STRONGLY NON-DEGENERATE LIE ALGEBRAS

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ABSTRACT. Let A be a semiprime 2 and 3-torsion free non-commutative associative algebra. We show that the Lie algebra $\mathcal{D}\mathrm{er}(A)$ of (associative) derivations of A is strongly non-degenerate, which is a strong form of semiprimeness for Lie algebras, under some additional restrictions on the center of A. This result follows from a description of the quadratic annihilator of a general Lie algebra inside appropriate Lie overalgebras. Similar results are obtained for an associative algebra A with involution and the Lie algebra $\mathrm{SDer}(A)$ of involution preserving derivations of A.

INTRODUCTION

This paper is concerned with the structure of the Lie algebra $\mathcal{D}\mathrm{er}(A)$ of associative derivations of an associative algebra A, which we will also assume to be 2 and 3-torsion free. It was proved in [10, Theorem 4 and Theorem 2] that if A is semiprime (respectively prime) then $\mathcal{D}\mathrm{er}(A)$ is a semiprime (respectively, prime) Lie algebra. We prove below that, if A is prime, this result can be strengthened to show that in fact $\mathcal{D}\mathrm{er}(A)$ is strongly non-degenerate (see below for the precise definitions).

The key result in the paper is Theorem 2.1 and has a technical flavour. Let L be a subalgebra of a Lie algebra Q. The quadratic annihilator of L inside Q is defined as the set $\{q \in Q \mid [q, [q, L]] = 0\}$. Roughly speaking, Theorem 2.1 allows to obtain non-zero elements in the quadratic annihilator of L in itself from non-zero elements in the quadratic annihilator of L inside Q whenever Q is a weak quotient algebra of L, i.e., $[L, q] \neq 0$ for every non-zero $q \in Q$. If L is strongly non-degenerate, then the quadratic annihilator

of L inside Q coincides with the annihilator of L in Q, and both are zero (Theorem 2.2 (ii)).

Another application of Theorem 2.1 leads to the proof of the fact that, if a Lie algebra L contains an essential ideal which is strongly non-degenerate, then the algebra L is itself strongly non-degenerate (Proposition 2.3 (ii)). This fact was already proved by Zelmanov in [17] by making use of the Kostrikin radical, while our proof is based on elements.

According to the above, and in order to obtain the result announced in the abstract (Theorem 2.5) we need to produce an essential ideal inside $\mathcal{D}\mathrm{er}(A)$ which is strongly non-degenerate. The natural candidate for this is the ideal $\mathrm{Inn}(A)$ of the so-called inner derivations of A, which can be identified with the quotient $A/Z(A)$, known to be strongly non-degenerate under appropriate mild hypotheses. However, this ideal might fail to be essential, and this is somehow measured by the ideal I_Z of those derivations that map A into the center $Z(A)$. Our result then asserts that $\mathcal{D}\mathrm{er}(A)/I_Z$ is strongly non-degenerate. In the particular case that the center $Z(A)$ of A does not contain associative ideals (e.g. if A is prime), one has that $I_Z = 0$, and then we do obtain that $\mathcal{D}\mathrm{er}(A)$ is strongly non-degenerate.

Our arguments can be subsequently adjusted with some extra effort to the case of a $*$-semiprime algebra A and the Lie algebra $\mathrm{SDer}(A)$ of those (associative) derivations of A that commute with the involution, that is, those $\delta \in \mathcal{D}\mathrm{er}(A)$ such that $\delta(a^*) = (\delta(a))^*$ for every $a \in A$ (Theorem 2.8).

1. Notation and preliminaries

Let Φ be a unital commutative ring. All algebras in this paper, associative or not, will be Φ-modules. Recall that a *Lie algebra* over Φ is a Φ-module L, together with a bilinear map $[\ ,\ ] \colon L \times L \to L$, denoted by $(x, y) \mapsto [x, y]$ and called the *bracket* of x and y such that the following axioms are satisfied:

(i) $[x, x] = 0$,

(ii) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ (*Jacobi identity*),

for every x, y, z in L.

The standard example is obtained by considering a (non-necessary unital) associative algebra A, with its same module structure and bracket given by $[x, y] = xy - yx$. Sometimes the notation A^- is used in order to emphasize the Lie structure of A.

Given an element x of a Lie algebra L, we may define a map $\mathrm{ad}\ x \colon L \to L$ by $\mathrm{ad}\ x(y) = [x, y]$ (which is a derivation of the Lie algebra L). We shall denote by $A(L)$ the associative subalgebra (possibly without identity) of $\mathrm{End}(L)$ generated by the elements $\mathrm{ad}\ x$ for x in L.

An element x in a Lie algebra L is an *absolute zero divisor* if $(\operatorname{ad} x)^2 = 0$. This is equivalent to saying that $[x, [x, L]] = 0$. The algebra L is said to be *strongly non-degenerate* (according to Kostrikin) if it does not contain non-zero absolute zero divisors.

Given a Lie algebra L, we say that L is *semiprime* if we have $I^2 \neq 0$ whenever I is a non-zero ideal. It is obvious from the definitions that strongly non-degenerate Lie algebras are semiprime, but the converse does not hold (see [15, Remark 1.1])

Next, L is said to be *prime* if $[I, J] \neq 0$ for any pair of non-zero ideals I, J of L. An ideal I of L is said to be *essential* if its intersection with any non-zero ideal is again a non-zero ideal.

For two subsets X, Y of a (Lie) algebra L we define the *annihilator of Y in X* as the set

$$\operatorname{Ann}_X(Y) := \{x \in X \mid [x, Y] = 0\},$$

and the *quadratic annihilator of Y in X* to be the set

$$\operatorname{QAnn}_X(Y) := \{x \in X \mid [x, [x, Y]] = 0\}.$$

When $X = L$, we write $\operatorname{Ann}(Y)$ or $\operatorname{Ann}_L(Y)$ (if no confusion may arise) and refer to it as the annihilator of Y. If $X = Y = L$, then $\operatorname{Ann}(L)$ is called the *centre* of L and usually denoted by $Z(L)$. In the case that $L = A^-$ for an associative algebra A, then $Z(A^-)$ agrees with the associative center Z of A. It is easy to check (by using the Jacobi identity) that $\operatorname{Ann}(X)$ is an ideal of L whenever X is an ideal of L. Therefore, for A associative we can form the Lie algebra A^-/Z. We will be primarily interested in this type of Lie algebras, and in Lie algebras that arise from associative algebras with involution. If A is associative and has an involution $*$, then the set of its *s*kew elements

$$K = K_A = \{x \in A \mid x^* = -x\}$$

is a subalgebra of A^-. The center $Z(K)$ of the Lie algebra K will be for brevity denoted by Z_K, and we will be interested in the Lie algebra K/Z_K.

The notion of the quadratic annihilator of an (arbitrary) algebra – defined in a similar way as we have done for Lie algebras – plays an important role: see, for example, Smirnov's paper [16]. Let us remark here that the quadratic annihilator need not be closed under sums in the case of an associative product (for an example, see [16]). The same phenomenon occurs in the Lie context, as is shown in the examples below.

Examples 1.1. (1) Consider a field F and the Lie algebra $L = \mathfrak{t}(3, F)$ (here we follow the terminology in [8]). Then $\operatorname{QAnn}(L) = \{a(e_{11} + e_{22} + e_{33}) + be_{13} + ce_{23} \mid a, b, c \in F\} \cup \{a(e_{11} + e_{22} + e_{33}) + be_{12} + ce_{13} \mid a, b, c \in F\}$, where, as usual, e_{ij} denotes the matrix in $\mathbb{M}_3(F)$ whose entries are all zero

except for the one in row i and column j. Then $\mathrm{QAnn}(L)$ is not closed under sums.

(2) Now, for L as before, consider the Lie algebra $\overline{L} := L/Z$. Then

$$\mathrm{QAnn}(\overline{L}) = \{\overline{ae_{13} + be_{23}} \mid a, b \in F\} \cup \{\overline{ae_{12} + be_{13}} \mid a, b \in F\},$$

where $\overline{x}$ denotes the class of an element x in L. Again we have that the quadratic annihilator of this algebra $\overline{L}$ is not closed under sums.

Let $L \subseteq Q$ be Lie algebras. When $[L, q] \neq 0$ for every non-zero $q \in Q$ we say that Q is a *weak algebra of quotients of* L (see [15]). The notion of algebra of quotients of an algebra (associative or not necessarily associative) has a long history and is an active research area, specially in recent years, following its development in the Lie and Jordan contexts. In the germ paper [15] the second author initiated the study of algebras of quotients of Lie algebras, by adapting some ideas from the associative and also Jordan ([13]) contexts. She introduced the notion of a general (abstract) algebra of quotients of a Lie algebra, and also the notion of the maximal algebra of quotients $Q_m(L)$ of a semiprime Lie algebra L. Follow up results can be found in [14, 4, 2].

Let B be a subalgebra of an associative algebra A. A linear map $\delta : B \to A$ is called a *derivation* if $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in B$. By a derivation of A we simply mean a derivation from A into A. Let $\mathcal{D}\mathrm{er}(A)$ denote the set of all derivations of A. It is clear that $\mathcal{D}\mathrm{er}(A)$ becomes a Φ-module under natural operations and it also becomes a Lie algebra by putting $[\delta, \mu] = \delta\mu - \mu\delta$, for every δ, μ in $\mathcal{D}\mathrm{er}(A)$. Any element x of A determines a map $\mathrm{ad}\, x \colon A \to A$ defined by $\mathrm{ad}\, x(y) = [x, y]$, which is a derivation of A. For every Lie ideal U of A, the restriction of the map $\mathrm{ad} : A \to \mathcal{D}\mathrm{er}(A)$ to U,

$$\begin{array}{ccc} U & \to & \mathcal{D}\mathrm{er}(A) \\ y & \mapsto & \mathrm{ad}\, y \end{array}$$

defines a Lie algebra homomorphism with kernel $\mathrm{Ann}(U)$, which allows us to identify $U/\mathrm{Ann}(U)$ with the subalgebra $\mathrm{ad}\,(U)$ of $\mathcal{D}\mathrm{er}(A)$. For any $y \in U$ and $\delta \in \mathcal{D}\mathrm{er}(A)$, $[\delta, \mathrm{ad}\, y] = \mathrm{ad}\, \delta(y)$, hence $\mathrm{ad}\,(U)$ is an ideal of $\mathcal{D}\mathrm{er}(A)$ whenever $\delta(U) \subseteq U$ for every $\delta \in \mathcal{D}\mathrm{er}(A)$. The ideal $\mathrm{ad}\,(A)$ of $\mathcal{D}\mathrm{er}(A)$ is usually denoted by $\mathrm{Inn}(A)$ and its elements are called *inner derivations* of A. Note that $A^-/Z \cong \mathrm{Inn}(A)$.

Let now A be an associative algebra with involution $*$. The set

$$\mathrm{SDer}(A) = \{\delta \in \mathcal{D}\mathrm{er}(A) \mid \delta(x^*) = \delta(x)^* \text{ for all } x \in A\}$$

is a Lie subalgebra of $\mathcal{D}\mathrm{er}(A)$. Denote by $\mathrm{ad}\,(K)$ the set of Lie derivations $\mathrm{ad}\, x : A \to A$ with x in K.

In what follows we will assume that 2 and 3 are invertible elements in Φ.

2. The results

By an *extension of Lie algebras* $L \subseteq Q$ we will mean that L is a (Lie) subalgebra of the Lie algebra Q. Let $L \subseteq Q$ be an extension of Lie algebras and let $A_Q(L)$ be the associative subalgebra of $A(Q)$ generated by $\{\text{ad } x : x \in L\}$.

For an extension $L \subseteq Q$ of Lie algebras, the condition that $\text{Ann}_L(Q) = 0$ ensures that the map $L \to A(Q)$ given by $x \mapsto \text{ad } x$ is a monomorphism of Lie algebras. Examples of extensions where $\text{Ann}_L(Q) = 0$ are the dense ones (see [3] for the definition of a dense extension and [14] for examples). If Q is a weak algebra of quotients of L, then $\text{Ann}_L(Q) = 0$.

Theorem 2.1. *Let $L \subseteq Q$ be an extension of Lie algebras such that the map $x \mapsto \text{ad } x$ from $L \to A(Q)$ is a monomorphism of Lie algebras. Let $a \in \text{QAnn}_Q(L)$. Then, for each $u \in L$ satisfying $x := [a, u] \in L$, we have that $z := [x, [x, v]]$ is in $\text{QAnn}_L(L)$, for every $v \in L$.*

Proof. In order to ease the notation in our computations, we shall temporarily get rid of the prefix ad and use capital letters X, Y, etc. instead of ad x, ad y, etc. Because of our assumption, we shall also identify an element x of L with its corresponding operator $X = \text{ad } x$ in $A(Q)$. An equation involving commutators on L is then translated into the corresponding equation with capital letters and commutators in $A(Q)$.

Let a be in $\text{QAnn}_Q(L)$. Then

$$[a, [a, y]] = 0 \text{ for every } y \in L. \tag{2.1}$$

This implies $[A, [A, Y]] = 0$ for every $Y \in \text{ad}(L) \subseteq A(Q)$, hence

$$A^2 Y + Y A^2 - 2AYA = 0 \text{ for every } Y \in \text{ad}(L). \tag{2.2}$$

By (2.1) we have

$$A^2 = 0 \text{ on } L, \tag{2.3}$$

and by (2.2) and $\frac{1}{2} \in \Phi$,

$$AYA = 0 \text{ on } L, \text{ for every } Y \in \text{ad}(L). \tag{2.4}$$

For $x = [a, u] \in L$, with $u \in L$, we have that

$$X^2 = (AU - UA)^2 = AUAU - AU^2A - UA^2U + UAUA = -AU^2A \tag{2.5}$$

on L, by using (2.3) and (2.4).

Note that $X^3 \overset{(2.5)}{=} (-AU^2A)(AU - UA) = -AU^2A^2U + AU^2AUA = 0$ by (2.3) and (2.4). Thus:

$$(2.6) \qquad\qquad X^3 = 0 \text{ on } L.$$

Next, take $z := [x, [x, v]]$, with $v \in L$, and compute that, on L,

$$(2.7) \qquad \begin{aligned} XVX &= (AU - UA)V(AU - UA) = AUV[A, U] - UAVAU + UAVUA \\ &\overset{(2.4)}{=} AUV[A, U] + UAVUA. \end{aligned}$$

We may now apply [11, Lemma 1.5.6] in order to obtain:

$$(2.8) \qquad XYX^2 = X^2YX \text{ on } L, \text{ for every } Y \in \mathrm{ad}\,(L), \text{ and}$$

$$(2.9) \qquad (YX^2)^2 = X^2Y^2X^2 \text{ on } L, \text{ for every } Y \in \mathrm{ad}\,(L).$$

Rewrite equation (2.8) taking (2.5) into account, so that we get

$$(AU - UA)Y(-AU^2A) = (-AU^2A)Y(AU - UA)$$

on L. This means that

$$(2.10)$$
$$-AUYAU^2A + UAYAU^2A = -AU^2AYAU + AU^2AYUA \overset{(2.4)}{=} AU^2AYUA.$$

Another usage of equation (2.4) shows that, since $x \in L$, we have $0 = AYAUX = AYAUAU - AYAU^2A$ on L, and thus

$$(2.11) \qquad AYAU^2A = AYAUAU = 0 \text{ on } L, \text{ for every } Y \in \mathrm{ad}\,(L).$$

This, together with equation (2.10), entails

$$(2.12) \qquad AUYAU^2A = -AU^2AYUA \text{ on } L, \text{ for every } Y \in \mathrm{ad}\,(L).$$

This last identity implies that for every Y in $\mathrm{ad}\,(L)$,

$$AUYAU^2A[[A, U], Y] = -AU^2AYUA[[A, U], Y]$$

on L, that is,

$$\begin{aligned} AUYAU^2A(AUY - UAY - YAU + YUA) &= \\ = -AU^2AYUA(AUY - UAY - YAU + YUA) \end{aligned}$$

on L. Apply (2.3) and (2.4) to obtain

$$(2.13)$$
$$AUYAU^2AYUA = -AU^2AYUAYUA \text{ on } L, \text{ for every } Y \in \mathrm{ad}\,(L).$$

Next, rewrite (2.9) taking equations (2.5) and (2.11) into account, so that

$$(2.14) \qquad AU^2AY^2AU^2A = YAU^2AYAU^2A = 0$$

on L and for every $Y \in$ ad (L).

Now, we shall find an expression for Z. By (2.7) and using (2.5), we find that

$$(2.15) \qquad \begin{aligned} Z &= [X, XV - VX] = X^2V + VX^2 - 2XVX \\ &= -AU^2AV - VAU^2A - 2AUV[A,U] - 2UAVUA \end{aligned}$$

on L.

Put $Y = U$ in equation (2.12). Since $\frac{1}{2} \in \Phi$,

$$(2.16) \qquad AU^2AU^2A = 0 \text{ on } L.$$

By (2.4), $0 = AYAVX = AYAVAU - AYAVUA = 0 - AYAVUA$ on L, hence

$$(2.17) \qquad AYAVUA = 0 \text{ on } L.$$

By (2.3), $0 = A^2V[U,A] = A^2VUA - A^2VAU \overset{(2.3)}{=} A^2VUA$ on L, that is,

$$(2.18) \qquad A^2VUA = 0 \text{ on } L.$$

In particular, if we let $V = U$ (since V is arbitrary),

$$(2.19) \qquad A^2U^2A = 0 \text{ on } L.$$

The identity (2.15) comes now into play. On L, we have:

$$Z^2 = -AU^2AVZ - VAU^2AZ - 2AUVAUZ + 2AUVUAZ - 2UAVUAZ,$$

which, expanding Z, yields

$$\begin{aligned} Z^2 = {}& AU^2AVAU^2AV + AU^2AV^2AU^2A + 2AU^2AVAUV[A,U] \\ &+ 2AU^2AVUAVUA + VAU^2A^2U^2AV + VAU^2AVAU^2A \\ &+ 2VAU^2A^2UV[A,U] + 2VAU^2AUAVUA + 2AUVAUAU^2AV \\ &+ 2AUVAUVAU^2A + 4AUVAUAUV[A,U] + 4AUVAU^2AVUA \\ &- 2AUVUA^2U^2AV - 2AUVUAVAU^2A - 4AUVUA^2UV[A,U] \\ &- 4AUVUAUAVUA + 2UAVUA^2U^2AV + 2UAVUAVAU^2A \\ &+ 4UAVUA^2UV[A,U] + 4UAVUAUAVUA. \end{aligned}$$

Use (2.11), (2.14) and (2.4) in the first line, (2.19), (2.11), (2.3) and (2.17) in lines two, four and five, and (2.17), (2.4) in the third one. Then

$$Z^2 = 2AU^2AVUAVUA + 2AUVAUVAU^2A + 4AUVAU^2AVUA \text{ on } L.$$

By (2.12), (2.13) and the fact that $AYZA = AZYA$ for every y, z in L (which is true on L by virtue of (2.4)),

$$AUVAUVAU^2A = -AUVAU^2AUVA = AU^2AUVAUVA$$

and

$$AUVAU^2AVUA = -AU^2AUVAVUA.$$

This implies $Z^2 = 0$, as desired. $\qquad\square$

Theorem 2.2. *Let $L \subseteq Q$ be an extension of Lie algebras with Q a weak algebra of quotients of L and L strongly non-degenerate. Then:*

(i) *Q is strongly non-degenerate ([15, Proposition 2.7(iii)]).*
(ii) *$\mathrm{Ann}_Q(L) = \mathrm{QAnn}_Q(L) = 0$.*

Proof. (i). Suppose that there exists a non-zero element $a \in \mathrm{QAnn}_Q(Q)$, and choose $u \in L$ satisfying $0 \neq x := [a, u] \in L$. Since L is strongly non-degenerate, $z := [x, [x, v]] \neq 0$ for some $v \in L$. But, by Theorem 2.1, z must be zero, a contradiction.

(ii). $\mathrm{Ann}_Q(L) = 0$ because Q is a weak algebra of quotients of L. For $a \in \mathrm{QAnn}_Q(L)$, $z := [x, [x, v]] \in \mathrm{QAnn}_L(L)$ whenever $x := [a, u] \in L$, with $u \in L$ (Theorem 2.1). Being L strongly non-degenerete implies $\mathrm{QAnn}_L(L) = 0$, whence x is zero. Since Q is a weak algebra of quotients of L, we obtain that $a = 0$. $\qquad\square$

Condition (ii) in the result below was proved by Zelmanov in [17, Corollary 2 in pg. 543] for strongly non-degenerate Lie algebras by using the Kostrikin radical. Our proof here is based on elements.

Proposition 2.3. *Let I be a strongly non-degenerate ideal of a Lie algebra L. Then:*

(i) *$\mathrm{Ann}_L(I) = \mathrm{QAnn}_L(I)$.*
(ii) *If $\mathrm{Ann}_L(I) = 0$, then the algebra L is strongly non-degenerate.*

Proof. (i). Clearly, $\mathrm{Ann}_L(I) \subseteq \mathrm{QAnn}_L(I)$. Conversely, consider $a \in \mathrm{QAnn}_L(I)$. The strongly non-degeneracy assumption on I implies that the map $I \to A(L)$ given by $y \mapsto \mathrm{ad}\, y$ is a monomorphism of Lie algebras. Since I is a strongly non-degenerate ideal of L, Theorem 2.1 implies that for every $u \in I$, the element $x = [a, u]$ is in $\mathrm{QAnn}_I(I) = 0$, hence $a \in \mathrm{Ann}_L(I)$.

(ii). In this case we have that L is a weak algebra of quotients of I. Apply Theorem 2.2 to obtain that L must be strongly non-degenerate too. $\qquad\square$

If A is an associative algebra, and since every derivation maps Z to Z, the set
$$I_Z = \{\delta \in \mathcal{D}\mathrm{er}(A) \mid \delta(A) \subseteq Z\}$$
is easily seen to be a Lie ideal of $\mathcal{D}\mathrm{er}(A)$ that contains the center of $\mathcal{D}\mathrm{er}(A)$. Indeed, for every $\delta \in Z(\mathcal{D}\mathrm{er}(A))$ and each $a \in A$, we have that $0 = [\delta, \mathrm{ad}\, a] = \mathrm{ad}\,(\delta(a))$, hence $\delta(a) \in Z$. Moreover, under certain conditions, $\mathrm{Inn}(A)$ can be seen as an essential ideal of $\mathcal{D}\mathrm{er}(A)/I_Z$.

Lemma 2.4. *Let A be a semiprime non-commutative associative algebra. Then:*

 (i) $\mathrm{Inn}(A)$ *is (isomorphic to) an essential ideal of $\mathcal{D}\mathrm{er}(A)/I_Z$, where I_Z is defined as before.*
 (ii) *If Z does not contain non-zero associative ideals (in particular, if A is prime), then $I_Z = 0$.*

Proof. (i). The map
$$\begin{array}{ccc} \mathrm{Inn}(A) & \to & \mathcal{D}\mathrm{er}(A)/I_Z \\ \mathrm{ad}\, a & \mapsto & \overline{\mathrm{ad}\, a} \end{array}$$
is a monomorphism of Lie algebras. This follows from the following fact:

(†) $\qquad\qquad [[a, A], A] = 0$, with $a \in A$, implies $a \in Z$.

Indeed, $[[a, A], A] = 0$ implies, by [6, Sublemma in pg. 5], $[a, A] = 0$, that is, $\mathrm{ad}\, a = 0$.

This allows us to identify $\mathrm{Inn}(A)$ with its image inside $\mathcal{D}\mathrm{er}(A)/I_Z$. The formula $[\delta, \mathrm{ad}\, a] = \mathrm{ad}\, \delta(a)$, where $a \in A$ and $\delta \in \mathcal{D}\mathrm{er}(A)$ ensures that $\mathrm{Inn}(A)$ is indeed an ideal of $\mathcal{D}\mathrm{er}(A)/I_Z$.

Now let J/I_Z be a non-zero ideal of $\mathcal{D}\mathrm{er}(A)$ and consider $\delta \in J \setminus I_Z$, that is, $[\delta(A), A] \neq 0$. A second usage of (†) allows us to conclude that $[[\delta(A), A], A] \neq 0$. Take a in A such that $[\delta(a), A] \not\subseteq Z$. Then $\overline{0} \neq \overline{[\delta, \mathrm{ad}\, a]} = \overline{\mathrm{ad}\, \delta(a)}$, and thus $\mathrm{Inn}(A)$ is essential in $\mathcal{D}\mathrm{er}(A)/I_Z$.

(ii). Take $\delta \in I_Z$ and $d \in \mathcal{D}\mathrm{er}(A)$. Put $\mu = [\delta, d]$. For every pair of elements $a, b \in A$ we have $\mu([a, b]) = [\mu(a), b] + [a, \mu(b)]$. Note that $[\mu(a), b] = [\delta d(a), b] - [d\delta(a), b] = -[d\delta(a), b] = 0$, because $d(Z) \subseteq Z$, and analogously $[a, \mu(b)] = 0$. It follows from this that $\mu([A, A]) = 0$.

Now let I be a non-central Lie ideal of A, and take y in $I \setminus Z$. Then $[y, A] \neq 0$ and by (†), we get $0 \neq [[y, A], A] \subseteq I \cap [A, A]$. Thus $[A, A]$ intersects non-trivially every non-central Lie ideal of A.

We claim that the subalgebra $\langle [A, A] \rangle$ generated by $[A, A]$ contains an essential associative ideal of A. Herstein's [7, Theorem 3] implies that $\langle [A, A] \rangle$ contains a non-zero associative ideal. By Zorn's Lemma, it is possible to find M maximal among all the associative ideals contained in $\langle [A, A] \rangle$. If $\mathrm{Ann}(M)$ were non-zero, we get by what we have just proved

that $\operatorname{Ann}(M) \cap [A, A] \neq 0$. Again by [7, Theorem 3], and since Z does not contain non-zero associative ideals, we have that $\langle \operatorname{Ann}(M) \cap [A, A] \rangle$ contains a non-zero associative ideal J. Then $M \oplus J \subseteq M \oplus \operatorname{Ann}(M) \cap \langle [A, A] \rangle$ and $M \subsetneq M \oplus J$, which contradicts the maximality of M.

By the first part of the proof we have $\mu([A, A]) = 0$, and so $\mu(I) = 0$, where I is an essential ideal of A contained in $\langle [A, A] \rangle$. This implies that $\mu = 0$. For, if $\mu(a) \neq 0$ for some $a \in A$, then the essentiality of I implies that there exists $y \in I$ such that $y\mu(a) \neq 0$. But $y\mu(a) = \mu(ya) - \mu(y)a = 0$, a contradiction. $\qquad \square$

Theorem 2.5. *Let A be a semiprime non-commutative associative algebra. Then:*

(i) *$\mathcal{D}\mathrm{er}(A)/I_Z$ is a strongly non-degenerate Lie algebra.*
(ii) *If Z does not contain non-zero associative ideals (in particular, if A is prime), then $\mathcal{D}\mathrm{er}(A)$ is a strongly non-degenerate Lie algebra.*

Proof. (i). Use [5, Lemma 5.2], Proposition 2.3 (ii) and Lemma 2.4 (i).
(ii) follows from (i) and condition (ii) in Lemma 2.4. $\qquad \square$

We now consider the case where our associative algebra A has an involution $*$. Under some additional mild assumptions on A in order to rule out algebras of low degrees, we obtain similar results on the non-degeneracy of $\mathrm{SDer}(A)$. Rather than proving them in full, we just indicate which changes are needed to adjust Lemma 2.4 and Theorem 2.5 to the current setting.

For every x in an algebra A we define $\deg(x)$ as the degree of algebraicity of x over the extended centroid $\mathcal{C}$ of A, provided that x is algebraic. If x is not algebraic, then we define $\deg(x) = \infty$. Put $\deg(A) = \sup\{\deg(x) \mid x \in A\}$. It is well-known that $\deg(A) < \infty$ if and only if A is a PI algebra. Furthermore, it is known that $\deg(A) = n < \infty$ if and only if A satisfies the standard polynomial identity of degreee $2n$, but does not satisfy any polynomial identity of degree $< 2n$, and this is further equivalent to the condition that A can be embedded into the matrix algebra $\mathbb{M}_n(F)$ for some field F (one can take, say, F to be the algebraic closure of $\mathcal{C}$), but cannot be embedded into $\mathbb{M}_{n-1}(R)$ for any commutative algebra R.

(a) The analogue of (†) in Lemma 2.4 is as follows:

   If A is a semiprime algebra with involution, then if $a \in K$ and $[[a, K], K] = 0$ we get $[a, K] = 0$. To prove this, suppose $a \in K$ such that $[a, K] \neq 0$. This means that $\bar{a} \neq 0$, where $\bar{a}$ denotes the class of a in K/Z_K. But then $[\bar{a}, K/Z_K] \neq 0$, since K/Z_K is semiprime by [9, Theorem 3], that is, $[[a, K], K] \neq 0$.

(b) The use of [7, Theorem 3] in the proof of condition (ii) in Lemma 2.4 must be changed to [12, Lemmas 2 and 3]. In order to

apply these results, certain restrictions on the degree of the algebra are needed. Concretely, $\deg(A/I) > 2$ for every $*$-prime ideal I of A.

Recall that an ideal I in an algebra A with involution $*$ is a $*$-*ideal* if I is invariant under the involution, that is, $I^* = I$. The algebra A is said to be $*$-*prime* if the product of two non-zero $*$-ideals is again non-zero. A $*$-ideal I is said to be $*$-*prime* if A/I is a $*$-prime algebra. The definition of a $*$-*semiprime* algebra is analogous.

Define the following Lie ideal of $\mathrm{SDer}(A)$:

$$I_{ZK} = \{\delta \in \mathrm{SDer}(A) \mid \delta(K) \subseteq Z\}.$$

In the current context, our Lemma 2.4 takes then the following form.

Lemma 2.6. *Let A be a semiprime non-commutative associative algebra with involution $*$. Then:*

(i) $\mathrm{Inn}(K)$ *is (isomorphic to) an essential ideal of* $\mathrm{SDer}(A)/I_{ZK}$, *where I_{ZK} is defined as before.*

(ii) *If $Z(K)$ does not contain non-zero associative $*$-ideals (in particular, if A is $*$-prime), then $I_{ZK} = 0$.*

The analogue of [5, Lemma 5.2] (used in the proof of condition (i) in Theorem 2.5) is the proposition below, which again requires conditions on the degree of the algebra. In particular, it generalizes [1, Theorem 2.13]. Recall that an involution $*$ in an associative algebra A is said to be of the *first kind* if it is the identity on the centroid of A. Otherwise it is called of the *second kind*.

Proposition 2.7. *Let A be a $*$-semiprime algebra. Assume that the involution is either of the second kind, or else it is of the first kind and $\deg(A/I) > 2$ for every $*$-prime ideal I of A. Then $[k,[k,K]] \subseteq Z(A)$, with $k \in K$, implies $k \in Z(A)$. In particular, $K/(K \cap Z)$ is a strongly non-degenerate Lie algebra.*

Proof. Let I be a $*$-ideal of A. It is clear that A/I also becomes a $*$-algebra with the natural involution.

On the other hand, if $\overline{x}$ denotes the class of an element x in A/I and $\overline{K} = \{\overline{k} \mid k \in K\}$, we have that $\overline{K} = K_{A/I}$. The containment $\overline{K} \subseteq K_{A/I}$ is clear, and for the converse, take $\overline{a}$ in $K_{A/I}$ and let $y \in I$ be such that $a^* + a = y$. Then $(a^* - \frac{1}{2}y)^* = y - a - \frac{1}{2}y = -a + \frac{1}{2}y$, that is, $a - \frac{1}{2}y \in K$, and $\overline{a} = \overline{a - \frac{1}{2}y}$.

Now, consider $k \in K$ satisfying $[k,[k,K]] \subseteq Z$. In particular, $(\mathrm{ad}\, k)^3(t) = 0$ for every $t \in K$. Arguing as in the proof of [5, Lemma 5.2], we obtain:

($\ddagger$) $$(\mathrm{ad}\, k)^2(t) = 0.$$

Let $\{I_\alpha\}_{\alpha \in \Lambda}$ be the collection of all $*$-prime ideals of A. Since A is $*$-semiprime, $\bigcap_{\alpha \in \Lambda} I_\alpha = 0$. Suppose $[\overline{k}, A/I_\beta] \neq 0$ for some $\beta \in \Lambda$. Since A/I_β is a $*$-prime algebra we may apply [2, Lemma 5.4] in order to conclude that $Z(\overline{K}) = Z(A/I_\beta) \cap \overline{K}$, and hence $[\overline{k}, \overline{K}] \neq 0$. Use [2, Theorem 5.3] if $*\colon A/I_\beta \to A/I_\beta$ is of the first kind, or [1, Theorem 2.13] if the involution is of the second kind, to conclude that $[\overline{k}, [\overline{k}, \overline{K}]] \neq 0$, in contradiction to ($\ddagger$). In consequence, $[\overline{k}, A/I_\alpha] = 0$ for every $\alpha \in \Lambda$, that is, $[k, A] \subseteq \bigcap_{\alpha \in \Lambda} I_\alpha = 0$. $\square$

Finally, the involutive version of Theorem 2.5 is the following:

Theorem 2.8. *Let A be a $*$-semiprime non-commutative associative algebra with $\deg(A/I) > 2$ for every $*$-prime ideal I of A. Then:*

(i) $\mathrm{SDer}(A)/I_{ZK}$ *is a strongly non-degenerate Lie algebra.*

(ii) *If $Z(K)$ does not contain non-zero associative $*$-ideals (in particular, if A is $*$-prime), then $\mathrm{SDer}(A)$ is a strongly non-degenerate Lie algebra.*

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