A RELATIVE DOUBLE COMMUTANT THEOREM FOR HEREDITARY SUB-C*-ALGEBRAS

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Abstract. We prove a double commutant theorem for hereditary subalgebras of a large class of C*-algebras, partially resolving a problem posed by Pedersen[8]. Double commutant theorems originated with von Neumann, whose seminal result evolved into an entire field now called von Neumann algebra theory. Voiculescu proved a C*-algebraic double commutant theorem for separable subalgebras of the Calkin algebra. We prove a similar result for hereditary subalgebras which holds for arbitrary corona C*-algebras. (It is not clear how generally Voiculescu's double commutant theorem holds.)

Résumé. Nous démontrons un théorème commutant double (d'après Voiculescu et von Neumann) pour des sous-C*-algèbres héréditaires dans une C*-algèbre ≪corona≫, c'est a dire $M(A)/A$.

1. Introduction

The most fundamental result in all of von Neumann algebra theory is perhaps von Neumann's double commutant theorem,† published in 1929 (see [12]). We phrase the theorem as follows:

Theorem 1.1. *Given a $*$-closed subalgebra of $B(H)$, the double commutant of the subalgebra is equal to the weak operator closure of its unitization.*

Approximately half a century later, Voiculescu proved[11, 10] a remarkable and unexpected C*-algebraic version of the above theorem:

Theorem 1.2. *Consider the Calkin algebra $B(H)/K(H)$ of a separable infinite-dimensional Hilbert space H. The double commutant of a separable sub-C^*-algebra is the unitization of that subalgebra.*

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†The commutant C' of a given algebra C is the set of all elements of some larger algebra that commute with all of C. Thus, commutants are always defined relative to some larger algebra, and when not clear from the context, the larger algebra can be specified by a subscript, as for example $C'_{B(H)}$. Iterating, $C'' := (C')'$ is by definition a double commutant.

The unitization mentioned in the above theorem is with respect to the unit of $B(H)/K(H)$.

Recall that the multiplier algebra $M(B)$ of a given C*-algebra B is the idealizer of B in the double dual B^{**}. Since the multiplier algebra of the compact operators $K(H)$ is $B(H)$, we may reasonably regard the corona algebra $M(B)/B$ as a sweeping generalization of the Calkin algebra considered by Voiculescu. At a conference in 1988, Pedersen posed the problem of generalizing Voiculescu's theorem to the setting of corona algebras[8].

In this note, we show that in most cases, the relative double commutant of a singly generated hereditary subalgebra H of a corona algebra is $H+Z$ where Z is the centre of the given corona algebra.

2. Full hereditary subalgebras

Theorem 2.1. *Let H be a full hereditary subalgebra of a unital C^*-algebra Q. Then an element x that commutes with H can be uniquely decomposed as $z+a$ where z is in the centre, $Z(Q)$, of Q and a is in the annihilator, $H^{\perp}$, of H.*

Proof. Let us first prove the uniqueness of the decomposition. If $x = z_1 + a_1 = z_2 + a_2$ with z_i in the centre of Q and a_i in the annihilator of H, then $c = z_1 - z_2 = a_2 - a_1$ is in both $Z(Q)$ and $H^{\perp}$. We are to show that c is zero. If not, then by the Gelfand–Naimark theorem there is an irreducible representation π of Q such that $\pi(c)$ is non-zero. Then as c is in the centre of Q, it follows that $\pi(c)$ is a multiple of the unit of $\pi(Q)$. But since c is also in $H^{\perp}$, it follows that the subalgebra $\pi(H^{\perp}) \subseteq \pi(Q)$ contains the unit of $\pi(Q)$. Since $\pi(H)\pi(H^{\perp}) = 0$, we have that $\pi(H) = 0$. This contradicts the assumption that H is full.

Let us now prove existence. Given x that commutes with all of H, we notice that if $h \in H$ then $xh = x_1 h x_2$ for any factoring $x = x_1 x_2$ in H, from which it follows that xh is in H. The case of action on the right is similar, and so x can be regarded as an element of $M(H)$. Clearly x is central as an element of $M(H)$.

By one of Pedersen's early results, as H is full, the natural map $t \mapsto t \cap H$ from $\mathrm{Prim}Q$ to $\mathrm{Prim}H$ is a homeomorphism[7]. Since this map is compatible with the map from $Z(Q)$ to $Z(M(H))$ constructed in the previous paragraph, it follows by the extension of the Dauns-Hofmann Theorem given in [2] that this map is an isomorphism of C*-algebras, and in particular is surjective.

Let us denote the (unique) pre-image of $x \in Z(M(H))$ under this isomorphism by $c \in Z(Q)$, and define $a := x - c$.

This element $a \in Q$ certainly multiplies H into itself, and as an element of $M(H)$ it is zero by construction. Thus a is in the annihilator of H, and $x = a + c$ is our desired decomposition. □

Now recall Pedersen's result[8]:

Theorem 2.2. *If H is a singly generated hereditary subalgebra of a corona algebra (of a σ-unital C^*-algebra), then $H^{\perp\perp} = H$.*

We have our first result on double commutants:

Theorem 2.3. *Suppose that H is a full and singly generated hereditary subalgebra of the corona C*-algebra of some σ-unital C*-algebra. Then the double relative commutant H'' is equal to $Z + H$, where Z is the centre of the corona.*

Proof. Let x be some element of H''. Note that x commutes with the annihilator $H^\perp$, since after all the elements of the annihilator commute with the elements of H. We may thus apply our Theorem 2.1 to decompose x as $a+z$ with a annihilating $H^\perp$ and z in the centre of the corona. But a is then in $H^{\perp\perp}$ and by Theorem 2.2 this algebra is equal to H. We conclude that H'' is contained in $Z + H$. On the other hand, it is routine to verify that both Z and H are contained in H''. □

3. The case of extremally disconnected primitive ideal space

We now remove the fullness condition on the given hereditary subalgebra, replacing it by a condition on the primitive ideal space of the corona algebra. This condition is that the space is *extremally disconnected*, meaning that the closure of every open set is open. For example, an extremally disconnected first countable Hausdorff space must be discrete, but of course primitive ideal spaces are not usually Hausdorff. The most important special case of relevance to C*-algebraic problems is probably the observation that prime C*-algebras have extremally disconnected primitive ideal space. To see this, recall that open sets in the primitive ideal space of a prime C*-algebra are either empty or dense. In either case, the closure of an open set is both open and closed.

We now partially characterize C*-algebras whose corona C*-algebra is prime:

Theorem 3.1. *The corona of a primitive σ-unital C*-algebra is prime. For separable C*-algebras with no unital ideals, the converse holds.*

Remark: One of the ideas in the proof is related to Theorem 2 of [4], where it is in effect shown that the multiplier algebra of a nonunital separable C*-algebra is strictly larger than the given C*-algebra. Also, the method of construction of corona ideals in the following proof is related to that of Theorem 3.1 of [13].

Proof. Akemann and Eilers[1] have shown that corona algebras of primitive σ-unital C*-algebras are prime. Conversely, if a separable C*-algebra A is not primitive, then it is not prime[3], and we can find a pair I, J of non-zero orthogonal ideals. The strict closures of these ideals in $M(A)$ are still orthogonal, and are unital (this follows from taking the strict limit of a suitable approximate unit). Being unital, they are not contained in A. Passing to the corona, we thus have a pair of non-zero orthogonal ideals, so that the corona algebra is not prime. □

On the other hand, the natural conjecture that the corona of a separable C^*-algebra with extremally disconnected primitive ideal space also has extremally disconnected primitive ideal space is false: consider the case $A = C_0(\mathbb{N})$. It can be shown [5] that the Stone-Čech corona of the natural numbers $\mathbb{N}$ is (surprisingly) not extremally disconnected. In this case, it is even true that the algebra and the multiplier algebra (and the corona algebra) have real rank zero.

Nevertheless, Theorem 3.1 gives a large supply of C*-algebras whose corona has extremally disconnected primitive ideal space, and it seems of interest that our Theorem 2.3 generalizes to coronas with this property.

The hypothesis on the primitive ideal space is applied by means of the following basically topological lemma:

Lemma 3.2. *The following conditions are equivalent, for a C^*-algebra A with primitive ideal space V :*

(i) *Any element of the centre of the multiplier algebra of an ideal comes from an element of the centre of the multiplier algebra $M(A)$ of A, and*
(ii) *The primitive ideal space V is extremally disconnected.*

The extension of a central multiplier from an ideal is unique if and only if the ideal is essential.

Proof. Let J be a given ideal and let z_0 be a given element of the centre of $M(J)$. By Dixmier's extension of the Dauns-Hofmann Theorem[2, 7], the element z_0 is a continuous bounded function on the primitive ideal space of J. Recall that the primitive ideal space of J is an open subset of V. Note that we may as well assume that J is essential, replacing J by $J + J^{\perp}$ and defining z_0 to be zero on $J^{\perp}$.

Now we apply the theorem that a space is extremally disconnected if and only if any continuous bounded function on a dense open set can be extended to a continuous bounded function on the whole space (see paragraphs 1.4

and 1.H.6 of [5]). Conversely, if property (i) holds for all essential ideals, we deduce again by Dixmier's extension of the Dauns-Hofmann Theorem that the primitive ideal space V has the requisite extension property.

The uniqueness stated in the last part of the lemma is straightforward. □

Theorem 3.3. *Let Q be a unital C^*-algebra with extremally disconnected primitive ideal space. If x commutes with a hereditary sub-C^*-algebra H, then $x = z + a$ for some a in $H^\perp$ and some central element $z \in Q$. The decomposition is unique if and only if the ideal generated by H in Q is essential.*

Proof. To show existence, we notice as before that x multiplies H into H. Denote by m the element of $Z(M(H))$ thus obtained (from x). By Dixmier's extension of the Dauns-Hofmann Theorem[2, 7], an element of $Z(M(H))$ is equivalently a continuous function on the open subset of Prim$Q$ that corresponds to the ideal I generated by H in Q, and this element is still a central multiplier. Applying Lemma 3.2, we obtain an element z of the centre of Q. Then $x = (x - z) + z$ is our desired decomposition. □

Specializing to the case of corona algebras and repeating the proof of Theorem 2.3, we have our main result:

Theorem 3.4. *Let Q be the corona algebra of some σ-unital C^*-algebra. Suppose that Q has extremally disconnected primitive ideal space. Then the double relative commutant of a singly generated hereditary subalgebra H is $H + Z$ where Z is the centre of the corona algebra.*

One particularly simple special case of the above theorem is as follows:

Corollary 3.5. *Let A be a σ-unital primitive C^*-algebra. If H is a singly generated hereditary subalgebra of $M(A)/A$, then H'' is equal to the unitization $\mathbb{C}1_{M(A)/A} + H$ of H.*

This is deduced by noting that by Theorem 3.1 the corona algebra $M(A)/A$ is prime, thus having trivial centre and extremally disconnected primitive ideal space.

Note that if A is simple it is certainly primitive, so in this case we recover our earlier result[6].

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