GENERATING TREES FOR PERMUTATIONS AVOIDING GENERALIZED PATTERNS

SERGI ELIZALDE

Abstract. We construct generating trees with with one, two, and three labels for some classes of permutations avoiding generalized patterns of length 3 and 4. These trees are built by adding at each level an entry to the right end of the permutation, which allows us to incorporate the adjacency condition about some entries in an occurrence of a generalized pattern. We use these trees to find functional equations for the generating functions enumerating these classes of permutations with respect to different parameters. In several cases we solve them using the kernel method and some ideas of Bousquet-Mélou [2]. We obtain refinements of known enumerative results and find new ones.

1. Introduction

1.1. **Generalized pattern avoidance.** We denote by $\mathcal{S}_n$ the symmetric group on $\{1, 2, \ldots, n\}$. Let n and k be two positive integers with $k \leq n$, and let $\pi = \pi_1\pi_2 \cdots \pi_n \in \mathcal{S}_n$ be a permutation. A generalized pattern σ is obtained from a permutation $\sigma_1\sigma_2 \cdots \sigma_k \in \mathcal{S}_k$ by choosing, for each $j = 1, \ldots, k-1$, either to insert a dash - between σ_j and σ_{j+1} or not. More formally, $\sigma = \sigma_1\varepsilon_1\sigma_2\varepsilon_2 \cdots \varepsilon_{k-1}\sigma_k$, where each ε_j is either the symbol - or the empty string. With this notation, we say that π *contains* (the generalized pattern) σ if there exist indices $i_1 < i_2 < \ldots < i_k$ such that (i) for each $j = 1, \ldots, k-1$, if ε_j is empty then $i_{j+1} = i_j + 1$, and (ii) for every $a, b \in \{1, 2, \ldots, k\}$, $\pi_{i_a} < \pi_{i_b}$ if and only if $\sigma_a < \sigma_b$. In this case, $\pi_{i_1}\pi_{i_2} \cdots \pi_{i_k}$ is called an *occurrence* of σ in π.

If π does not contain σ, we say that π *avoids* σ, or that it is σ*-avoiding*. For example, the permutation $\pi = 3542716$ contains the pattern 12-4-3 because it has the subsequence 3576. On the other hand, π avoids the pattern 12-43. We denote by $\mathcal{S}_n(\sigma)$ the set of permutations in $\mathcal{S}_n$ that avoid σ. More generally, if $\Sigma = \{\sigma_1, \sigma_2, \ldots\}$ is a collection of generalized patterns, we say that a permutation π is Σ-avoiding if it avoids all the patterns in Σ simultaneously. We denote by $\mathcal{S}_n(\Sigma)$ the set of Σ-avoiding permutations in $\mathcal{S}_n$.

1.2. **Generating trees.** Generating trees are a useful technique to enumerate classes of pattern-avoiding permutations (see for example [12, 13]). The nodes at each level of the generating tree are indexed by permutations of a given length. It is common in the literature to define the children of a permutation π of length n to be those permutations that are obtained by inserting the entry $n+1$ in $\pi = \pi_1\pi_2\cdots\pi_n$ in such a way that the new permutation is still in the class. In this paper we consider a variation of this definition. Here, the children of a permutation π of length n are obtained by appending an entry to the right of π, and shifting up by one all the entries in π that were greater than or equal to the new entry. For example, if the entry 3 is appended to the right of $\pi = 24135$, the child that we obtain is 251463. Adding the new entry to the right of the permutation makes these trees well-suited to enumerate permutations avoiding generalized patterns, as we will see in Sections 2, 3, and 4. We will refer to these trees as *rightward generating trees*. This kind of generating trees have been used in [1] to enumerate permutations avoiding sets of three generalized patterns of length three with one dash, such as $\{1\text{-}23, 2\text{-}13, 1\text{-}32\}$.

For some classes of permutations, a label can be associated to each node of the tree in such a way that the number of children of a permutation and their labels depend only on the label of the parent. For example, in the tree for 1-2-3-avoiding permutations, we can label each node π with $m = \min\{\pi_i : \exists j < i \text{ with } \pi_j < \pi_i\}$ (or $m = n+1$ if $\pi = n\cdots 21$). Then, the children of a permutation with label (m) have labels $(m+1), (2), (3), \ldots, (m)$, corresponding to the appended entry being $1, 2, 3, \ldots, m$, respectively. This succession rule, together with the fact that the root ($\pi = 1 \in \mathcal{S}_1$) has label (2), completely determines the tree. From this rule one can derive a functional equation for the generating function that enumerates the permutations by their length and the label of the corresponding node in the tree. For generating trees with one label, these equations are well understood and their solutions are algebraic series. This is the case of the generating trees obtained in [1], for example.

In other cases, however, one label is not enough to describe the generating tree in terms of a succession rule. In Section 3 we consider some classes of permutations avoiding generalized patterns where to describe the generating tree we need to assign two labels to each node. Generating trees with two labels were used in [2] to enumerate restricted permutations. In fact, the inspiration for the present paper and many of the ideas used come from Bousquet-Mélou's work. One difference is that here trees are constructed by adding at each level an entry to the right end of the permutation, which allows us to keep track of elements occurring in adjacent positions. In Section 4 we consider some classes of permutations whose rightward generating tree has three labels for each node.

Given a permutation $\pi \in \mathcal{S}_n$, we will write $r(\pi) = \pi_n$ to denote the rightmost entry of π. In all our generating functions, the variable t will be used to mark the size of the permutation.

2. Generating trees with one label

In this section we enumerate classes of pattern-avoiding permutations whose rightward generating trees can be described by a succession rule involving only one label for each node. For some of these classes, rightward generating trees are not the only way to obtain the results, but they are a tool that works in all these cases.

The classes in this section avoid the pattern 2-1-3. Note that avoiding this pattern is equivalent to avoiding the generalized pattern 2-13. Indeed, if π contains an occurrence of 2-1-3, say $\pi_i\pi_j\pi_k$ with $\pi_j < \pi_i < \pi_k$, then there must be some index l with $j \leq l < k$ such that $\pi_l < \pi_i$ and $\pi_{l+1} > \pi_i$, so $\pi_i\pi_l\pi_{l+1}$ is an occurrence of 2-13. For any class of permutations that avoid this pattern, the corresponding rightward generating tree has the property that the appended entry at each level can never be more than one unit larger than the entry appended at the previous level.

2.1. $\{2\text{-}1\text{-}3, \bar{2}\text{-}31\}$-avoiding permutations.

Recall that a permutation π is said to avoid the *barred* pattern $\bar{2}$-31 if every descent in π (an occurrence of 21) is part of an occurrence of 2-31; equivalently, for any index i such that $\pi_i > \pi_{i+1}$ there is an index $j < i$ such that $\pi_i > \pi_j > \pi_{i+1}$. The bar indicates that the 2 is forced whenever a 31 occurs. For example, the permutation 4627513 avoids $\bar{2}$-31, but 2475613 does not.

We use M_n to denote the n-th Motzkin number. Recall that $\sum_{n\geq 0} M_n t^n = \frac{1-t-\sqrt{1-2t-3t^2}}{2t^2}$. The next result seems to be a new interpretation of the Motzkin numbers.

Proposition 2.1. *The number of* $\{2\text{-}1\text{-}3, \bar{2}\text{-}31\}$*-avoiding permutations of size n is M_{n-1}.*

Proof. Consider the rightward generating tree for $\{2\text{-}1\text{-}3, \bar{2}\text{-}31\}$-avoiding permutations. Labeling each permutation with its rightmost entry $r = r(\pi)$, this tree is described by the succession rule

$$\begin{array}{l}(1)\\ (r) \longrightarrow (1)\ (2)\ \cdots\ (r-1)\ (r+1).\end{array}$$

Indeed, the new entry appended to the right of π cannot be greater than $\pi_n + 1$ in order for the new permutation to be 2-1-3-avoiding, and it cannot be π_n because then it would create an occurrence of 21 that is not part of an occurrence of 2-31.

Defining $D(t,u) = \sum_{n\geq 1} \sum_{\pi\in\mathcal{S}_n(2\text{-}1\text{-}3,\overline{2}\text{-}31)} u^{r(\pi)}t^n = \sum_{r\geq 1} D_r(t)u^r$, the succession rule above gives the following equation for the generating function:

$$\begin{aligned} D(t,u) &= tu + t\sum_{r\geq 1} D_r(t)(u+u^2+\cdots+u^{r-1}+u^{r+1}) \\ &= tu + \frac{t}{u-1}[D(t,u)-uD(t,1)] + tuD(t,u), \end{aligned}$$

which can be written as

$$\left(1-\frac{t}{u-1}-tu\right)D(t,u) = tu - \frac{tu}{u-1}D(t,1). \tag{1}$$

Now we apply the kernel method. The values of u as a function of t that cancel the term multiplying $D(t,u)$ on the left hand side are $u_0 = u_0(t) = \frac{1+t\pm\sqrt{1-2t-3t^2}}{2t}$, of which the one with the minus sign is a well-defined formal power series in t. Substituting $u = u_0$ in (1) gives

$$D(t,1) = u_0 - 1 = \frac{1-t-\sqrt{1-2t-3t^2}}{2t},$$

which is the generating function for the Motzkin numbers with the indices shifted by one. □

There is also a bijective proof of Proposition 2.1. Given a permutation $\pi = \pi_1\pi_2\cdots\pi_n \in \mathcal{S}_n(2\text{-}1\text{-}3,\overline{2}\text{-}31)$, we can construct a Dyck path of size n (i.e., a sequence of n U's and n D's so that no prefix contains more D's than U's) as follows. Recall that a right-to-left maximum of π is an entry π_i such that $\pi_i > \pi_j$ for all $j > i$. Let $\pi_{i_1}, \pi_{i_2}, \ldots, \pi_{i_m}$ be the right-to-left maxima of π, with $i_1 < i_2 < \cdots < i_m = n$. Consider the Dyck path

$$U^{i_1}D^{\pi_{i_1}-\pi_{i_2}}U^{i_2-i_1}D^{\pi_{i_2}-\pi_{i_3}}U^{i_3-i_2}\ldots D^{\pi_{i_{m-1}}-\pi_{i_m}}U^{i_m-i_{m-1}}D^{\pi_{i_m}},$$

where exponentiation indicates repetition of a step. The condition that π is $\{2\text{-}1\text{-}3,\overline{2}\text{-}31\}$-avoiding implies that this path contains no three consecutive steps UDU. It is not hard to see that this map is in fact a bijection between $\mathcal{S}_n(2\text{-}1\text{-}3,\overline{2}\text{-}31)$ and UDU-free Dyck paths of size n. Such paths are in bijection with Motzkin paths of length $n-1$, as shown in [4]. The composition of these two bijections completes the bijective proof of Proposition 2.1.

2.2. **$\{2\text{-}1\text{-}3, \overline{2}^o\text{-}31\}$-avoiding permutations.** Extending the notion of barred patterns, we say that a permutation π avoids the pattern $\overline{2}^o$-31 if every descent in π is the '31' part of an odd number of occurrences of 2-31; equivalently, for any index i such that $\pi_i > \pi_{i+1}$, the number of indices $j < i$ such that $\pi_i > \pi_j > \pi_{i+1}$ is odd.

Proposition 2.2. *The number of $\{2\text{-}1\text{-}3, \overline{2}^o\text{-}31\}$-avoiding permutations of size n is*

$$|\mathcal{S}_n(2\text{-}1\text{-}3, \overline{2}^o\text{-}31)| = \begin{cases} \frac{1}{2k+1}\binom{3k}{k} & \text{if } n = 2k, \\ \frac{1}{2k+1}\binom{3k+1}{k+1} & \text{if } n = 2k+1. \end{cases} \tag{2}$$

Proof. The rightward generating tree for $\{2\text{-}1\text{-}3, \overline{2}^o\text{-}31\}$-avoiding permutations is given by the succession rule

$$\begin{array}{l} (1) \\ (r) \longrightarrow (r+1)\ (r-1)\ (r-3)\cdots, \end{array}$$

that is, the labels of the children of a node labeled r are the numbers $1 \leq j \leq r+1$ such that $r-j$ is odd. Let $J(t,u) = \sum_{n\geq 1} \sum_{\pi\in\mathcal{S}_n(2\text{-}1\text{-}3,\overline{2}^o\text{-}31)} u^{r(\pi)}t^n = \sum_{r\geq 1} J_r(t)u^r$, and let $J^o(t,u) = \sum_{r \text{ odd}} J_r(t)u^r$, $J^e(t,u) = \sum_{r \text{ even}} J_r(t)u^r$. The succession rule implies that

$$\begin{aligned} J(t,u) &= tu + t\sum_{r \text{ odd}} J_r(t)(u^2+u^4+\cdots+u^{r-1}+u^{r+1}) \\ &\quad + t\sum_{r \text{ even}} J_r(t)(u+u^3+\cdots+u^{r-1}+u^{r+1}) \\ &= tu + \frac{tu}{u^2-1}\left[\sum_{r \text{ odd}} J_r(t)(u^{r+2}-u) + \sum_{r \text{ even}} J_r(t)(u^{r+2}-1)\right] \\ &= tu + \frac{tu}{u^2-1}\left[u^2J^o(t,u) - uJ^o(t,1) + u^2J^e(t,u) - J^e(t,1)\right], \end{aligned}$$

so we get the following functional equation:

$$\left(1 - \frac{tu^3}{u^2-1}\right)J(t,u) = tu - \frac{tu^2}{u^2-1}J(t,1) + \frac{tu(u-1)}{u^2-1}J^e(t,1). \tag{3}$$

The kernel $1 - \frac{tu^3}{u^2-1}$ as a function in the variable u has three zeroes, two of which are complex conjugates. Denote them by $u_1 = a(t) + b(t)i$ and $u_2 = \bar{u_1} = a(t) - b(t)i$. Adding the equations $0 = u_i^2 - 1 - u_iJ(t,1) + (u_i-1)J^e(t,1)$ for $i = 1, 2$, we get

$$a(t)J(t,1) = a(t)^2 - b(t)^2 - 1 + (a(t)-1)J^e(t,1),$$

and subtracting them gives

$$J(t,1) = 2a(t) + J^e(t,1).$$

Solving this system of equations for J, we get that $J(t,1) = 2a(t) - a(t)^2 - b(t)^2 - 1$. Plugging the values of $a(t)$ and $b(t)$ yields the expression

$$J(t,1) = \frac{(2-3t)f(t)^2+(9t-2-g(t))f(t)+(2-6t)g(t)+54t^2-18t-4}{3tf(t)^2},$$

where $g(t) = \sqrt{3(27t^2-4)}$ and $f(t) = [12tg(t) - 108t^2 + 8]^{1/3}$. It is easy to check that $J = J(t,1)$ is a root of the polynomial $tJ^3+(3t-2)J^2+(3t-1)J+t = 0$. Using the Lagrange inversion formula, one sees that its coefficients are given by (2), which is sequence A047749 from the On-Line Encyclopedia of Integer Sequences [11]. Observe that we can also obtain an expression for $J(t,u)$ using (3) and the fact that $J^e(t,1) = -a(t)^2 - b(t)^2 - 1$. □

It is also possible to give a direct bijective proof of Proposition 2.2 that does not use rightward generating trees. A well-known combinatorial interpretation of the numbers (2) is that they enumerate lattice paths from $(0,0)$ to $(n, \lfloor n/2 \rfloor)$ with steps $E = (1,0)$ and $N = (0,1)$ that never go above the line $y = x/2$. Here is a bijection from $\mathcal{S}_n(2\text{-}1\text{-}3, \overline{2}^o\text{-}31)$ to these paths. Let $\pi = \pi_1\pi_2\cdots\pi_n \in \mathcal{S}_n(2\text{-}1\text{-}3, \overline{2}^o\text{-}31)$. Let $\pi_{i_1}, \pi_{i_2}, \ldots, \pi_{i_m}$ be the right-to-left maxima of π, with $i_1 < i_2 < \cdots < i_m = n$. The condition that π is $\{2\text{-}1\text{-}3, \overline{2}^o\text{-}31\}$-avoiding guarantees that all the differences $\pi_{i_j} - \pi_{i_{j+1}}$ are even. For $j = 1, \ldots, m-1$, let $a_j = (\pi_{i_j} - \pi_{i_{j+1}})/2$. Let $a_m = \lfloor \pi_{i_m}/2 \rfloor$. We map π to the following path from $(0,0)$ to $(n, \lfloor n/2 \rfloor)$:

$$E^{i_1}N^{a_1}E^{i_2-i_1}N^{a_2}E^{i_3-i_2}N^{a_3}\cdots E^{i_m-i_{m-1}}N^{a_m}.$$

It can be checked that this map is a bijection. For example, if $\pi = 4675123$, we have $\pi_{i_1}\pi_{i_2}\pi_{i_3} = \pi_3\pi_4\pi_7 = 753$, so the corresponding path from $(0,0)$ to $(7,3)$ is $EEENENEEEN$.

Analogously to the definition for the pattern $\overline{2}^o$-31, we say that a permutation π avoids the pattern $\overline{2}^e$-31 if every occurrence of 21 in π is part of an even number of occurrences of 2-31. We can also enumerate $\{2\text{-}1\text{-}3, \overline{2}^e\text{-}31\}$-avoiding permutations.

Proposition 2.3. *The number of* $\{2\text{-}1\text{-}3, \overline{2}^e\text{-}31\}$*-avoiding permutations of size* n *is*

$$\frac{1}{n}\sum_{k=0}^{\lfloor n/2 \rfloor}\left[2\binom{n}{2k}\binom{n-k}{k-1} + \frac{n}{n-k}\binom{n}{2k+1}\binom{n-k}{k}\right].$$

Proof. Let $Q(t) = \sum_{n\geq 1} |\mathcal{S}_n(2\text{-}1\text{-}3, \overline{2}^e\text{-}31)|\ t^n$. An argument similar to the proof of Proposition 2.2 shows that $Q(t)$ equals

$$\frac{(2-4t)\tilde{f}(t)^2 + (-2+12t-7t^2-\tilde{g}(t))\tilde{f}(t) + (2-8t)\tilde{g}(t) + 8t^3 + 46t^2 - 8t - 4}{3t\tilde{f}(t)^2},$$

where $\tilde{g}(t) = \sqrt{3(-5t^4 + 24t^3 - 4t^2 + 12t - 4)}$ and $\tilde{f}(t) = [4(3t\tilde{g}(t) - 11t^3 - 12t^2 - 6t + 2)]^{1/3}$. It follows that $Q = Q(t)$ is a root of the polynomial $tQ^3 + (4t-2)Q^2 + (4t-1)Q + t = 0$. Applying Lagrange inversion we get the stated formula. □

2.3. **{2-1-3, 2-3-41, 3-2-41}-avoiding permutations.** The rightward generating tree for this class of permutations has a simple succession rule. This allows us to enumerate them easily. Let

$$K(t,u) = \sum_{n\geq 1} \sum_{\pi\in\mathcal{S}_n(2\text{-}1\text{-}3,2\text{-}3\text{-}41,3\text{-}2\text{-}41)} u^{r(\pi)} t^n = \sum_{r\geq 1} K_r(t) u^r.$$

Proposition 2.4. *The generating function for* {2-1-3, 2-3-41, 3-2-41}*-avoiding permutations where u marks the value of the rightmost entry is*

$$K(t,u) = \frac{1 - t - 2tu - \sqrt{1-2t-3t^2}}{2t(\frac{1}{u} + 1 + u) - 2}.$$

Proof. The succession rule for this class of permutations is

$$\begin{array}{l} (1) \\ (r) \longrightarrow \begin{cases} (r-1)\ (r)\ (r+1) & \text{if } r > 1, \\ (r)\ (r+1) & \text{if } r = 1. \end{cases} \end{array}$$

This translates into the functional equation

$$\left[1 - t\left(\frac{1}{u} + 1 + u\right)\right] K(t,u) = tu - tK_1(t). \tag{4}$$

Applying the kernel method we find that $K_1(t) = \frac{1-t-\sqrt{1-2t-3t^2}}{2t}$, and substituting back into (4) we get the expression for $K(t,u)$. □

The generating function $K(t,1)$ enumerates also {1-3-2, 123-4}-avoiding permutations, as shown in [10, Example 2.6]. However, no direct bijection between $\mathcal{S}_n$(2-1-3, 2-3-41, 3-2-41) and $\mathcal{S}_n$(1-3-2, 123-4) seems to be known.

3. Generating trees with two labels

Here we enumerate some classes of permutations whose rightward generating tree has a succession rule that can be described using a pair of labels for each node. These trees give rise to functional equations with three variables. Even though no method is known to solve them in general, in this section we present special cases where we have been able to solve the corresponding equations.

In a few cases, one of the two labels is the length of the permutation. When that happens, the functional equations have only two variables, but the variable t appears multiplied by another variable, which makes these equations more difficult than the ones in Section 2.

Note that for the classes that we consider in this section, the enumeration of the permutations by their length has already been done by different authors. Our contribution is a refined enumeration of these permutations by several parameters, and also the fact that our results are obtained using the unifying framework of rightward generating trees.

3.1. **$\{$2-1-3, 12-3$\}$-avoiding permutations.** It was shown by Claesson [5] that $|\mathcal{S}_n(\text{1-3-2, 1-23})| = M_n$. A bijection between $\mathcal{S}_n(\text{1-3-2, 1-23})$ and the set of Motzkin paths of length n was given in [7]. Clearly the sets $\mathcal{S}_n(\text{1-3-2, 1-23})$ and $\mathcal{S}_n(\text{2-1-3, 12-3})$ are equinumerous, since a permutation $\pi_1\pi_2\cdots\pi_n$ is $\{$1-3-2, 1-23$\}$-avoiding exactly when $(n+1-\pi_n)\cdots(n+1-\pi_2)(n+1-\pi_1)$ is $\{$2-1-3, 12-3$\}$-avoiding. In this section we recover the formula for $|\mathcal{S}_n(\text{2-1-3, 12-3})|$ using a generating tree with two labels. This method provides a refined enumeration of $\{$2-1-3, 12-3$\}$-avoiding permutations by two new parameters: the value of the last entry and the smallest value of the top of an ascent.

Let $\mathcal{T}_1$ be the rightward generating tree for the set of $\{$2-1-3, 12-3$\}$-avoiding permutations. Given any $\pi \in \mathcal{S}_n$, define the parameter

$$l(\pi) = \begin{cases} n+1 & \text{if } \pi = n(n-1)\cdots 21, \\ \min\{\pi_i : i > 1,\ \pi_{i-1} < \pi_i\} & \text{otherwise.} \end{cases} \tag{5}$$

Let each permutation π be labeled by the pair $(l, r) = (l(\pi), r(\pi))$. Note that if π avoids $\{$2-1-3, 12-3$\}$, then necessarily $l \geq r$.

Lemma 3.1. *The rightward generating tree $\mathcal{T}_1$ for $\{$2-1-3, 12-3$\}$-avoiding permutations is specified by the following succession rule on the labels:*

$$(2,1)$$
$$(l,r) \longrightarrow \begin{cases} (l+1,1)\ (l+1,2)\ \cdots\ (l+1,l) & \text{if } l = r, \\ (l+1,1)\ (l+1,2)\ \cdots\ (l+1,r)\ (r+1,r+1) & \text{if } l > r. \end{cases}$$

Proof. The permutation obtained by appending an entry to the right of $\pi \in \mathcal{S}_n(\text{2-1-3, 12-3})$ is 2-1-3-avoiding if and only if the appended entry is at most $r(\pi)+1$, and it is 12-3-avoiding if and only if the appended entry is at most $l(\pi)$. The labels of the children are obtained by looking at how the values of (l, r) change when the new entry is added. □

We will use this generating rule to obtain a formula for the generating function

$$M(t,u,v) := \sum_{n\geq 1} \sum_{\pi \in \mathcal{S}_n(\text{2-1-3,12-3})} u^{l(\pi)} v^{r(\pi)}\ t^n.$$

For fixed l and r, let $M_{l,r}(t) = \sum_{n\geq 1} |\{\pi \in \mathcal{S}_n(\text{2-1-3}, \text{12-3}) : l(\pi) = l, r(\pi) = r\}|\ t^n$. Note that $M(t,u,v) = \sum_{l,r} M_{l,r}(t) u^l v^r$.

Proposition 3.2. *The generating function $M(t,u,v)$ for $\{\text{2-1-3}, \text{12-3}\}$-avoiding permutations where u and v mark the parameters l and r defined above, respectively, equals*

$$\frac{[(1-u)v+c_1t+c_2t^2+c_3t^3+c_4t^4-((1-u)v+tu+t^2u^2v)\sqrt{1-2t-3t^2})]u^2v}{2(1-u-tu(1-u)+t^2u^2)(1-uv+tuv+t^2u^2v^2)},$$

where $c_1 = 2-u-v-uv+2u^2v$, $c_2 = u(-1+(2-u)v+2(u-1)v^2)$, $c_3 = u^2v(-3+2v-2uv)$, and $c_4 = -2u^3v^2$.

Substituting $u = v = 1$ in the above expression we recover the generating function for the Motzkin numbers.

Proof. The coefficient of t^n in $M(t,u,v)$ is the sum of $u^l v^r$ over all the pairs (l,r) of labels that appear at level n of the tree. By Lemma 3.1, the children of a node with labels (l,l) contribute $u^{l+1}v + u^{l+1}v^2 + \cdots + u^{l+1}v^l$ to the next level, and the children of a node with labels (l,r) with $l > r$ contribute $u^{l+1}v + u^{l+1}v^2 + \cdots + u^{l+1}v^r + u^{r+1}v^{r+1}$. It follows that

$$\begin{aligned}(6)\ \ M(t,u,v) &= tu^2v + t\sum_{l} M_{l,l}(t)u^{l+1}(v+v^2+\cdots+v^l)\\ &\quad + t\sum_{l>r} M_{l,r}(t)[u^{l+1}(v+v^2+\cdots+v^r)+u^{r+1}v^{r+1}].\end{aligned}$$

It will be convenient to define

$$\begin{aligned} M_{>}(t,u,v) &:= \sum_{n\geq 1} \sum_{\substack{\pi\in\mathcal{S}_n(\text{2-1-3},\text{12-3})\\ \text{with } l(\pi)>r(\pi)}} u^{l(\pi)}v^{r(\pi)}\ t^n,\\ M_{=}(t,u,v) &:= \sum_{n\geq 1} \sum_{\substack{\pi\in\mathcal{S}_n(\text{2-1-3},\text{12-3})\\ \text{with } l(\pi)=r(\pi)}} (uv)^{l(\pi)}\ t^n,\end{aligned}$$

so that $M(t,u,v) = M_>(t,u,v) + M_=(t,u,v)$. Taking from (6) only the pairs (l,r) with $l > r$, we get

$$\begin{aligned} M_>(t,u,v) &= tu^2v + t\sum_l M_{l,l}(t)u^{l+1}(v+v^2+\cdots+v^l) \\ &\quad + t\sum_{l>r} M_{l,r}(t)[u^{l+1}(v+v^2+\cdots+v^r)] \\ &= tu^2v + t\sum_l M_{l,l}(t)u^{l+1}\frac{v^{l+1}-v}{v-1} + t\sum_{l>r} M_{l,r}(t)u^{l+1}\frac{v^{r+1}-v}{v-1} \\ &= tu^2v + \frac{tuv}{v-1}\left[M_=(t,u,v) - M_=(t,u,1) + M_>(t,u,v) - M_>(t,u,1)\right]. \end{aligned} \tag{7}$$

Similarly, taking from (6) only the pairs (l,r) with $l = r$,

$$M_=(t,u,v) = t\sum_{l>r} M_{l,r}(t)u^{r+1}v^{r+1} = tuv\sum_{l>r} M_{l,r}(t)(uv)^r = tuv\ M_>(t,1,uv). \tag{8}$$

Using in (7) the expression of $M_=$ in terms of $M_>$ given in (8), we get

$$\begin{aligned} M_>(t,u,v) = tu^2v + \frac{tuv}{v-1}&\left[tuv\ M_>(t,1,uv) - tu\ M_>(t,1,u)\right. \\ &\left. + M_>(t,u,v) - M_>(t,u,1)\right]. \end{aligned}$$

Substituting $u = 1$ in this equation and collecting the terms in $M_>(t,1,v)$, we have

$$\left(1 - \frac{t^2v^2}{v-1} - \frac{tv}{v-1}\right) M_>(t,1,v) = tv - \frac{t(t+1)v}{v-1} M_>(t,1,1). \tag{9}$$

Now we apply the kernel method, with the substution $v = v_0 = v_0(t) = \frac{1-t-\sqrt{1-2t-3t^2}}{2t^2}$ in (9), which gives

$$M_>(t,1,1) = \frac{v_0 - 1}{t+1} = \frac{1 - t - 2t^2 - \sqrt{1-2t-3t^2}}{2t^2(t+1)}.$$

Plugging this expression for $M_>(t,1,1)$ back into (9) we get that

$$M_>(t,1,v) = \frac{(1 - t - 2t^2v - \sqrt{1-2t-3t^2})v}{2t(1 - v + tv + t^2v^2)}. \tag{10}$$

If we write equation (9) as

$$\begin{aligned} &\left(1 - \frac{tuv}{v-1}\right) M_>(t,u,v) \\ &\quad = tu^2v + \frac{tuv}{v-1}\left[tuv\ M_>(t,1,uv) - tu\ M_>(t,1,u) - M_>(t,u,1)\right], \end{aligned}$$

we can apply again the kernel method to it, taking $v = v_1 = v_1(t,u) = \frac{1}{1-tu}$. This cancels the left hand side and gives

$$M_>(t,u,1) = \frac{[2(1-u)+u^2 - t(1+2t)u^2+(1-2u)\sqrt{1-2t-3t^2})]tu^2}{2(1-u+tu+t^2u^2)(1-u-tu(1-u)+t^2u^2)}$$

using (10). Substituting back into (11) and using (10) again we get that $M_>(t,u,v)$ equals

$$\frac{[2-u-uv+u^2v+tu(v-1)-t(1+2t)u^2v+(1-2u)\sqrt{1-2t-3t^2})]tu^2v}{2(1-u-tu(1-u)+t^2u^2)(1-uv+tuv+t^2u^2v^2)}.$$

Finally, combining it with the fact that $M(t,u,v) = M_>(t,u,v) + M_=(t,u,v) = M_>(t,u,v) + tuv\, M_>(t,1,uv)$, we obtain the desired expression for $M(t,u,v)$. □

3.2. $\{$2-1-3, 32-1$\}$**-avoiding permutations.** It is already known [5] that $|\mathcal{S}_n(\text{2-1-3, 32-1})| = 2^{n-1}$. Here we use rightward generating trees with two labels to recover this simple fact, and to refine it with two parameters: the value of the last entry and the largest value of the bottom of a descent. Let $\mathcal{T}_2$ be the rightward generating tree for the set of $\{$2-1-3, 32-1$\}$-avoiding permutations. Given any $\pi \in \mathcal{S}_n$, define the parameter

$$h(\pi) = \begin{cases} 0 & \text{if } \pi = 12\cdots n, \\ \max\{\pi_i : i > 1,\ \pi_{i-1} > \pi_i\} & \text{otherwise.} \end{cases}$$

To each node π of $\mathcal{T}_2$ we assign the pair of labels $(h,r) = (h(\pi), r(\pi))$. Note that if π avoids $\{$2-1-3, 32-1$\}$, then necessarily $h \le r$.

Lemma 3.3. *The rightward generating tree $\mathcal{T}_2$ for $\{$2-1-3, 32-1$\}$-avoiding permutations is specified by the following succession rule on the labels:*

$$\begin{array}{l} (0,1) \\ (h,r) \longrightarrow (h+1,h+1)\ (h+2,h+2)\ \cdots\ (r,r)\ (h,r+1). \end{array}$$

Proof. When we append an entry i to a $\{$2-1-3, 32-1$\}$-avoiding permutation, the new permutation is 2-1-3-avoiding if and only if $i \le r(\pi)+1$, and it is 32-1-avoiding if and only if $i \ge h(\pi)$. The list of labels of the children obtained by appending an i satisfying these two conditions is the right hand side of the rule. □

This generating rule can be used to obtain a formula for the generating function

$$N(t,u,v) := \sum_{n\ge1} \sum_{\pi\in\mathcal{S}_n(\text{2-1-3,32-1})} u^{h(\pi)} v^{r(\pi)}\, t^n.$$

For fixed h and r, let $N_{h,r}(t) = \sum_{n\ge1} |\{\pi \in \mathcal{S}_n(\text{2-1-3, 32-1}) : h(\pi) = h, r(\pi) = r\}|\, t^n$. Note that $N(t,u,v) = \sum_{h,r} N_{h,r}(t) u^h v^r$.

Proposition 3.4. *The generating function for* $\{2\text{-}1\text{-}3, 32\text{-}1\}$*-avoiding permutations where* u *and* v *mark the parameters* h *and* r *defined above is*

$$N(t,u,v) = \frac{tv(1-t+tu-tuv)}{(1-tv)(1-t-tuv)}.$$

Proof. By Lemma 3.3, the children of a node with labels (h,r) contribute $u^{h+1}v^{h+1}+u^{h+1}v^{h+2}+\cdots+u^r v^r+u^h v^{r+1}$ to the next level. It follows that

$$\begin{aligned} N(t,u,v) &= tv + \sum_{h,r} N_{h,r}(t)\left[\frac{(uv)^{r+1}-(uv)^{h+1}}{uv-1}+u^h v^{r+1}\right] \\ &= tv + tvN(t,u,v) + \frac{tuv[N(t,1,uv)-N(t,uv,1)]}{uv-1}. \end{aligned} \tag{11}$$

Substituting $u=1$ we get

$$\left(1-tv-\frac{tv}{v-1}\right)N(t,1,v) = tv - \frac{tv}{v-1}N(t,v,1), \tag{12}$$

and substituting $v=1$ in (11) gives

$$\left(1-t+\frac{tu}{u-1}\right)N(t,u,1) = t + \frac{tu}{u-1}N(t,1,u). \tag{13}$$

Combining (12) and (13), using the same variable w for both u and v, we get

$$N(t,1,w) = \frac{tw}{1-t-tw}, \qquad N(t,w,1) = \frac{t}{1-t-tw}.$$

Using these expressions in (11) we get the desired formula for $N(t,u,v)$. □

The above result can indeed be obtained as well without using rightward generating trees. The recursive structure of 2-1-3-avoiding permutations (i.e., they are of the form $\sigma 1 \tau$, where σ and τ are 2-1-3-avoiding and any entry in σ is larger than any entry in τ) can be used to obtain an equation satisfied by $N(t,u,v)$ and deduce the above formula without much difficulty.

3.3. **$\{2\text{-}1\text{-}3, 34\text{-}21\}$-avoiding permutations.** The labels that will be convenient to use to describe the rightward generating tree for this class are $(s,r) = (s(\pi), r(\pi))$, where

$$s(\pi) = \begin{cases} 0 & \text{if } \pi = n(n-1)\cdots 21, \\ \max\{\pi_i :\ \pi_i < \pi_{i+1}\} & \text{otherwise,} \end{cases} \tag{14}$$

and $r(\pi) = \pi_n$ as usual.

Lemma 3.5. *The rightward generating tree for* $\{2\text{-}1\text{-}3, 34\text{-}21\}$*-avoiding permutations is specified by the following succession rule on the labels:*

$$(0,1)$$
$$(s,r) \longrightarrow \begin{cases} (s{+}1,1)\ (s{+}1,2)\ \cdots\ (s{+}1,s)\ (s,s{+}1)\ (r,r+1) & \textit{if } s<r, \\ (s+1,r+1) & \textit{if } s>r. \end{cases}$$

Proof. First note that the 2-1-3-avoiding condition implies that new entry appended to π has to be at most $r{+}1$. If $h<r$ and π_i is an entry to the right of h, then $\pi_i>h$, otherwise $h\pi_i r$ would be an occurrence of 2-1-3. In fact, we also know that $\pi_i \neq h+1$, unless $\pi_i = r = h+1$, because otherwise the entry following $h+1$ would be greater than it, contradicting the definition of h. So, unless $r=h+1$, the entry $h+1$ precedes h, so the appended entry cannot be greater than $h+1$, otherwise it would create a 2-1-3. This explains the labels in the case $h<r$. If $h>r$, the appended entry has to be greater than r for the new permutation to be 34-21-avoiding. □

Let
$$K(t,u,v) := \sum_{n\geq 1} \sum_{\pi\in\mathcal{S}_n(\text{2-1-3,34-21})} u^{s(\pi)} v^{r(\pi)}\, t^n = \sum_{s,r} K_{s,r}(t) u^s v^r,$$
and let $K_<(t,u,v)$ and $K_>(t,u,v)$ be defined similarly, with the sum running only over permutations with $s(\pi)<r(\pi)$ and $s(\pi)>r(\pi)$, respectively, so that $K(t,u,v)=K_<(t,u,v)+K_>(t,u,v)$.

Proposition 3.6. *The generating function for* $\{$2-1-3, 34-21$\}$*-avoiding permutations where u and v mark the parameters s and r defined above is*

$$K(t,u,v) = \frac{tv[1-(1+u+uv)t+(u^2+uv+u^2v)t^2]}{(1-t-tu)(1-t-tuv)(1-tuv)}. \tag{15}$$

Proof. By Lemma 3.5, the generating functions $K_<$ and $K_>$ satisfy
$$\begin{aligned} K_<(t,u,v) &= tv + tv[K_<(t,uv,1)+K_<(t,1,uv)], \\ K_>(t,u,v) &= \frac{tuv}{v-1}[K_<(t,uv,1)+K_<(t,u,1)] + tuvK_>(t,u,v). \end{aligned}$$

Substituting first $u=1$ and then $v=1$ in the first equation, we get two equations involving $K_<(t,1,w)$ and $K_<(t,w,1)$ that can be easily solved, from where we get
$$K_<(t,u,v) = \frac{tv}{1-t-tuv}.$$
The second equation then implies that
$$K_>(t,u,v) = \frac{u^2vt^3}{(1-t-tu)(1-t-tuv)(1-tuv)},$$
and the proposition follows. □

Corollary 3.7. *The number of* $\{$2-1-3, 34-21$\}$*-avoiding permutations of size* n *is* $(n-1)2^{n-2}+1$.

Proof. Taking $u = v = 1$ in (15), we get that

$$K(t,1,1) = \frac{t(1-3t+3t^2)}{(1-t)(1-2t)^2}.$$

The coefficient of t^n in the series expansion of this rational function is $(n-1)2^{n-2}+1$. $\square$

It is not difficult to show that $\mathcal{S}_n(\text{2-1-3}, \text{34-21}) = \mathcal{S}_n(\text{2-1-3}, \text{3-4-2-1}) = \mathcal{S}_n(\text{1-3-2}, \text{3-4-2-1})$. This last set of permutations had been enumerated by West [13], and Corollary 3.7 agrees with his result.

3.4. **$\{$1-2-34, 2-1-3$\}$-avoiding permutations.** The generating function for these permutations appears in [10]. It is easy to see that $\mathcal{S}_n(\text{1-2-34}, \text{2-1-3}) = \mathcal{S}_n(\text{1-2-3-4}, \text{2-1-3})$, and the latter set of permutations was studied in [13], where it is shown that they are counted by the Fibonacci numbers F_{2n-1}. Here we derive the generating function and obtain a refinement of it using a rightward generating tree with two labels.

Let the labels of a permutation π be the pair $(m,r) = (m(\pi), r(\pi))$, where $r(\pi) = \pi_n$ and

$$m(\pi) = \begin{cases} n+1 & \text{if } \pi = n(n-1)\cdots 21, \\ \min\{\pi_i : \exists j < i \text{ with } \pi_j < \pi_i\} & \text{otherwise.} \end{cases}$$

Note that we always have $m(\pi) \le r(\pi)$ unless $r = 1$.

Lemma 3.8. *The rightward generating tree for $\{$1-2-34, 2-1-3$\}$-avoiding permutations is specified by the following succession rule on the labels:*

$$(2,1)$$
$$(m,r) \longrightarrow \begin{cases} (m{+}1,1)\ (2,2) & \textit{if } r=1, \\ (m{+}1,1)\ (2,2)\ (3,3)\ \cdots\ (m,m)\ (m,m{+}1) & \textit{if } m=r, \\ (m{+}1,1)\ (2,2)\ (3,3)\ \cdots\ (m,m)\ (m,m{+}1)\ \cdots\ (m,r) & \\ & \textit{if } m<r. \end{cases}$$

Proof. As usual, the appended entry has to be at most $r+1$ for the permutation to avoid 2-1-3. In the case that $m < r$, this entry cannot be greater than r in order to avoid 1-2-34. The labels are now obtained by looking at how the parameter m changes after appending the new entry. $\square$

Let $H(t,u,v) := \sum_{n\ge1} \sum_{\pi\in\mathcal{S}_n(\text{1-2-34},\text{2-1-3})} u^{m(\pi)} v^{r(\pi)}\ t^n$, and let $H_1(t,u,v)$, $H_=(t,u,v)$, and $H_<(t,u,v)$ be defined similarly, with the summation restricted to permutations with $r(\pi)=1$, $m(\pi)=r(\pi)$, and $m(\pi) < r(\pi)$, respectively, so that $H(t,u,v) = H_1(t,u,v) + H_=(t,u,v) + H_<(t,u,v)$.

Proposition 3.9. *The generating function for* $\{1\text{-}2\text{-}34, 2\text{-}1\text{-}3\}$*-avoiding permutations where* u *and* v *mark the parameters* m *and* r *defined above is*

$$H(t,u,v) = \frac{tu^2v[1+(v-3)t+(1-v+v^2)t^2]}{(1-3t+t^2)(1-tuv)}.$$

Proof. From Lemma 3.8 we get the following functional equations defining H_1, $H_=$, and $H_<$.

$$\begin{aligned}
H_1(t,u,v) &= tu^2v + tuvH(t,u,1)\\
H_=(t,u,v) &= tu^2v^2H_1(t,1,1) + \frac{tuv}{uv-1}[H_=(t,uv,1) - uvH_=(t,1,1)\\
&\qquad + H_<(t,uv,1) - uvH_<(t,1,1)]\\
H_<(t,u,v) &= tvH_=(t,uv,1) + \frac{tv}{v-1}[H_<(t,u,v) - H_<(t,uv,1)]
\end{aligned}$$

Combining them appropriately, we obtain this functional equation for H:

$$\begin{aligned}
H(t,u,v) &= tu^2v + t^2u^2v^2 + \left(tuv + \frac{tv(1-tuv)}{v-1}\right)H(t,u,v)\\
&\quad + (1-tuv)tv\left(\frac{u(1+tv)}{uv-1} - \frac{1}{v-1}\right)H(t,uv,1)\\
&\quad + tu^2v^2(1+tv)\left(t - \frac{1-t}{uv-1}\right)H(t,1,1).
\end{aligned} \tag{16}$$

It is convenient to introduce a variable $w = uv$, and consider the function $\tilde{H}(t,w,v) = H(t,\frac{w}{v},v)$. After collecting the terms with $\tilde{H}(t,w,v)$, (16) becomes

$$\begin{aligned}
&\left(1 - tw - \frac{tv(1-tw)}{v-1}\right)\tilde{H}(t,w,v)\\
&\quad = \frac{tw^2}{v} + t^2w^2 + (1-tw)tv\left(\frac{w(1+tv)}{v(w-1)} - \frac{1}{v-1}\right)\tilde{H}(t,w,1)\\
&\qquad\qquad + tw^2(1+tv)\left(t - \frac{1-t}{w-1}\right)\tilde{H}(t,1,1).
\end{aligned} \tag{17}$$

Taking $v = \frac{1}{1-t}$, the kernel on the left hand side is canceled, and we get the equation

$$0 = tw^2 + \frac{tw^2(1-tw)}{(1-t)(1-w)}\tilde{H}(t,1,1) + \frac{(1-tw)(1-w-t+2tw)}{(1-t)(w-1)}\tilde{H}(t,w,1) \tag{18}$$

Now we apply the kernel method again, this time taking $w=\frac{1+t}{1-2t}$ to cancel the term multiplying $\tilde{H}(t,w,1)$. This gives the formula for $\tilde{H}(t,1,1)$, namely

$$\tilde{H}(t,1,1)=\frac{t(1-t)}{1-3t+t^2}.$$

Substituting this expression back into (18) we get the formula for $\tilde{H}(t,w,1)$, which then we use in (17) to obtain

$$\tilde{H}(t,w,v)=\frac{tw^2[1+(v-3)t+(1-v+v^2)t^2]}{v(1-3t+t^2)(1-tw)},$$

and now we just use that $H(t,u,v)=\tilde{H}(t,uv,v)$. □

3.5. **$\{12\text{-}34, 2\text{-}1\text{-}3\}$-avoiding permutations.** It was proved by Mansour [10] that the generating function for permutations avoiding $\{12\text{-}34, 2\text{-}1\text{-}3\}$ is $\frac{1-2t-t^2-\sqrt{1-4t+2t^2+t^4}}{2t^2}$. Using the labels (l,r) defined as in (5), we can construct a generating tree with two labels for this class of permutations. The proof of the following lemma is straightforward and analogous to that of Lemma 3.1.

Lemma 3.10. *The rightward generating tree for $\{12\text{-}34, 2\text{-}1\text{-}3\}$-avoiding permutations is specified by the following succession rule on the labels:*

$$\begin{array}{l}(2,1)\\ (l,r)\longrightarrow\begin{cases}(l{+}1,1)\ (l{+}1,2)\ \cdots\ (l{+}1,r)\ (r{+}1,r{+}1) & \text{if } l>r,\\ (l{+}1,1)\ (l{+}1,2)\ \cdots\ (l{+}1,l)\ (l,l{+}1) & \text{if } l=r,\\ (l{+}1,1)\ (l{+}1,2)\ \cdots\ (l{+}1,l)\ (l,l{+}1)\ (l,l{+}2)\ \cdots\ (l,r) & \\ & \text{if } l<r.\end{cases}\end{array}$$

Let $F(t,u,v):=\sum_{n\geq 1}\sum_{\pi\in\mathcal{S}_n(12\text{-}34,2\text{-}1\text{-}3)}u^{l(\pi)}v^{r(\pi)}\,t^n$, and let $F_>(t,u,v)$, $F_=(t,u,v)$, and $F_<(t,u,v)$ be defined similarly, with the summation restricted to permutations with $l(\pi)>r(\pi)$, $l(\pi)=r(\pi)$, and $l(\pi)<r(\pi)$, respectively. By definition, $F(t,u,v)=F_>(t,u,v)+F_=(t,u,v)+F_<(t,u,v)$.

Proposition 3.11. *The generating function for $\{12\text{-}34, 2\text{-}1\text{-}3\}$-avoiding permutations where u and v mark the parameters l and r defined above is*

$$F(t,u,v)=\frac{u^2v[p_1(t,u,v)+p_2(t,u,v)\sqrt{1-4t+2t^2+t^4}]}{2[(1+tuv)^2-uv-t-uvt^2][1+(u+t)(tu-1)]},$$

where $p_1(t,u,v)=(1-u)v+(2-u-4v+2uv+v^2+2u^2v-uv^2)t+(-4+u+6v+uv-3v^2-6u^2v+3u^2v^2)t^2+(2+u-4v-5uv+3v^2+4uv^2-4u^2v^2-2u^3v^2+2u^2v^3-2v^3u+4u^2v)t^3+(-u+v+4uv-u^2v^2-4uv^2-2u^3v^3-v^2+2u^3v^2+2v^3u)t^4-uv(v-1)(2uv-1)t^5$, *and* $p_2(t,u,v)=(u-1)v+[(u-1)v(v-2)-u]t+(u-v+v^2-u^2v^2)t^2+uv(1-v)t^3$.

Note that this expression becomes much simpler if we ignore the parameter r, that is,

$$F(t,u,1) = \frac{u^2(1-2tu-t^2-\sqrt{1-4t+2t^2+t^4})}{2[1+(u+t)(tu-1)]},$$

and coincides with the result from [10] if we ignore both parameters:

$$F(t,1,1) = \frac{1-2t-t^2-\sqrt{1-4t+2t^2+t^4}}{2t^2}.$$

Proof. From Lemma 3.10 we get the following functional equations defining $F_>$, $F_=$, and $F_<$.

$$\begin{aligned}
(19)\; F_>(t,u,v) &= tu^2v + \frac{tuv}{v-1}[F_>(t,u,v) - F_>(t,u,1) + F_=(t,u,v) \\
&\quad -F_=(t,u,1) + F_<(t,uv,1) - F_<(t,u,1)], \\
(20)\; F_=(t,u,v) &= tuvF_>(t,1,uv), \\
(21)\; F_<(t,u,v) &= tvF_=(t,u,v) + \frac{tv}{v-1}[F_<(t,u,v) - F_<(t,uv,1)].
\end{aligned}$$

We can introduce a variable $w = uv$ in (21), and apply the kernel method with $v = \frac{1}{1-t}$ to get that

$$F_<(t,w,1) = \frac{t^2wF_>(t,1,w)}{1-t}.$$

Using this expression together with (20) in (19), we get an equation that involves only $F_>$:

$$(22) \qquad \left(1 - \frac{tuv}{v-1}\right) F_>(t,u,v)$$
$$= tu^2v - \frac{tuv}{v-1}\left[F_>(t,u,1) + \frac{tuv}{1-t}F_>(t,1,uv) - \frac{tu}{1-t}F_>(t,1,u)\right].$$

Substituting $u = 1$, it becomes

(23)
$$\frac{(1-v-t+2tv-vt^2+v^2t^2)}{(1-v)(1-t)}F_>(t,1,v) = tv + \frac{tv}{(1-v)(1-t)}F_>(t,1,1).$$

We apply the kernel method again, this time with $v = \frac{1-2t+t^2-\sqrt{1-4t+2t^2+t^4}}{2t^2}$ to cancel the left hand side of (23), which yields

$$F_>(t,1,1) = \frac{1-3t+t^2+t^3+(t-1)\sqrt{1-4t+2t^2+t^4}}{2t^2}.$$

Now we can use (23) to obtain a formula for $F_>(t,1,v)$. Applying again the kernel method in (22), with $v = \frac{1}{1-tu}$, we get an expression for $F_>(t,u,1)$ in terms of $F_>(t,1,u)$ and $F_>(t,1,\frac{u}{1-tu})$, and therefore a formula for $F_>(t,u,1)$.

Substituting back into (22), we get a formula for $F_>(t,u,v)$. From this it is straightforward to obtain formulas for $F_<(t,u,v)$ and $F_=(t,u,v)$ as well, and the result follows. □

3.6. **1-23-avoiding permutations.** It was shown in [5] that the number of 1-23-avoiding permutations of size n is the n-th Bell number. In [9] the authors use an approach similar to ours to enumerate these permutations. Their so-called *scanning-elements algorithm* finds a system of recurrence relations for the number of permutations that start with a particular entry. Then, from these equations they obtain a functional equation satisfied by the generating function for 1-23-avoiding permutations.

We show here how essentially the same functional equation is obtained using generating trees with two labels. The labels are particularly easy in this case because we can take one of them to be just the length n of the permutation. The labels of $\pi \in \mathcal{S}_n$ are then (r,n), where $r=\pi_n$ as usual. The advantage of having one of the labels be n is that we do not need an extra variable for this label in the generating function, since it is already encoded in the exponent of the variable t.

Lemma 3.12. *The rightward generating tree for* 1-23*-avoiding permutations is specified by the following succession rule on the labels:*

$$\begin{array}{l} (1,1) \\ (r,n) \longrightarrow \begin{cases} (1,n+1)\ (2,n+1)\ \cdots\ (n+1,n+1) & \text{if } r=1, \\ (1,n+1)\ (2,n+1)\ \cdots\ (r,n+1) & \text{if } r>1. \end{cases} \end{array}$$

Proof. If the rightmost entry of π is 1, there is no restriction on the elements that can be appended to it. However, if the rightmost entry is greater than 1, then it cannot be smaller than the new appended entry. □

We define $G(t,u) := \sum_{n\geq 1} \sum_{\pi\in\mathcal{S}_n(\text{1-23})} u^{r(\pi)}\, t^n$. For fixed r, let $g_{n,r} = |\{\pi \in \mathcal{S}_n(\text{1-23}) : r(\pi)=r\}|$, and let $G_r(t)=\sum_{n\geq 1} g_{n,r}t^n$. Note that $G(t,u)=\sum_r G_r(t)u^r = \sum_{n,r} g_{n,r}u^r t^n$.

Proposition 3.13. *The generating function $G(t,u)$ for* 1-23*-avoiding permutations where u marks the value of the rightmost entry is*

$$\frac{1-u}{1-u+tu}\left[tu+t^2u^2+\frac{tu}{u-1}\sum_{k\geq 1}\left(\frac{t^{k+1}u^{k+2}}{(1-tu)(1-2tu)\cdots(1-ktu)} - \frac{(1+tu)t^k}{(1-t)(1-2t)\cdots(1-kt)}\right)\right].$$

Proof. Lemma 3.12 gives the following recurrence for G.

$$G(t,u) = tu + t\left[\sum_{r>1} G_r(t)\frac{u^{r+1}-u}{u-1} + \sum_{n\geq 1} g_{n,1}t^n \frac{u^{n+2}-u}{u-1}\right] \tag{24}$$

$$= tu + \frac{tu}{u-1}\left[(G(t,u)-uG_1(t))-(G(t,1)-G_1(t))+uG_1(tu)-G_1(t)\right].$$

Since every node has exactly one child with $r=1$, we have that $G_1(t) = t + t\,G(t,1)$. Using this in (24) and collecting the terms with $G(t,u)$, we get

$$\left(1-\frac{tu}{u-1}\right)G(t,u) = tu + t^2u^2 + \frac{tu}{u-1}\left[tu^2G(tu,1)-(1+tu)G(t,1)\right]. \tag{25}$$

Applying the kernel method with $u=\frac{1}{1-t}$ yields the functional equation

$$G(t,1) = \frac{t}{1-t}\left(1+G(\frac{t}{1-t},1)\right),$$

which agrees with [9]. By iterated application of this formula,

$$G(t,1) = \frac{t}{1-t}\left(1+\frac{t}{1-2t}\left(1+\frac{t}{1-3t}\left(1+\frac{t}{1-4t}\left(1+\cdots\right)\right)\right)\right)$$
$$= \sum_{k\geq 1}\frac{t^k}{(1-t)(1-2t)\cdots(1-kt)},$$

which is the well-known ordinary generating function for the Bell numbers. Substituting this expression back into (25) we get the desired formula for $G(t,u)$. □

3.7. **$\{$1-23, 3-12$\}$-avoiding permutations.** An argument very similar to the one in the previous section gives us the generating function for the number of $\{$1-23, 3-12$\}$-avoiding permutations. These permutations were studied in [6], where they showed that if we let $b_n = |\mathcal{S}_n(\text{1-23, 3-12})|$, then these numbers satisfy the recurrence $b_{n+2} = b_{n+1} + \sum_{k=0}^{n}\binom{n}{k}b_k$. Here we obtain a generating function without going through the recurrence. The labels that we assign to $\pi \in \mathcal{S}_n$ are again (r,n), where $r = \pi_n$. The following lemma is analogous to Lemma 3.12.

Lemma 3.14. *The rightward generating tree for $\{$1-23, 3-12$\}$-avoiding permutations is specified by the following succession rule on the labels:*

$$(1,1)$$
$$(r,n) \longrightarrow \begin{cases} (1,n+1)\ (n+1,n+1) & \text{if } r=1,\\ (1,n+1)\ (2,n+1)\ \cdots\ (r,n+1) & \text{if } r>1.\end{cases}$$

Let $P(t,u) := \sum_{n\geq 1}\ \sum_{\pi\in\mathcal{S}_n(\text{1-23,3-12})} u^{r(\pi)}\ t^n = \sum_r P_r(t)u^r$.

Proposition 3.15. *The generating function $P(t,1)$ for $\{1\text{-}23, 3\text{-}12\}$-avoiding permutations is*

$$\sum_{k\geq 1} \frac{t^{2k-1}(1-(k-1)t)}{(1-t)^2(1-2t)^2\cdots(1-kt)^2}.$$

Proof. From Lemma 3.14 we get

$$(26) P(t,u) = tu+\frac{tu}{u-1}(P(t,u)-uP_1(t)-P(t,1)+P_1(t))+tu(P_1(t)+P_1(tu)).$$

Using that $P_1(t) = t + t\ P(t,1)$ and collecting the terms with $P(t,u)$, we get

$$(27) \left(1-\frac{tu}{u-1}\right)P(t,u)=tu+t^2u^2+t^2u^2P(tu,1)+\left(t^2u+\frac{tu(t-1-tu)}{u-1}\right)P(t,1).$$

Substituting $u = \frac{1}{1-t}$ gives

$$P(t,1) = \frac{t}{(1-t)^2}\left(1+tP(\frac{t}{1-t},1)\right),$$

and by iterated application of this formula,

$$\begin{aligned} P(t,1) &= \frac{t}{(1-t)^2}\left(1+\frac{t^2(1-t)}{(1-2t)^2}\left(1+\frac{t^2(1-2t)}{(1-t)(1-3t)^2}\right.\right.\\ &\qquad\left.\left.\left(1+\frac{t^2(1-3t)}{(1-2t)(1-4t)^2}(1+\cdots)\right)\right)\right)\\ &= \frac{t}{(1-t)^2}+\frac{t^3}{(1-t)(1-2t)^2}+\frac{t^5}{(1-t)^2(1-2t)(1-3t)^2}\\ &\qquad+\frac{t^7}{(1-t)^2(1-2t)^2(1-3t)(1-4t)^2}+\cdots, \end{aligned}$$

which is the formula above. We could substitute this expression back into (27) we get the refined formula for $P(t,u)$. $\square$

3.8. 123**-avoiding permutations.** Permutations avoiding the consecutive pattern 123 were studied in [8], where the authors give their exponential generating function. We can describe a generating tree with two labels for this class of permutations, where one label is just the length of the permutation. These labels are *colored*, meaning that a pair of labels consists of its two components plus one of two possible "colors". To each $\pi \in \mathcal{S}_n(123)$ we assign the pair (π_n, n) if $\pi_{n-1} > \pi_n$ or $n = 1$, and the pair $(\pi_n, n)'$ if $\pi_{n-1} < \pi_n$ (some authors [1] call this a *colored* succession rule).

Lemma 3.16. *The rightward generating tree for* 123*-avoiding permutations is specified by the following succession rule on the labels:*

$$\begin{array}{lll} (1,1) & & \\ (r,n) & \longrightarrow & (1,n+1)\ (2,n+1)\cdots(r,n+1) \\ & & \qquad (r+1,n+1)'\ (r+2,n+1)'\cdots(n+1,n+1)' \\ (r,n)' & \longrightarrow & (1,n+1)\ (2,n+1)\cdots(r,n+1). \end{array}$$

Proof. If $\pi_{n-1} > \pi_n$, we can append any entry to π without creating a 123. If $\pi_{n-1} < \pi_n$, the appended entry cannot be larger than $r = \pi_n$. □

Let

$$A(t,u) := tu + \sum_{n\geq 2} \sum_{\substack{\pi\in\mathcal{S}_n(123)\\ \text{with } \pi_{n-1}>\pi_n}} u^{r(\pi)}\, t^n, \quad B(t,u) := \sum_{n\geq 2} \sum_{\substack{\pi\in\mathcal{S}_n(123)\\ \text{with } \pi_{n-1}<\pi_n}} u^{r(\pi)}\, t^n,$$

and let $C(t,u) = A(t,u) + B(t,u)$. Lemma 3.16 provides these functional equations defining A and B:

$$\begin{aligned} A(t,u) &= tu + \frac{tu}{u-1}[C(t,u) - C(t,1)], \\ B(t,u) &= \frac{tu}{u-1}[uA(tu,1) - A(t,u)]. \end{aligned}$$

The ideas in the rest of this proof are due by Mireille Bousquet-Mélou [3]. The second equation implies that $C(t,u) = A(t,u) + \frac{tu}{u-1}[uA(tu,1) - A(t,u)]$, and substituting this back into the first equation, we get

$$\frac{(1-u)^2 + tu(1-u) + t^2u^2}{(u-1)^2}\, A(t,u) = tu + \frac{t^2u^3}{(u-1)^2}\, A(tu,1) - \frac{tu}{u-1}\, C(t,1). \tag{28}$$

The kernel of this equation is $(1-u)^2 + t(u-u^2) + t^2u^2$. Solving for u, the two solutions are

$$u_1 = \frac{1}{1-\alpha t},\ u_2 = \frac{1}{1-\bar{\alpha} t},\ \text{where } \alpha = \frac{1+i\sqrt{3}}{2},\ \bar{\alpha} = \frac{1-i\sqrt{3}}{2}.$$

Substituting $u = u_j$ in (28), we get

$$C(t,1) = \frac{tu_j^2}{u_j - 1}\, A(tu_j, 1) + u_j - 1,$$

for $j = 1, 2$. We can use the equation for $j = 1$ to write $A(x,1)$ in terms of $C(x/u_1, 1)$, and then substitute the result in the equation for $j = 2$. After simplifying, we get the following expression that involves only $C(\ ,1)$:

$$C(t,1) = \frac{3+i\sqrt{3}}{2(3t - i\sqrt{3})}\, C\left(\frac{t}{1+i\sqrt{3}t}, 1\right) - \frac{3(2t+1-i\sqrt{3})t}{(2t-1-i\sqrt{3})(3t-i\sqrt{3})}.$$

Extracting the coefficient of t^n in both sides of the equation, we obtain a recurrence relation for the coefficients $|\mathcal{S}_n(123)|$ of $C(t,1)$. This recurrence can be translated into an equation for the exponential generating function of these coefficients. We get that

$$\sum_{n\geq 0} |\mathcal{S}_n(123)| \frac{t^n}{n!} = \frac{\sqrt{3}}{2} \frac{e^{t/2}}{\cos(\frac{\sqrt{3}}{2}t + \frac{\pi}{6})}.$$

4. Generating trees with three labels

In this section we include two instances of permutations avoiding generalized patterns where the rightward generating tree can be described by a succession rule with three labels. One of these labels is the length of the permutation, so that the functional equations that we obtain have three variables instead of four. However, the fact that the variable t appears multiplied by another variable adds some difficulty to the equations.

To the best of our knowledge, the two classes of restricted permutations considered in this section have never been enumerated before.

4.1. $\{$1-23, 3-12, 34-21$\}$-avoiding permutations.

Given a permutation $\pi \in \mathcal{S}_n$, let $s(\pi)$ be defined as in (14). We associate to π the triple of labels $(s,r,n) = (s(\pi), r(\pi), n)$.

Lemma 4.1. *The rightward generating tree for* $\{$1-23, 3-12, 34-21$\}$*-avoiding permutations is specified by the following succession rule on the labels:*

$$(0,1)$$

$$(s,r,n) \longrightarrow \begin{cases} (s{+}1,1,n{+}1)\ (s{+}1,2,n{+}1)\ \cdots\ (s{+}1,s,n{+}1) \\ \qquad (s,s{+}1,n{+}1)\ (s,s{+}2,n{+}1)\ \cdots\ (s,r,n{+}1) \\ \qquad\qquad\qquad\qquad\qquad\qquad \textit{if } s<r\neq 1, \\ (0,1,n{+}1)\ (1,n{+}1,n{+}1) \\ \qquad\qquad\qquad\qquad\qquad\qquad \textit{if } (s,r)=(0,1), \\ (s,n{+}1,n{+}1) \qquad\qquad\qquad\qquad \textit{if } s>r=1, \\ \emptyset \qquad\qquad\qquad\qquad\qquad\qquad \textit{if } s>r>1. \end{cases}$$

Proof. If $r>1$, the appended entry has to be at most r for the new permutation to avoid 1-23. If $s>r$, it has to be at least $r+1$ for the new permutation to avoid 34-21. Finally, if $r=1$, the appended entry has to be $n+1$ for the permutation to avoid 3-12, unless $s=0$, which means that π is the decreasing permutation. Combining these conditions we get the four possible cases and the new labels in each case. □

The four cases in the succession rule above suggest dividing the set Θ of values that the pair (s,r) can take into four disjoint sets: $\Theta_1 = \{(s,r) : s < r \neq 1\}$, $\Theta_2 = \{(0,1)\}$, $\Theta_3 = \{(s,r) : s > r = 1\}$, $\Theta_4 = \{(s,r) : s > r > 1\}$. For $i = 1,2,3,4$, let

$$R_i(t,u,v) := \sum_{n\geq 1} \sum_{\substack{\pi\in\mathcal{S}_n(\text{1-23,3-12,34-21}) \\ \text{with } (s(\pi),r(\pi))\in\Theta_i}} u^{s(\pi)} v^{r(\pi)}\ t^n,$$

and let $R(t,u,v) = R_1(t,u,v) + R_2(t,u,v) + R_3(t,u,v) + R_4(t,u,v)$.

Proposition 4.2. *The generating function for* $\{1\text{-}23, 3\text{-}12, 34\text{-}21\}$*-avoiding permutations where* u *marks the parameter* s *defined above is*

$$1 + R(t,u,1) = \sum_{k\geq 0} \frac{t^{2k}u^k(1+ktu)}{(1-(k+1)t)\prod_{j=1}^{k-1}(1-jt)}. \tag{29}$$

Proof. Lemma 4.1 translates into the following equations for the generating functions R_i:

$$\begin{aligned} R_1(t,u,v) &= tuvR_2(tv,1,1) + tvR_3(tv,u,1) \\ &\quad + \frac{tv}{v-1}[R_1(t,u,v) - R_1(t,uv,1)], \\ R_2(t,u,v) &= \frac{tv}{1-t}, \\ R_3(t,u,v) &= tuvR_1(t,u,1), \quad (30) \\ R_4(t,u,v) &= \frac{tuv}{v-1}[R_1(t,uv,1) - vR_1(t,u,1)]. \quad (31) \end{aligned}$$

Combining them we get an equation involving only R_1:

$$R_1(t,u,v) = \frac{t^2uv^2}{1-tv} + t^2uv^2R_1(tv,u,1) + \frac{tv}{v-1}[R_1(t,u,v) - R_1(t,uv,1)].$$

If we collect on one side the terms with $R_1(t,u,v)$, the kernel of the equation is $1 - \frac{tv}{v-1}$. Introducing a new variable $w = uv$ and then canceling the kernel with $v = \frac{1}{1-t}$, we obtain an expression that involves $R_1(t,w,1)$ and $R_1(\frac{t}{1-t}, (1-t)w, 1)$, which can be simplified to

$$R_1(t,w,1) = tw^2\left[\frac{1}{1-2t} + \frac{1}{1-t}R_1\left(\frac{t}{1-t}, (1-t)w, 1\right)\right].$$

By iterated application of this formula,

$$\begin{aligned}
R_1(t,u,1) &= t^2u\left(\frac{1}{1-2t}+\frac{t^2u}{1-t}\left(\frac{1}{1-3t}+\frac{t^2u}{1-2t}\left(\frac{1}{1-4t}+\cdots\right)\right)\right)\\
&= \frac{t^2u}{1-2t}+\frac{(t^2u)^2}{(1-t)(1-3t)}+\frac{(t^2u)^3}{(1-t)(1-2t)(1-4t)}\\
&\quad+\frac{(t^2u)^4}{(1-t)(1-2t)(1-3t)(1-5t)}+\cdots\\
&= \sum_{k\geq 1}\frac{t^{2k}u^k}{(1-(k+1)t)\prod_{j=1}^{k-1}(1-jt)}.
\end{aligned}$$

Equation (30) gives now an expression for $R_3(t,u,1)$, and (31) implies that

$$R_4(t,u,v)=\sum_{k\geq 1}\frac{t^{2k+1}u^{k+1}(v^2+v^3+\cdots+v^k)}{(1-(k+1)t)\prod_{j=1}^{k-1}(1-jt)}.$$

Adding up the four generating functions $R(t,u,1)=R_1(t,u,1)+R_2(t,u,1)+R_3(t,u,1)+R_4(t,u,1)$ we get (29). □

The first coefficients of $R(t,1,1)$ are $1,2,4,8,19,47,125,\ldots$

4.2. **$\{1\text{-}23, 34\text{-}21\}$-avoiding permutations.** The derivation of the generating function for this class of permutations is very similar to the previous subsection. The labels that we associate to a permutation are again (s,r,n). The proof of the next lemma is analogous to that of Lemma 4.1.

Lemma 4.3. *The rightward generating tree for $\{1\text{-}23, 34\text{-}21\}$-avoiding permutations is specified by the following succession rule on the labels:*

$$\begin{array}{l}
(0,1)\\
(s,r,n)\longrightarrow\begin{cases}
(s+1,1,n+1)\ (s+1,2,n+1)\ \cdots\ (s+1,s,n+1)\\
\qquad (s,s+1,n+1)\ (s,s+2,n+1)\ \cdots\ (s,r,n+1) & \\
 & \textit{if } s<r\neq 1,\\
(0,1,n+1)\ (1,2,n+1)\ (1,3,n+1)\ \cdots\ (1,n+1,n+1) & \\
 & \textit{if } (s,r)=(0,1),\\
(s+1,2,n+1)\ (s+1,3,n+1)\ \cdots\ (s+1,s,n+1)\\
\qquad (s,s+1,n+1)\ (s,s+2,n+1)\ \cdots\ (s,n+1,n+1) & \\
 & \textit{if } s>r=1,\\
\emptyset & \textit{if } s>r>1.
\end{cases}
\end{array}$$

Divide the set Θ of values that the pair (s, r) can take into four disjoint sets Θ_i, $i = 1, 2, 3, 4$ as before, and let

$$T_i(t,u,v) := \sum_{n\geq 1} \sum_{\substack{\pi\in\mathcal{S}_n(\text{1-23,34-21}) \\ \text{with } (s(\pi),r(\pi))\in\Theta_i}} u^{s(\pi)} v^{r(\pi)}\ t^n$$

and $T(t,u,v) = T_1(t,u,v) + T_2(t,u,v) + T_3(t,u,v) + T_4(t,u,v)$.

Proposition 4.4. *The generating function for* $\{$1-23, 34-21$\}$*-avoiding permutations where* u *marks the parameter* s *defined above is*

$$T(t,u,1) = \sum_{k\geq 0} \frac{t^{k+1}u^k(1+ktu)}{(1+tu)^k(1-kt)(1-(k+1)t)}. \tag{32}$$

Proof. The equations that follow from Lemma 4.3 are now

$$\begin{aligned}
T_1(t,u,v) &= \frac{tuv}{v-1}[T_2(tv,1,1) - T_2(t,1,1)] \\
&\quad + \frac{tv}{v-1}[vT_3(tv,u,1) - T_3(t,uv,1)] \\
&\quad + \frac{tv}{v-1}[T_1(t,u,v) - T_1(t,uv,1)], \\
T_2(t,u,v) &= \frac{tv}{1-t}, \\
T_3(t,u,v) &= tuvT_1(t,u,1), \qquad (33) \\
T_4(t,u,v) &= \frac{tuv}{v-1}[T_1(t,uv,1) + T_3(t,uv,1) \qquad (34) \\
&\qquad -vT_1(t,u,1) - vT_3(t,u,1)].
\end{aligned}$$

From them we can get an equation involving only T_1:

$$\begin{aligned}
T_1(t,u,v) = \frac{t^2uv^2}{v-1}\left(\frac{v}{1-tv} - \frac{1}{1-t}\right) + \frac{t^2uv^2}{v-1}[vT_1(tv,u,1) - T_1(t,uv,1)] \\
+ \frac{tv}{v-1}[T_1(t,u,v) - T_1(t,uv,1)].
\end{aligned}$$

Letting $w = uv$ and canceling the kernel multiplying $T_1(t,u,v)$ with $v = \frac{1}{1-t}$, we get that

$$T_1(t,w,1) = \frac{tw}{(1+tw)(1-t)}\left[\frac{t}{1-2t} + T_1\left(\frac{t}{1-t}, (1-t)w, 1\right)\right].$$

Iterating this formula, we see that

$$T_1(t,u,1) = \sum_{k\geq 1} \frac{t^{k+1}u^k}{(1+tu)^k(1-kt)(1-(k+1)t)}.$$

Using (33) and (35) we get expressions for $T_3(t, u, 1)$ and $T_4(t, u, v)$. Finally, the sum $T(t, u, 1) = T_1(t, u, 1) + T_2(t, u, 1) + T_3(t, u, 1) + T_4(t, u, 1)$ gives the formula (32). □

The first coefficients of $T(t, 1, 1)$ are $1, 2, 5, 14, 42, 138, 492, \ldots$, which teaches us not to judge a sequence by looking only at its first five terms.

5. Concluding remarks

The choice of the sets of generalized patterns that have been studied in this paper may seem somewhat arbitrary. We have considered some sets of patterns for which the rightward generating tree of the class of permutations avoiding them have a simple succession rule, once appropriate labels are chosen. This is just a small sample of the sets of patterns that we could study using the same method.

We have also encountered several instances where we can construct similar generating trees with two or three labels and obtain a functional equation for the generating function, but we have not been able to solve the equation. We have not included these examples here.

We expect that this technique of rightward generating trees with two labels, together with the kernel method and other ad-hoc tools for solving the functional equations that are obtained, will lead to many more enumerative results for classes of permutations avoiding generalized patterns.

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Department of Mathematics, Dartmouth College, Hanover, NH 03755 / Centre de Recerce Matemàtica, E-08193 Bellaterra, Spain

E-mail address: `sergi.elizalde@dartmouth.edu`