitely generated, a contradiction.

Thus every finitely generated projective R_Σ-module is free. The uniqueness of the rank follows from the fact that R_Σ has dual Goldie dimension 2 (see [7, Sect. 2.8] for definition of dual Goldie dimension). □

In particular, every finitely generated projective R_Σ-module has dimension vector (n, n), where n is its rank as a free module. An easy corollary is that R_Σ has no nontrivial idempotents (and it is difficult to imagine how to prove this corollary otherwise).

Corollary 6.4. *R_Σ has no nontrivial idempotents.*

Proof. Suppose that e is a nontrivial idempotent of R_Σ. Then $R_\Sigma = eR_\Sigma \oplus (1 - e)R_\Sigma$ is a proper direct sum decomposition of R_Σ. Since $\dim(R_\Sigma) = (1, 1)$, it follows that $\dim(eR_\Sigma)$ must be $(1, 0)$ or $(0, 1)$, a contradiction. □

Now we would like to classify projective right R_Σ-modules and we know that they are determined by their dimensions. The following is a standard way to reduce this classification to one particular case (again see [19] and [17]). Let Q be a projective right R-module. Since by Kaplansky's theorem Q is a direct sum of countably generated modules, we may assume that Q is countably generated. If $\dim(Q) = (\omega, \omega)$, comparing dimensions, we see that $Q \cong R_\Sigma^{(\omega)}$ is a free module of (an infinite) countable rank. Otherwise we may assume that $\alpha(Q)$ or $\beta(Q)$ is finite. Subtracting finitely many copies of R_Σ (that is, using projective covers) we may further assume that $\alpha(Q) = 0$ or $\beta(Q) = 0$. If $\beta(P) = 0$, then, comparing dimensions, we see that $Q \cong P^{(\alpha)}$, whether α is finite or not.

It remains to consider the case when $\dim(Q) = (0, \beta)$. If $\beta = n$ is finite, then $P^k \oplus Q \cong R_\Sigma^n$, hence P is finitely generated, a contradiction. Thus we are left with the case $\dim(Q) = (0, \omega)$. Comparing dimensions, we see that $P^{(\omega)} \oplus Q \cong R_\Sigma^{(\omega)}$. We will give an indirect proof that R_Σ possesses no such projective Q. But first we should collect some information about idempotent ideals of R_Σ.

Recall that an ideal I of a ring is said to be *idempotent* if $I^2 = I$.

Lemma 6.5. *Suppose that I is a (proper) idempotent ideal of R_Σ. Then either $I = \langle x\rangle$, or $I = \langle y\rangle$, or $I \subseteq \mathrm{Jac}(R_\Sigma)$.*

Proof. Since $x = (x + y)^{-1}x^2$ and $y = y^2(x + y)^{-1}$, both ideals $\langle x\rangle$ and $\langle y\rangle$ are idempotent. Suppose that I is an idempotent ideal which is not a subset of $\mathrm{Jac}(R_\Sigma)$. Since (by Proposition 5.3) $\mathrm{Jac}(R_\Sigma) = (y - 1)R_\Sigma \cap R_\Sigma\,(x - 1)$, by symmetry we may assume that I is not contained in $(y - 1)R_\Sigma$, and we prove that $I = \langle y\rangle$. Since $(y-1)R_\Sigma$ is a maximal right ideal, it follows that $(y - 1)R_\Sigma + I = R_\Sigma$, hence $(y - 1)s + t = 1$ for some $s \in R_\Sigma$ and $t \in I$. In R_Σ/I we obtain the equality $\overline{y-1} \cdot \bar{s} = \bar{1}$, hence $\overline{y-1}$ is invertible (on the right, therefore on the left, since R_Σ/I is a semilocal ring). Furthermore, since $1 - x - y \in \mathrm{Jac}(R_\Sigma)$, it follows that $\bar{1} - \bar{x} - \bar{y} \in \mathrm{Jac}(R_\Sigma/I)$, therefore $\bar{x} = (\bar{1} - \bar{y}) - (\bar{1} - \bar{x} - \bar{y})$ is also invertible in R_Σ/I. Then $\bar{y}\bar{x} = 0$ yields $\bar{y} = 0$, hence $y \in I$. Since I is idempotent and proper, looking at Figure 1, we conclude that $I = \langle y\rangle$. □

Question 6.6. *Does there exist a nonzero idempotent ideal $I \subseteq \mathrm{Jac}(R_\Sigma)$?*

Recall that the trace of the projective module P is $\langle x\rangle$. We will consider possible values for $J = \mathrm{Tr}(Q)$, where Q is a(n imaginary) projective R_Σ-module with $\dim(Q) = (0, \omega)$. By general theory (see Section 2) we know that J is an idempotent ideal such that $QJ = Q$ and J is not a subset of $\mathrm{Jac}(R_\Sigma)$. Since Q is not a generator (because $\alpha(Q)$=0), Lemma 6.5 implies that $J = \langle x\rangle$ or $J = \langle y\rangle$. The first possibility would lead to $Q = Q\langle x\rangle$, therefore $Q = Q(y - 1)R_\Sigma$ yielding $\beta(P) = 0$, a contradiction. Thus we must have $\mathrm{Tr}(Q) = \langle y\rangle$, in particular $Q(x - 1) = Q$. We will show, that it is not possible by specializing R_Σ in a ring S.

The ring S was constructed (in [18]) as the endomorphism ring of a uniserial module M. In fact the construction could start from any field k thus making S into an k-algebra. For a detailed analysis of properties of S see [19] and [6]. First of all (this is true for endomorphism rings of most uniserial modules, see [7, Sect. 9.1]) S has exactly two maximal (two-sided and one-sided) ideals: I, consisting of non-monomorphisms, and K consisting of non-epimorphisms. In particular, $\mathrm{Jac}(S) = I \cap K$ consists of endomorphisms of M that are neither mono nor epi. Furthermore, I is a principal right ideal (generated by any $g \in I \setminus K$), and K is a principal left ideal generated by any $f \in K \setminus I$. Also S is a projective-free ring, K is a non-finitely generated projective right module with trace K and dimension vector $(1, 0)$; and every projective right S-module is a direct sum of copies of K_S and S_S. We also need a pair of elements $f, g \in S$ satisfying the following properties: $f, 1-g \in K \setminus I$ (that is, they are mono not epi), $g, 1-f \in I \setminus K$ (that is, they are epi not mono), and also $gf = 0$ (the pair constructed in [6] has the additional property $\mathrm{im}(f) = \ker(g)$, but we do need to be so precise). It readily follows that $1 - f - g \in \mathrm{Jac}(S)$.

Now we are ready to construct a specialization. Define a map $\beta : R \to S$ by sending y to g, x to f and extending it by k-linearity. Since $gf = 0$ in S, this map is correctly defined.

Proposition 6.7. *β inverts all matrices in Σ, hence is uniquely extended to a morphism $\bar{\beta} : R_\Sigma \to S$.*

$$\begin{tikzcd} R \arrow[r, "\lambda"] \arrow[d, "\beta"'] & R_\Sigma \arrow[dl, "\bar{\beta}"] \\ S & \end{tikzcd}$$

Proof. Suppose that $A \in \Sigma$ and we have to prove that $\beta(A)$ is an invertible matrix over S. It is well known that a matrix over a ring is invertible if and only if it is invertible modulo the Jacobson radical. In our case $\mathrm{Jac}(S) = I \cap K$, so it suffices to prove the invertibility modulo I and modulo K. To do this modulo I recall that $I = gS$ and $1 - f - g \in \mathrm{Jac}(S) \subseteq I$. Thus calculating the inverse of $\beta(A)$ in S/I we may assume that $g = 0$ and $f = 1$ (since $1 - f - g = 0$ in S/I). Since y goes to g and x goes to f this is the same as to first substitute $y = 0$ and $x = 1$ in A getting $A(1, 0)$ and then take to the image of this matrix in S/I. Because $A \in \Sigma$, the matrix $A(1, 0)$ is invertible over k, hence its image is invertible in S/I.

Similarly, the image of A in S/K is invertible, because $A(0, 1)$ is invertible. $\square$

Now we are in a position to complete the classification of projective R_Σ-modules. Recall that P is a non-finitely generated projective right R_Σ-module of dimension $(1, 0)$.

Theorem 6.8. *Every projective right R_Σ-module is a direct sum of copies of P and R_Σ.*

Proof. By what we have already said it suffices to show that there exists no projective right R_Σ-module Q of dimension $(0, \omega)$. If such a module Q exists, then $Q = Q(x-1)$. Consider the induced (via $\bar{\beta}$) projective right S-module $Q' = Q \otimes_{R_\Sigma} S$. We claim that $Q'I = I$, hence $\mathrm{Tr}(Q') \subseteq I$. Namely, if $m = q \otimes s \in Q'$, then $q = q'(x-1)$ for some $q' \in Q$, hence $m = q'(x-1) \otimes s = q' \otimes (f-1)s \in Q'I$, because $f - 1 \in I$. But, by the classification of projective right S-modules, every such (nonzero) module has either K or S as its trace.

It follows that $Q' = 0$. Recall (see after Corollary 6.4) that $P^{(\omega)} \oplus Q \cong R_\Sigma^{(\omega)}$. Tensoring by S on the right we obtain $P'^{(\omega)} \cong S^{(\omega)}$, where $P' = P \otimes_{R_\Sigma} S$. But $P = P\langle x \rangle$ implies $P' = P'f$, hence $Sf = S$, a contradiction. □

Note that a similar classification takes place for projective left R_Σ-modules. Namely, by symmetry (see a remark after Corollary 2.6), there is a non-finitely generated projective left R_Σ-module Q of dimension $(0, 1)$ whose trace is generated by y. Then (by symmetric arguments) every projective left R_Σ-module is isomorphic to a direct sum of copies of Q and R_Σ.

Let us turn back to the map $\bar{\beta} : R_\Sigma \to S$ and try to say something about the image S' of this map. The reader is referred to [4, Sect. 7.1] for the definitions of the rational closure, and the division closure. Recall also, that a morphism of rings is said to be *local* if non-units are sent to non-units, and *matrix local*, if all induced maps of $n \times n$ matrix rings are also local. For an example of a local morphism which is not matrix local see [8, p. 189].

Proposition 6.9. $\ker(\bar{\beta}) \subseteq \mathrm{Jac}(R_\Sigma)$, *hence $S' = \bar{\beta}(R_\Sigma)$ is a semilocal ring with two maximal (two-sided and one-sided) ideals. Furthermore, S' is the division closure of $1, f$ and g in S.*

Proof. It follows from the proof of Proposition 6.7 that a square matrix A over R has an invertible image $\beta(A)$ in S if and only if $A \in \Sigma$. Thus $S' = \bar{\beta}(R_\Sigma)$ is the rational closure of R in S as defined in [4, p. 382], that is, S' consists of entries of inverses of matrices $\beta(A)$, $A \in \Sigma$.

By standard arguments using Cramer's rule (see [8, Thm. 3.3]) we conclude that $\bar{\beta}$ is a matrix local morphism (that is, every R_Σ-matrix whose image is invertible, is already invertible in R_Σ). In particular (see [8, L. 3.1])

$\ker(\bar{\beta}) \subseteq \mathrm{Jac}(R_\Sigma)$, therefore S' is a semilocal ring with exactly two maximal ideals. Furthermore, since $\bar{\beta}$ is a matrix local morphism, the inclusion $S' \subset S$ is matrix local.

Let S'' denote the division closure of $\beta(R)$ in S, that is, the smallest subalgebra of S containing $\beta(R)$ and closed under taking inverses in S. Thus S'' consists of 'rational functions' of $1, f$ and g in S, of which $g(f+g)^{-1}f$ is a particular example.

Clearly $S'' \subseteq S'$ and we prove that $S'' = S'$. It is obvious that the inclusion $S'' \subseteq S$ is local. Since $S/\mathrm{Jac}(S)$ is a direct sum of two skew fields, then [9, Prop. 2.5] implies that this inclusion is matrix local. By the description of R_Σ (or see [4, Thm. 7.1.2 d)]) every element of S' can be written as $\beta(p)\beta(A)^{-1}\beta(q)$, where p is a row over R, q is a column over R and $A \in \Sigma$. Since $S'' \subseteq S$ is a matrix local inclusion, the entries of all matrices $\beta(A)^{-1}$, $A \in \Sigma$ belong to S'', hence $S'' = S'$. □

It is essentially easier to calculate in S than in R_Σ. However to take a real advantage of this we should address the following question.

Question 6.10. *Is $\bar{\beta} : R_\Sigma \to S'$ an isomorphism? Equivalently, is $\bar{\beta}$ an embedding?*

Unfortunately we do not know the answer to this question even for the particular choice of f and g, as in [6, Exam. 4.3].

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N. Dubrovin,
Department of Mathematics, Vladimir State University, Gorky Street 87, Vladimir, 600000, Russia, E-mail: ndubrovin@rambler.ru

P. Příhoda,
Centre de Recerca Matemàtica, Apartat 50, 08193, Bellaterra (Barcelona), Spain, E-mail: pprihoda@crm.cat

G. Puninski,
School of Mathematics, Manchester University, Lamb Building, Booth Street East, Manchester, M13 9PL, UK, E-mail: gpuninski@maths.man.ac.uk