INFINITE TIME AGGREGATION FOR THE CRITICAL PATLAK-KELLER-SEGEL MODEL IN $\mathbb{R}^2$

ADRIEN BLANCHET, JOSÉ A. CARRILLO, AND NADER MASMOUDI

Abstract. We analyze the two-dimensional parabolic-elliptic Patlak-Keller-Segel model in the whole Euclidean space $\mathbb{R}^2$. Under the hypotheses of integrable initial data with finite second moment and entropy, we first show local in time existence for any mass of "free-energy solutions", namely weak solutions with some free energy estimates. We also prove that the solution exists as long as the entropy is controlled from above. The main result of the paper is to show the global existence of free-energy solutions with initial data as before for the critical mass $8\pi/\chi$. Actually, we prove that solutions blow-up as a delta dirac at the center of mass when $t \to \infty$ keeping constant their second moment at any time. Furthermore, all moments larger than 2 blow-up as $t \to \infty$ if initially bounded.

Contents

1. Introduction

The Patlak-Keller-Segel (PKS) model describes the collective motion of cells which are attracted by a self-emitted chemical substance. A model organism for this type of behavior is the *dictyostelium discoideum* which segregates *cyclic adenosine monophosphate* attracting themselves in starvation conditions. It is observed that after the appearance of a suitable number of mixamoebae, they aggregate to form a multi-cellular organism called pseudo-plasmoid. This rudimentary phenomenon could be a step toward the understanding of cell differentiation.

There have been many mathematical modelling sources for chemotaxis. Historically, the first mathematical model was introduced in 1953 by C. S. Patlak in [44] and E. F. Keller and L. A. Segel in [31] in 1970. Here, we focus on the minimal Patlak-Keller-Segel model-type introduced by V. Nanjundiah in [41], which is:

$$\begin{cases} \dfrac{\partial n}{\partial t}(x,t) = \Delta n(x,t) - \chi \nabla\cdot(n(x,t)\nabla c(x,t)) & x \in \mathbb{R}^2\,,\ t>0\,, \\ -\Delta c(x,t) = n(x,t) & x \in \mathbb{R}^2\,,\ t>0\,, \\ n(x,t=0) = n_0 \geq 0 & x \in \mathbb{R}^2\,. \end{cases} \tag{1}$$

Here $(x,t) \mapsto n(x,t)$ represents the cell density, and $(x,t) \mapsto c(x,t)$ is the concentration of chemo-attractant. The first equation takes into account that the motion of cells is driven by the steepest increase in the concentration of chemo-attractant while following a Brownian motion due to external interactions. In fact, this equation is a standard drift-diffusion equation obtained from the underlying stochastic dynamical system. The second equation takes into account that cells are producing themselves the chemo-attractant while this is diffusing onto the environment. In fact, this second equation has an additional time derivative of c that in this model has been neglected assuming that the relaxation of the concentration is much quicker than the time scale of cell movement.

The constant $\chi > 0$ is the *sensitivity* of the bacteria to the chemo-attractant. Mathematically, it measures the non-linearity of the system. Since the total mass of cells is assumed to be preserved, it is usual to impose no-flux boundary conditions if these equations were posed in bounded domains. Here, we are not interested in boundary effects and for this reason we are going to consider the system in the full space without boundary conditions. The dimension 2 is critical when we consider the problem in L^1 because in $\mathbb{R}^2$ the Green kernel associated with $-\Delta c(x,t) = n(x,t)$ has a logarithmic singularity. In $\mathbb{R}^d$, for $d > 2$, the critical space is $L^{d/2}(\mathbb{R}^d)$, see [20, 21].

In [19], S. Childress and J. K. Percus conjectured that the aggregation or *chemotactic collapse*, if any, should proceed by the formation of a delta dirac at the center of mass of cell density. Concerning the understanding of this phenomena we refer to the seminal work of M. A. Herrero and J.

J. L. Velázquez, [27] and subsequent works of the last author [51, 52, 53]. Numerical evidence of this collapse has also been reported in [37]. The literature on these models is huge and it is out of the scope of this paper to give exhaustive references. For further bibliography on the Patlak-Keller-Segel system and related models we refer the interested reader to the surveys [28, 29, 45].

Since the solution of the Poisson equation $-\Delta c = n$ is given up to a harmonic function, we will define the concentration of the chemo-attractant directly by

$$c(x,t) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \log|x-y|\, n(y,t)\, dy\ .$$

Hence, for $n(t) \in \mathcal{M}_+(\mathbb{R}^2)$, the space of positive bounded total variation measures, $\nabla c(t) \in L^{2,\infty}(\mathbb{R}^2)$, where $L^{2,\infty}(\mathbb{R}^2)$ is the Lorenz space given by

$$f \in L^{2,\infty}(\mathbb{R}^2) \quad \text{if and only if} \quad \forall\, t>0,\ \text{meas}\{|f(x)|>t\} \le \frac{C}{t^2}.$$

Thanks to this remark the Patlak-Keller-Segel system (1) can be viewed as a parabolic equation with a non-local interaction term. Initial data are assumed to verify

$$(1+|x|^2)\, n_0 \in L^1_+(\mathbb{R}^2) \quad \text{and} \quad n_0 \log n_0 \in L^1(\mathbb{R}^2)\ . \tag{2}$$

If we assume the existence of smooth fast-decaying at infinity non-negative $L^1([0,T]\times\mathbb{R}^2)$ solutions n, for all $T>0$, to the Patlak-Keller-Segel system (1) with initial data satisfying assumptions (2), then the solutions satisfy the formal conservations of the total mass of the system

$$M := \int_{\mathbb{R}^2} n_0(x)\, dx = \int_{\mathbb{R}^2} n(x,t)\, dx$$

and its center of mass

$$M_1 := \int_{\mathbb{R}^2} x\, n_0(x)\, dx = \int_{\mathbb{R}^2} x\, n(x,t)\, dx\ .$$

Due to the conservation of the center of mass and the translational invariance of (1), we may assume $M_1 = 0$ without loss of generality.

There is a competition between the tendency of cells to spread all over $\mathbb{R}^2$ by diffusion and the tendency to aggregate because of the drift induced by the chemo-attractivity. The balance between these two mechanisms happens precisely at the critical mass $\chi\, M = 8\,\pi$.

In fact, in the case $\chi\, M > 8\,\pi$, under assumptions (2) on n_0, it is easy to see, using moment estimates, that solutions to the Patlak-Keller-Segel system (1) blow-up in finite time. J. Dolbeault and B. Perthame announced in [22] that if $\chi\, M < 8\pi$ there is global existence of solution for the Patlak-Keller-Segel system (1) in a weak sense. This result was further completed and improved in [13] where the existence of "free-energy solutions" is proved under the hypothesis (2) for the subcritical case $\chi M < 8\pi$. Furthermore, the asymptotic behavior in the subcritical case is shown to be given by unique self-similar profiles of the system. We also refer to [40] for radially symmetric results concerning self-similar behavior.

The critical case $\chi M = 8\pi$ has a family of explicit stationary solutions [51] of the form

$$n_b(x) = \frac{8b}{\chi(b+|x|^2)^2} \tag{3}$$

with $b > 0$. All of these stationary solutions have critical mass and infinite second moment and they played an important role in the matched-asymptotic expansions done by J.J.L. Velázquez in [51] to show the existence of blowing-up solutions with delta dirac formation for the supercritical case $\chi M > 8\pi$ and subsequent improvements [52, 53].

In the case $\chi M = 8\pi$, P. Biler, G. Karch, P. Laurençot and T. Nadzieja [9] prove the existence of global radially symmetric solutions to (1) for initial data with finite or infinite second moment. They also show that all the stationary solutions n_b given in (3) have an attraction region for radial symmetric solutions with initial data of infinite second moment in a suitable sense [9, Proposition 3.6]. However, the asymptotic behavior of radially symmetric solutions with initial data of finite second moment was left open and the radial stationary solution given by the $\frac{8\pi}{\chi}\delta_0$ was not proved to attract finite second moment radial solutions.

The main aim of this paper is to show the existence of global in time "free-energy solutions", defined below, for the Patlak-Keller-Segel system (1) with critical mass $\chi M = 8\pi$ and initial data satisfying assumptions (2). Moreover, we will show that these solutions blow-up in infinite time converging towards a delta dirac distribution at the center of mass. The main tool for the proof of global existence is the free energy functional:

$$t \mapsto \mathcal{F}[n](t) := \int_{\mathbb{R}^2} n(x,t)\log n(x,t)\,dx - \frac{\chi}{2}\int_{\mathbb{R}^2} n(x,t)\,c(x,t)\,dx\,.$$

The free energy functional has a long history in kinetic modelling and its diffusive approximations, see [1] for a modern perspective on the use of free energy or entropy functionals in nonlinear diffusion and kinetic models. In fact, it is important to point out that the PKS model has also been considered for a long time by the kinetic theory community as a diffusive limit of gravitational kinetic models with the name of gravitational drift-diffusion-Poisson or Smoluchowsky-Poisson system, see [3, 43, 10, 54, 11, 23] for instance. It was introduced for chemotactic models by T. Nagai, T. Senba and K. Yoshida in [38], by P. Biler in [7] and by H. Gajewski and K. Zacharias in [25]. We also refer to [47, 50] and references therein. Moreover, the free energy functional and related functional inequalities played an essential role in improving the range of existence of global in time solutions up to the critical mass in [22, 13] for the PKS system (1), for nonlinear diffusion and chemotactic models in [32, 14] and for related models [15]. Let us finally mention that the PKS system has been derived from kinetic models of chemotaxis [49, 42, 17].

We will work with "improved weak solutions" to the PKS system (1). We first remind that the notion of weak solutions n in the space $C^0\big([0,T);L^1_{\text{weak}}(\mathbb{R}^2)\big)$, with fixed $T>0$, using the symmetry in x, y for the concentration gradient was introduced in [46]. This notion is capable of handling measure solutions. We shall say that $n\in C^0\big([0,T);L^1_{\text{weak}}(\mathbb{R}^2)\big)$ is a weak solution to the PKS system (1) if for all test functions $\psi\in\mathcal{D}(\mathbb{R}^2)$,

$$\frac{d}{dt}\int_{\mathbb{R}^2}\psi(x)\,n(x,t)\,dx =$$

$$\int_{\mathbb{R}^2}\Delta\psi(x)n(x,t)\,dx - \frac{\chi}{4\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}[\nabla\psi(x)-\nabla\psi(y)]\cdot\frac{x-y}{|x-y|^2}\,n(x,t)n(y,t)dxdy$$

holds in the distributional sense in $(0,T)$ and $n(0)=n_0$. We now define the concept of "free-energy solution" that we consider in this paper:

Definition 1.1 (Free-energy solution). *Given $T>0$, the function n is a* free-energy solution *to the Patlak-Keller-Segel system* (1) *with initial data n_0 on $[0,T]$ if $(1+|x|^2+|\log n|)\,n\in L^\infty((0,T),L^1(\mathbb{R}^2))$, n satisfies* (1) *in the above weak sense and*

$$\mathcal{F}[n](t)+\int_0^t\int_{\mathbb{R}^2} n(x,t)\left|\nabla\left(\log n(x,t)\right)-\chi\nabla c(x,t)\right|^2\,dx\,ds\le\mathcal{F}[n_0]$$

for almost every $t\in(0,T)$.

The concept of free-energy solutions is necessary to apply entropy methods to analyze the asymptotic behaviour of solutions. Our first result shows local in time existence of free-energy solutions for all masses, and characterizes the maximal time of existence of free-energy solutions. This result extends the known existence theory of free-energy solutions obtained in [13] for subcritical mass.

Proposition 1.2 (Maximal Free-energy Solutions). *Under assumptions* (2) *on the initial data n_0, there exists a maximal time $T^*>0$ of existence of a free-energy solution to the PKS system* (1). *Moreover, if $T^*<\infty$ then*

$$\lim_{t\nearrow T^*}\int_{\mathbb{R}^2} n(x,t)\log n(x,t)\,dx=+\infty.$$

The main result of this paper describes the behaviour of the solution to the Patlak-Keller-Segel system (1) with critical mass $M=8\pi/\chi$.

Theorem 1.3 (Infinite Time Aggregation)**.** *If $\chi M = 8\pi$, under assumptions* (2) *on the initial data n_0, there exists a global in time non-negative free-energy solution of the Patlak-Keller-Segel system* (1) *with initial data n_0. Moreover if $\{t_p\}_{p\in\mathbb{N}} \to \infty$ as $p \to \infty$, then $t_p \mapsto n(x,t_p)$ converges to a delta dirac of mass $8\,\pi/\chi$ concentrated at the center of mass of the initial data weakly-* as measures as $p \to \infty$.*

As mentioned before, the main tool is the free energy $\mathcal{F}[n](t)$ which is related to its time derivative, the Fisher information, in the following way: consider a non-negative solution $n \in C^0([0,T), L^1(\mathbb{R}^2))$ of the Patlak-Keller-Segel system (1) such that $n(1+|x|^2)$, $n\log n$ are bounded in $L^\infty((0,T), L^1(\mathbb{R}^2))$, $\nabla\sqrt{n} \in L^1((0,T), L^2(\mathbb{R}^2))$ and $\nabla c \in L^\infty((0,T)\times\mathbb{R}^2)$. Then

$$\frac{d}{dt}\mathcal{F}[n](t) = -\int_{\mathbb{R}^2} n(x,t)\left|\nabla \log n(x,t) - \chi\nabla c(x,t)\right|^2\,dx\,. \tag{4}$$

We will make a fundamental use of the Logarithmic Hardy-Littlewood-Sobolev inequality.

Proposition 1.4 (Logarithmic Hardy-Littlewood-Sobolev inequality)**.** [16, 4] *Let f be a non-negative function in $L^1(\mathbb{R}^2)$ such that $f\log f$ and $f\log(1+|x|^2)$ belong to $L^1(\mathbb{R}^2)$. If $\int_{\mathbb{R}^2} f\,dx = M$, then*

$$\int_{\mathbb{R}^2} f\log f\,dx + \frac{2}{M}\iint_{\mathbb{R}^2\times\mathbb{R}^2} f(x)f(y)\log|x-y|\,dx\,dy \geq -\,C(M)\,, \tag{5}$$

with $C(M) := M(1+\log\pi - \log M)$.

Let us point out that the Logarithmic Hardy-Littlewood-Sobolev inequality remains true in bounded domains just by multiplying f by the corresponding characteristic function of the domain. Note that, in the case $\chi M = 8\,\pi$, as a direct consequence of the Logarithmic Hardy-Littlewood-Sobolev inequality, the free energy $\mathcal{F}[n](t)$ is bounded from below.

Section 2 is devoted to some remarks on the necessary condition of existence of free-energy solution to the Patlak-Keller-Segel system (1). We prove the local in time existence of free-energy solution to the Patlak-Keller-Segel system (1) for any mass and the characterization of the maximal time of existence, see Proposition 1.2. Section 3 is devoted to the critical mass case $\chi M = 8\,\pi$. We prove first that if the blow-up occurs the blow-up profile is a delta dirac of mass $8\,\pi/\chi$ concentrated at the center of mass, then that blow-up cannot happen in finite time and finally that blow-up does happen in infinite time as a delta dirac of mass $8\,\pi/\chi$ concentrated at the center of mass.

2. Characterization of the Maximal Time of Existence

First, we introduce a regularized system and prove that it is enough to show that the entropy of the regularized cell density n^ϵ,

$$\mathcal{S}[n^\epsilon] := \int_{\mathbb{R}^2} n^\epsilon(x,t)\,\log n^\epsilon(x,t)\,dx$$

is bounded from above uniformly in ϵ on $[0,T)$ to prove that there exists a free-energy solution to the Patlak-Keller-Segel system (1) in $[0,T)$. Section (2.3) is devoted to the proof of the existence of maximal free-energy solutions to the Patlak-Keller-Segel system (1) for any mass (Proposition 1.2).

2.1. **Regularized system.** We introduce the truncated convolution kernel $\mathcal{K}^\epsilon$ to be such that

$$\mathcal{K}^\epsilon(z) := \mathcal{K}^1\left(\frac{|z|}{\epsilon}\right) - \frac{1}{2\pi}\log\epsilon$$

where $\mathcal{K}^1$ is a radial monotone non-increasing smooth function satisfying

$$\begin{cases} \mathcal{K}^1(|z|) = -\dfrac{1}{2\pi}\log|z| & \text{if } |z| \geq 4\,, \\ \mathcal{K}^1(|z|) = 0 & \text{if } |z| \leq 1\,. \end{cases}$$

Moreover, we assume that

$$\left|\nabla\mathcal{K}^1(z)\right| \leq \frac{1}{2\pi\,|z|}\,, \quad \mathcal{K}^1(z) \leq -\frac{1}{2\pi}\log|z| \quad \text{and} \quad -\Delta\mathcal{K}^1(z) \geq 0$$

for any $z \in \mathbb{R}^2$. Since $\mathcal{K}^\epsilon(z) = \mathcal{K}^1(|z|/\epsilon)$, we also have

$$|\nabla\mathcal{K}^\epsilon(z)| \leq \frac{1}{2\pi\,|z|} \quad \forall\, z \in \mathbb{R}^2\,. \tag{6}$$

We consider the following regularized version of (1)

$$\begin{cases} \dfrac{\partial n^\epsilon}{\partial t}(x,t) = \Delta n^\epsilon(x,t) - \chi\nabla\cdot(n^\epsilon(x,t)\nabla c^\epsilon(x,t)) & \\ c^\epsilon(x,t) = (\mathcal{K}^\epsilon * n^\epsilon)(x,t) & x \in \mathbb{R}^2\,,\ t > 0\,, \\ n_0^\epsilon(x) := \min\{n_0, \epsilon^{-1}\}(x) & \end{cases} \tag{7}$$

written in the distribution sense. For any fixed positive ϵ, under assumptions (2) on the initial data, it is proved (see Proposition 2.8 in [13]) that there exists a global solution in $L^2([0,T],H^1(\mathbb{R}^2)) \cap C([0,T],L^2(\mathbb{R}^2))$ for the regularized version of the Patlak-Keller-Segel system (7). The proof of this result uses the Schauder's fixed point theorem and the Lions-Aubin's compactness method [2, 36, 48].

2.2. **On the Local in time Existence Proof.** An attentive reading of the proof of Theorem 1.1 in [13] permits to see that the existence of free-energy solution is valid as long as $\mathcal{S}[n^\epsilon](t)$ and the 2-momentum are uniformly bounded from above in ϵ and $t \in [0,T)$. Since the proof follows the same lines as in [13, Theorem 1.1], we will just sketch the proof by underlining the most relevant aspects for reader's sake.

Proposition 2.1 (Criterion for Local Existence)**.** *Let n^ϵ be the solution of* (7) *and $T^* > 0$. If $\mathcal{S}[n^\epsilon](t)$ is bounded from above uniformly in ϵ for any $t \in (0,T^*)$, then the cluster points of $\{n^\epsilon\}_{\epsilon>0}$, in a suitable topology, are non-negative free-energy solutions of the Patlak-Keller-Segel system* (1) *with initial data n_0 on $[0,T^*)$.*

Proof. We will make use of the Gagliardo-Nirenberg-Sobolev inequality:

$$\|u\|^2_{L^p(\mathbb{R}^2)} \le C_{\mathrm{GNS}}(p)\, \|\nabla u\|^{2-4/p}_{L^2(\mathbb{R}^2)}\, \|u\|^{4/p}_{L^2(\mathbb{R}^2)}\ \forall\, u \in H^1(\mathbb{R}^2)\ ,\ \forall\, p \in [2,\infty)\ . \tag{8}$$

In a first step we show *a priori* estimates on the solution n^ϵ to the regularized version of the Patlak-Keller-Segel system (7). These estimates shall allow us to prove in the second step the existence of solutions in the distribution sense to the Patlak-Keller-Segel system (1) by passing to the $\epsilon \to 0$ limit. To prove that we get a free-energy solution as defined in Definition 1.1, we need to prove further regularity in the third step.

Step 1.- *A priori* estimates on n^ϵ. This step corresponds to Lemma 2.11 in [13]. We start by estimating the second moment of solutions independently of ϵ. Since $\mathcal{K}^\varepsilon$ is a radial non-increasing function satisfying (6), we have

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^2} |x|^2\, n^\varepsilon(x,t)\, dx &= \\ &= 4M + 2\chi \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} n^\varepsilon(x,t)\, n^\varepsilon(y,t)\, (x\cdot \nabla\mathcal{K}^\varepsilon(x-y))\, dx\, dy \\ &= 4M + \chi\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} n^\varepsilon(x,t)\, n^\varepsilon(y,t)\, ((x-y)\cdot\nabla\mathcal{K}^\varepsilon(x-y))\, dx\, dy \le 4M,\end{aligned} \tag{9}$$

from which $(1+|x|^2)n^\epsilon \in L^\infty((0,T), L^1(\mathbb{R}^2))$ uniformly in ϵ.

Let us now obtain a useful and classical remark for different arguments. The following usual notation will be adopted. For any real-valued function $f = f(x)$, we define its positive and negative part as $f^+(x) = \max\{f(x),0\}$ and $f^-(x) = (-f)^+(x)$, so that $f = f^+ - f^-$.

Lemma 2.2 (Control of the Negative Part of the Entropy)**.** *For any g such that $(1+|x|^2)g \in L^1_+(\mathbb{R}^2)$, we have $g\log^- g \in L^1(\mathbb{R}^2)$ and*

$$\int_{\mathbb{R}^2} g(x)\log^- g(x)\,dx \le \frac{1}{2}\int_{\mathbb{R}^2}|x|^2 g(x)\,dx + \log(2\pi)\int_{\mathbb{R}^2} g(x)\,dx + \frac{1}{e}\,.$$

Proof. Let $u := g\mathbb{1}_{\{g\le 1\}}$ and $m = \int_{\mathbb{R}^2} u\,dx \le M = \int_{\mathbb{R}^2} g\,dx$. Then

$$\int_{\mathbb{R}^2} u\Big(\log u + \frac{1}{2}|x|^2\Big)\,dx = \int_{\mathbb{R}^2} U\log U\,d\mu - m\log(2\pi)$$

where $U := u/\mu$, $d\mu(x) = \mu(x)dx$ and $\mu(x) = (2\pi)^{-1}e^{-|x|^2/2}$. By Jensen's inequality,

$$\int_{\mathbb{R}^2} U\log U\,d\mu \ge \Big(\int_{\mathbb{R}^2} U\,d\mu\Big)\ \log\Big(\int_{\mathbb{R}^2} U\,d\mu\Big) = m\log m\,,$$

$$\int_{\mathbb{R}^2} u\log u\,dx \ge m\log\Big(\frac{m}{2\pi}\Big) - \frac{1}{2}\int_{\mathbb{R}^2}|x|^2 u\,dx \ge -\frac{1}{e} - M\log(2\pi) - \frac{1}{2}\int_{\mathbb{R}^2}|x|^2 u\,dx$$

completing the proof. □

The previous lemma implies that

$$\int_{\mathbb{R}^2} n^\epsilon(x,t)\,|\log n^\epsilon(x,t)|\,dx \le \int_{\mathbb{R}^2} n^\epsilon(x,t)\,\Big(\log n^\epsilon(x,t) + |x|^2\Big)\,dx + 2\log(2\pi)M + \frac{2}{e}\,,$$

and thus, the entropy term verifies $n^\epsilon\,\log n^\epsilon \in L^\infty((0,T),L^1(\mathbb{R}^2))$ uniformly in ϵ whenever $(1+|x|^2)\,n^\epsilon \in L^\infty((0,T),L^1(\mathbb{R}^2))$ uniformly in ϵ.

By the monotonicity of $\mathcal{F}$ (see (4)) and the Logarithmic Hardy-Littlewood-Sobolev inequality (5)

$$-C(M) \le \mathcal{S}[n^\epsilon](t) - \frac{\chi}{2}\int_{\mathbb{R}^2} n^\epsilon(x,t)\,c^\epsilon(x,t)\,dx \le \mathcal{F}[n_0]\,.$$

Hence, the map

$$t \mapsto \int_{\mathbb{R}^2} n^\epsilon(x,t)\,c^\epsilon(x,t)\,dx \tag{10}$$

is bounded uniformly in ϵ in $L^\infty(0,T)$.

The main *a priori* estimate is the $L^2((0,T)\times\mathbb{R}^2)$ estimate on $\sqrt{n^\epsilon}\,\nabla c^\epsilon$. For the sake of simplicity we show only the formal computations for the solutions of the Patlak-Keller-Segel system (1), referring to [13] for details. The main difficulty, when proving the same estimates for the solutions to the regularized Patlak-Keller-Segel problem (7), is that we do not have $\Delta c = -n$. Nevertheless, $-\Delta\mathcal{K}^\epsilon$ converges to a delta dirac as $\epsilon \to 0$, and thus, the result remains true for ϵ small enough up to technical computations.

Given $K > 1$, by the Gagliardo-Nirenberg-Sobolev inequality (8):

$$\int_{\mathbb{R}^2}(n-K)_+^2\,dx \le C_{\rm GNS}^2\int_{\mathbb{R}^2}\Big|\nabla\sqrt{(n-K)_+}\Big|^2\,dx\int_{\mathbb{R}^2}(n-K)_+\,dx\,.$$

The left hand side can be made as small as desired by taking K large enough and using

$$\int_{\mathbb{R}^2} (n-K)_+ \, dx \le \frac{1}{\log K} \int_{\mathbb{R}^2} (n-K)_+ \, \log n \, dx \le \frac{C}{\log K} =: \eta(K) \, ,$$

where C is the bound on $n \log n \in L^\infty((0,T), L^1(\mathbb{R}^2))$ obtained above. We differentiate in time $\mathcal{S}[n](t)$. By using an integration by parts, the Gagliardo-Nirenberg-Sobolev's inequality (8) and the equation for c, we obtain:

$$\begin{aligned}
\frac{d}{dt}\mathcal{S}[n](t) &= -4 \int_{\mathbb{R}^2} |\nabla \sqrt{n(t)}|^2 \, dx + \chi \int_{\mathbb{R}^2} n(t)^2 \, dx \\
&\le -4 \int_{\mathbb{R}^2} |\nabla \sqrt{n(t)}|^2 \, dx + \chi (M+2) K + 2\chi \int_{\mathbb{R}^2} (n(t)-K)_+^2 \, dx \\
&\le \left(-4 + 2\chi \, \eta(K) \, C_{\mathrm{GNS}}^2\right) \int_{\mathbb{R}^2} |\nabla \sqrt{n(t)}|^2 \, dx + \chi (M+2) K \, .
\end{aligned}$$

The factor $\left(-4 + 2\chi \, \eta(K) \, C_{\mathrm{GNS}}^2\right)$ can be made non-positive for K large enough and therefore $\nabla\sqrt{n}$ is bounded in $L^2((0,T)\times\mathbb{R}^2)$. This idea will be used in the forthcoming proof of Proposition 1.2 and the interested reader can refer to it for full details applied to the regularized problem.

Now, we assume that we have derived a uniform in ϵ estimate for $\nabla\sqrt{n^\epsilon}$ in $L^2((0,T)\times\mathbb{R}^2)$. As a consequence of the $L^2((0,T)\times\mathbb{R}^2)$-estimate on $\nabla\sqrt{n^\epsilon}$ and of the computation

$$\frac{d}{dt}\mathcal{S}[n^\epsilon](t) = -4 \int_{\mathbb{R}^2} \left|\nabla\sqrt{n^\epsilon}\right|^2 \, dx + \chi \int_{\mathbb{R}^2} n^\epsilon (-\Delta c^\epsilon) \, dx,$$

the function $n^\epsilon \Delta c^\epsilon$ is bounded in $L^1([0,T]\times\mathbb{R}^2)$. A computation shows that

$$\frac{1}{2}\frac{d}{dt} \int_{\mathbb{R}^2} n^\epsilon \, c^\epsilon \, dx = \int_{\mathbb{R}^2} n^\epsilon \, \Delta c^\epsilon \, dx + \chi \int_{\mathbb{R}^2} n^\epsilon \, |\nabla c^\epsilon|^2 \, dx \, ,$$

proving finally that $\sqrt{n^\epsilon}\, \nabla c^\epsilon \in L^2((0,T)\times\mathbb{R}^2)$ where the boundedness of the map (10) was used. In this way, we have obtained estimates on the two terms appearing in the dissipation of the free energy in (4).

Step 2.- Passing to the limit. We will use the Aubin-Lions *compactness method*, (see [36], Ch. IV, §4 and [2], and [48] for more recent references). A simple statement goes as follows: *Take $T > 0$, $p \in (1,\infty)$ and let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence of functions in $L^p(0,T;H)$ where H is a Banach space. If $(f_n)_{n\in\mathbb{N}}$ is bounded in $L^p(0,T;V)$, where V is compactly imbedded in H and $\partial f_n/\partial t$ is bounded in $L^p(0,T;V')$ uniformly with respect to $n \in \mathbb{N}$, then $(f_n)_{n\in\mathbb{N}}$ is relatively compact in $L^p(0,T;H)$.*

- **Bound on $\|n^\epsilon\|_{L^2}$:** As $\mathcal{S}(n^\epsilon) \in L^\infty((0,T), L^1(\mathbb{R}^2))$ the first equation in (7) has the hyper-contractivity property [13, Theorem 3.5]. It means that for any $p \in (1,\infty)$, there exists a continuous function h_p on $(0,T)$ such that for almost any $t \in (0,T)$, $\|n^\epsilon(\cdot,t)\|_{L^p(\mathbb{R}^2)} \le h_p(t)$. Hence $n^\epsilon \in L^\infty((\delta,T), L^p(\mathbb{R}^2))$, $p \in (1,\infty)$, for any $\delta \in (0,T)$.
- **Bound on $\|\nabla n^\epsilon\|_{L^2}$:** Recall the Hardy-Littlewood-Sobolev inequality: *For all $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$, $1 < p,\ q < \infty$ such that $\frac{1}{p}+\frac{1}{q}+\frac{\lambda}{d} = 2$ and $0 < \lambda < d$, there exists a constant $C = C(p,q,\lambda) > 0$ such that*
$$\left|\int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{1}{|x-y|^\lambda} f(x)\, g(y)\, dx\, dy\right| \le C\|f\|_{L^p(\mathbb{R}^d)}\, \|g\|_{L^q(\mathbb{R}^d)}\ .$$
As a consequence, for any $(q,p) \in (2,+\infty)\times(1,2)$ with $\frac{1}{q} = \frac{1}{p} - \frac{1}{2}$, there exists a constant $C > 0$ such that
$$\|\nabla c^\epsilon(t)\|_{L^q(\mathbb{R}^2)} \le C\, \|n^\epsilon(t)\|_{L^p(\mathbb{R}^2)}\ . \tag{11}$$
And using Hölder's inequality, we can write
$$\begin{aligned}\|n^\epsilon(t)\, \nabla c^\epsilon(t)\|_{L^2(\mathbb{R}^2)} &\le \|n^\epsilon(t)\|_{L^{q/(q-1)}(\mathbb{R}^2)} \|\nabla c^\epsilon(t)\|_{L^q(\mathbb{R}^2)}\\ &\le C\, \|n^\epsilon(t)\|_{L^{q/(q-1)}(\mathbb{R}^2)} \|n^\epsilon(t)\|_{L^p(\mathbb{R}^2)}\ .\end{aligned}$$
Hence $n^\epsilon\, \nabla c^\epsilon$ is bounded in $L^\infty((\delta,T), L^2(\mathbb{R}^2))$.
Now the following computation
$$\frac{d}{dt}\int_{\mathbb{R}^2} |n^\epsilon|^2\ dx = -2\int_{\mathbb{R}^2} |\nabla n^\epsilon|^2\ dx + 2\chi\int_{\mathbb{R}^2} n^\epsilon(\nabla n^\epsilon \cdot \nabla c^\epsilon)\ dx$$
shows that $X := \|\nabla n^\epsilon\|_{L^2((\delta,T)\times\mathbb{R}^2)}$ satisfies the estimate
$$2\, X^2 - 2\,\chi\, \|n^\epsilon\, \nabla c^\epsilon\|_{L^\infty((\delta,T),L^2(\mathbb{R}^2))}\, X \le 2\|n^\epsilon\|^2_{L^\infty((\delta,T),L^2(\mathbb{R}^2))}.$$
This implies that ∇n^ϵ is bounded in $L^2((\delta,T)\times\mathbb{R}^2)$.
- **V is relatively compact in H:** The last estimate is given by Hölder's inequality
$$\int_{\mathbb{R}^2} |x||n^\epsilon|^2(x,t)\, dx \le \left(\int_{\mathbb{R}^2} |x|^2 n^\epsilon(x,t)\, dx\right)^{1/2} \left(\int_{\mathbb{R}^2} |n^\epsilon|^3(x,t)\, dx\right)^{1/2}\ .$$
This bound allows to consider only compact sets, on which compactness holds by Sobolev's imbeddings.

Aubin-Lions Lemma applies to prove that n^ϵ is relatively compact in $L^2((\delta,T)\times\mathbb{R}^2))$. Let us denote n the limit of the sub-sequence $\{n^{\epsilon_k}\}_k$. Note that the $L^{4/3}((\delta,T)\times\mathbb{R}^2)$-bound of n^ϵ and (11) with $q = 4$ implies that $n^{\epsilon_k}\nabla c^{\epsilon_k}$ converges to $n\nabla c$ in the distribution sense.

Step 3.- Free energy estimates. By convexity of $n \mapsto \int_{\mathbb{R}^2} |\nabla\sqrt{n}|^2\,dx$, see [5, 6], and weak semi-continuity we have

$$\iint_{(\delta,T)\times\mathbb{R}^2} |\nabla\sqrt{n}|^2\,dx\,dt \le \liminf_{k\to\infty} \iint_{(\delta,T)\times\mathbb{R}^2} |\nabla\sqrt{n^{\epsilon_k}}|^2\,dx\,dt\;,$$

$$\iint_{(\delta,T)\times\mathbb{R}^2} n\,|\nabla c|^2\,dx\,dt \le \liminf_{k\to\infty} \iint_{(\delta,T)\times\mathbb{R}^2} n^{\epsilon_k}\,|\nabla c^{\epsilon_k}|^2\,dx\,dt\;.$$

Moreover, it can be proved as in [13, Lemma 4.6] that the regularized entropy converges to the limiting entropy for almost every $t>0$, *i.e.*,

$$\mathcal{S}[n^\epsilon](t) \to \mathcal{S}[n](t) \qquad \text{as } \epsilon \to 0. \tag{12}$$

This proves the free energy estimate using the strong convergence of $\{n^{\epsilon_k}\}$ in $L^2((\delta,T)\times\mathbb{R}^2)$ and

$$\iint_{(\delta,T)\times\mathbb{R}^2} n\left|\nabla\,(\log n) - \chi\nabla c\right|^2\,dx\,dt \;=\; 4\iint_{(\delta,T)\times\mathbb{R}^2} |\nabla\sqrt{n}|^2\,dx\,dt$$
$$+\,\chi^2 \iint_{(\delta,T)\times\mathbb{R}^2} n\,|\nabla c|^2\,dx\,dt - 2\chi \iint_{(\delta,T)\times\mathbb{R}^2} n^2\,dx\,dt\;.$$

This ends the proof of Proposition 2.1. Full details of these final arguments are given in [13, Corollary 3.6]. □

2.3. **Proof of Proposition 1.2.** In the next lemma, we characterize the maximal time of existence of free-energy solutions.

Lemma 2.3 (Maximal Free-energy Solutions)**.** *Under the assumptions* (2) *on the initial data, if there exists a time $t^*\ge 0$ such that $\mathcal{S}[n^\epsilon](t)$ is bounded uniformly in ϵ and $0\le t\le t^*$, then there is $\tau>0$ independent of ϵ for which $\mathcal{S}[n^\epsilon](t)$ is uniformly bounded in ϵ for $t\in[t^*,t^*+\tau)$. Moreover, the free-energy solution of the Patlak-Keller-Segel system* (1) *constructed above can be extended up to $t^*+\tau$. Here, τ depends only on the uniform estimate on $[1+|x|^2+|\log n^\epsilon(t^*)|]n^\epsilon(t^*)$ in $L^1(\mathbb{R}^2)$.*

Proof. Let n^ϵ be the global solution of the regularised version of the Patlak-Keller-Segel system (7). We compute

$$\frac{d}{dt}\mathcal{S}[n^\epsilon](t) = -4\int_{\mathbb{R}^2}\left|\nabla\sqrt{n^\epsilon(x,t)}\right|^2\,dx + \chi\,[(\mathrm{I})+(\mathrm{II})+(\mathrm{III})]$$

with

$$(\mathrm{I}) := -\int_{\{n^\epsilon\le K\}} n^\epsilon(x,t)\cdot\Delta[\mathcal{K}^\epsilon * n^\epsilon(x,t)],$$

$$(\mathrm{II}) := -\int_{\{n^\epsilon>K\}} n^\epsilon(x,t)\cdot\Delta[\mathcal{K}^\epsilon * n^\epsilon(x,t)] - (\mathrm{III}) \quad\text{and}\quad (\mathrm{III}) = \int_{\{n^\epsilon>K\}} |n^\epsilon(x,t)|^2\;.$$

We define the non-negative normalised function ϕ by

$$\frac{1}{\varepsilon^2}\phi\Big(\frac{\cdot}{\varepsilon}\Big) = -\Delta\mathcal{K}^\epsilon \ .$$

The term (I) is easy to estimate using

$$(\mathrm{I}) \leq \int_{\{n^\epsilon \leq K\}} K \int_{\mathbb{R}^2} \frac{1}{\varepsilon^2}\phi\left(\frac{|x-y|}{\varepsilon}\right) n^\epsilon(y,t)\, dy\, dx = M\, K \ .$$

We have

$$\begin{aligned}
(\mathrm{II}) \quad &= \quad \int_{\{n^\epsilon > K\}} n^\epsilon(x,t) \int_{\mathbb{R}^2} \left[n^\epsilon(x-\epsilon y,t) - n^\epsilon(x,t)\right]\, \phi(y)\, dy\, dx \\
&\leq \quad \int_{\{n^\epsilon > K\}} -n^\epsilon(x,t) \int_{\mathbb{R}^2} \left[\sqrt{n^\epsilon(x-\epsilon y,t)} - \sqrt{n^\epsilon(x,t)}\right]\, \sqrt{\phi(y)} \\
&\qquad \times \left[\sqrt{n^\epsilon(x-\epsilon y,t)} - \sqrt{n^\epsilon(x,t)} + 2\sqrt{n^\epsilon(x,t)}\right]\, \sqrt{\phi(y)}\, dy\, dx \ .
\end{aligned}$$

Using the Cauchy-Schwarz' inequality and $(a+2b)^2 \leq 2a^2 + 8b^2$ we obtain

$$\begin{aligned}
(\mathrm{II}) \leq &\int_{\{n^\epsilon > K\}} n^\epsilon(x,t) \left[\|\phi\|_{L^\infty(\mathbb{R}^2)} \int_{\frac{1}{2}\leq y \leq 2} \left|\sqrt{n^\epsilon(x-\epsilon y,t)} - \sqrt{n^\epsilon(x,t)}\right|^2\, dy\right]^{1/2} \\
&\cdot \left[\int_{\{n^\epsilon > K\}} \left[2\, \left|\sqrt{n^\epsilon(x-\epsilon y,t)} - \sqrt{n^\epsilon(x,t)}\right|^2 + 8\, n^\epsilon(x,t)\right]\, \phi(y) dy\right]^{1/2} dx.
\end{aligned}$$

By the Poincaré's inequality we have

$$\begin{aligned}
&(\mathrm{II}) \leq \|\phi\|^{1/2}_{L^\infty(\mathbb{R}^2)}\, C_P\, \|\nabla\sqrt{n^\epsilon(x,t)}\|_{L^2(\mathbb{R}^2)} \\
&\int_{\{n^\epsilon > K\}} n^\epsilon(x,t) \cdot \sqrt{2}\left[\, \|\phi\|^{1/2}_{L^\infty(\mathbb{R}^2)}\, C_P\, \|\nabla\sqrt{n^\epsilon(x,t)}\|_{L^2(\mathbb{R}^2)} + 2\sqrt{n^\epsilon(x,t)}\right]\, dx \ ,
\end{aligned}$$

which can be written as

$$\begin{aligned}
(\mathrm{II}) \leq\ &\sqrt{2}\|\phi\|_{L^\infty(\mathbb{R}^2)}\, C_P^2\, \|\nabla\sqrt{n^\epsilon(x,t)}\|^2_{L^2(\mathbb{R}^2)} \int_{\{n^\epsilon > K\}} n^\epsilon(x,t)\, dx \\
&+ 2^{3/2}\, \|\phi\|^{1/2}_{L^\infty(\mathbb{R}^2)}\, C_P\, \|\nabla\sqrt{n^\epsilon(x,t)}\|_{L^2(\mathbb{R}^2)} \int_{\{n^\epsilon > K\}} [n^\epsilon(x,t)]^{3/2}\, dx \ .
\end{aligned}$$

By the Cauchy-Schwarz inequality we obtain

$$
\begin{aligned}
(\mathrm{II}) \le \sqrt{2}\|\phi\|_{L^\infty(\mathbb{R}^2)}\, C_P^2 \,\|\nabla\sqrt{n^\epsilon(x,t)}\|^2_{L^2(\mathbb{R}^2)} \int_{\{n^\epsilon>K\}} n^\epsilon(x,t)\,dx \\
+ 2^{\frac{3}{2}}\,\|\phi\|^{\frac12}_{L^\infty(\mathbb{R}^2)}\, C_P\,\|\nabla\sqrt{n^\epsilon(x,t)}\|_{L^2(\mathbb{R}^2)}\cdot \\
\cdot\left(\int_{\{n^\epsilon>K\}} n^\epsilon(x,t)\,dx\right)^{\frac12}\cdot\left(\int_{\{n^\epsilon>K\}} [n^\epsilon(x,t)]^2\,dx\right)^{\frac12}
\end{aligned}
$$

and by the Gagliardo-Nirenberg-Sobolev's inequality (8)

$$
(\mathrm{II}) \le B\int_{\{n^\epsilon>K\}} [n^\epsilon(x,t)]^2\,dx
$$

with

$$
B := \sqrt{2}\|\phi\|_{L^\infty(\mathbb{R}^2)}\left(\frac{C_P}{C_{\mathrm{GNS}}}\right)^2 + 2^{\frac32}\,\|\phi\|^{\frac12}_{L^\infty(\mathbb{R}^2)}\,\frac{C_P}{C_{\mathrm{GNS}}}\;.
$$

Hence

$$
(\mathrm{II}) + (\mathrm{III}) \le [B+1]\int_{\{n^\epsilon>K\}} [n^\epsilon(x,t)]^2\,dx\;.
$$

Combining the previous results we have

$$
\frac{d}{dt}\mathcal{S}[n^\epsilon](t) \le -4\int_{\mathbb{R}^2}\left|\nabla\sqrt{n^\epsilon(x,t)}\right|^2\,dx + \chi\,M\,K + \chi\,[B+1]\int_{\{n^\epsilon>K\}}[n^\epsilon(x,t)]^2\,dx\;.
$$

And using the Gagliardo-Nirenberg-Sobolev's inequality (8) in the set $\{n^\epsilon > K\}$ we obtain

$$
\frac{d}{dt}\mathcal{S}[n^\epsilon](t) \le \chi\,M\,K + \left[-4 + \chi\,C^2_{\mathrm{GNS}}\,(B+1)\int_{\{n^\epsilon>K\}} n^\epsilon(x,t)dx\right]\int_{\mathbb{R}^2}\left|\nabla\sqrt{n^\epsilon(x,t)}\right|^2 dx.
$$

Given $K>1$, using Lemma 2.2 and (9), we have

$$
\begin{aligned}
\int_{\{n^\epsilon>K\}} n^\epsilon(x,t)\,dx &\le \frac{1}{\log K}\int_{\{n^\epsilon>K\}} n^\epsilon(x,t)\,\log n^\epsilon(x,t)\,dx \\
&\le \frac{1}{\log K}\int_{\{n^\epsilon>K\}} n^\epsilon(x,t)\,|\log n^\epsilon(x,t)|\,dx \le \frac{1}{\log K}\left(\mathcal{S}[n^\epsilon](t) + \tilde{C}(t)\right)
\end{aligned}
$$

with

$$
\tilde{C}(t) := \int_{\mathbb{R}^2}|x|^2 n^\epsilon(x,t^*)dx + 4M(t-t^*) + 2\log(2\pi)M + 2e^{-1} := 4M(t-t^*) + \tilde{C}.
$$

Hence we actually proved that

$$
\frac{d}{dt}\mathcal{S}[n^\epsilon](t) \le \chi\,M\,K + A(t)\int_{\mathbb{R}^2}\left|\nabla\sqrt{n^\epsilon(x,t)}\right|^2\,dx
$$

with
$$A(t) := \left[-4 + \chi\, \frac{C_{\rm GNS}^2\,(B+1)}{\log K}\, \left(\mathcal{S}[n^\epsilon](t) + \tilde{C}(t)\right)\right] .$$
At time $t = t^*$, it implies that
$$\frac{d}{dt}\mathcal{S}[n^\epsilon](t)\Big|_{t=t^*} \le \chi\, M\, K + A(t^*) \int_{\mathbb{R}^2} \left|\nabla \sqrt{n^\epsilon(x,t^*)}\right|^2\, dx .$$
By hypothesis, there exists $\mathcal{S}^* > 0$ such that $\mathcal{S}[n^\epsilon](t^*) \le \mathcal{S}^*$ since $\mathcal{S}[n^\epsilon](t^*)$ is bounded uniformly in ϵ. We can thus choose K large enough such that the coefficient $A(t^*)$ is non-positive. It implies that there exists $\tau_\epsilon > 0$ such that
$$\mathcal{S}[n^\epsilon](t) \le \mathcal{S}^* + M\, K\, (t - t^*)$$
in $[t^*, t^* + \tau_\epsilon]$. But in this interval $t \in [t^*, t^* + \tau_\epsilon]$,
$$A(t) \le \left[-4 + \chi\, \frac{C_{\rm GNS}^2\,(B+1)}{\log K}\, \left(\mathcal{S}^* + M\, K\, (t - t^*) + \tilde{C}(t)\right)\right]$$
is non-positive in $[t^*, t^* + \tau]$ with
$$\tau \le \frac{1}{MK + 4M} \left[\frac{4\, \log K}{\chi\, C_{\rm GNS}^2\,(B+1)} - \mathcal{S}^* - \tilde{C}\right] ,$$
which is independent from ϵ and positive for K large enough. Therefore, the previous procedure can be continued in the time interval $[t^*, t^* + \tau]$ showing finally that $[t^*, t^* + \tau] \subset [t^*, t^* + \tau_\epsilon]$. We finally make use of Proposition 2.1 to conclude the existence of a free-energy solution of the PKS system (1) in $[t^*, t^* + \tau]$. □

Proof of Proposition (1.2). The local in time existence is a direct consequence of Lemma 2.3. Assume by contradiction that $\lim_{t \nearrow T^*} \mathcal{S}[n](t) < \infty$. Taking into account the convergence of regularized entropies (12) and Lemma 2.3, there exists $\tau > 0$ independent from ϵ such that $\mathcal{S}[n^\epsilon](t)$ is bounded uniformly in ϵ for any $t \in [0, T^* + \tau)$. By Proposition 2.1 it implies that the limit n is a nonnegative solution of the Patlak-Keller-Segel system (1) with initial data n_0 in $[0, T^* + \tau)$ contradicting the choice of T^*. □

Let us finally point out the following conservations for free-energy solutions of the PKS system (1).

Lemma 2.4 (Mass Centering and Variance Evolution)**.** *Let n be a free-energy solution on the time interval $[0, T^*)$ to system* (1) *with initial data*

n_0 verifying assumptions (2). Then the center of mass is preserved

$$M_1 = \int_{\mathbb{R}^2} x\, n(x,t)\, dx = \int_{\mathbb{R}^2} x\, n_0(x)\, dx$$

and the variance of the cell density verifies

$$\int_{\mathbb{R}^2} |x|^2\, n(x,t)\, dx = \int_{\mathbb{R}^2} |x|^2\, n_0(x)\, dx + 4M \left(1 - \frac{\chi M}{8\pi}\right) t \tag{13}$$

for all $t \in (0, T^*)$.

Proof. For any $\epsilon < 1$, we can find a C^∞ radial compactly supported cut-off function $\psi_\varepsilon(x)$ such that $0 \le \psi_\varepsilon \le 1$, $\psi_\varepsilon = 1$ for $|x| \le \varepsilon^{-1}$, $\psi_\varepsilon = 0$ for $|x| \ge \varepsilon^{-1} + 1$ and $\|\psi_\varepsilon\|_{C^2(\mathbb{R}^2)} \le C_1$ for all $\varepsilon \le 1$.

Taking $\varphi_\varepsilon(x) = x_i\, \psi_\varepsilon(x)$, for fixed $i \in \{1, 2\}$, as test function in the weak solution concept (1.1), we obtain

$$\frac{d}{dt} \int_{\mathbb{R}^2} \varphi_\varepsilon n\, dx = \tag{14}$$
$$= \int_{\mathbb{R}^2} \Delta\varphi_\varepsilon n\, dx - \frac{\chi}{4\pi} \int_{\mathbb{R}^2} \frac{(\nabla\varphi_\varepsilon(x) - \nabla\varphi_\varepsilon(y))\cdot(x-y)}{|x-y|^2} n(x,t)\, n(y,t)\, dxdy\,.$$

Since $\Delta\varphi_\varepsilon$ is bounded and $\nabla\varphi_\varepsilon$ is Lipschitz continuous uniformly in ε, we deduce that

$$\frac{d}{dt} \int_{\mathbb{R}^2} \varphi_\varepsilon(x)\, n(x,t)\, dx \le c_2 \int_{\mathbb{R}^2} n_0(x)\ dx\,,$$

where c_2 is some positive constant, from which

$$\int_{\mathbb{R}^2} \varphi_\varepsilon(x)\, n(x,t)\, dx \le c_1 + c_2 t\,,$$

with c_1 suitable constant, and thus

$$\int_{\mathbb{R}^2} x_i\, n(x,t)\, dx < \infty, \quad \forall\, t \in (0, T)\,.$$

We can now pass to the limit $\varepsilon \to 0$ in the integral version of (14) using Lebesgue's dominated convergence theorem together with $\Delta\varphi_\varepsilon \to 0$ and $(\nabla\varphi_\varepsilon(x) - \nabla\varphi_\varepsilon(y)) \to 0$ for almost every x, y in $\mathbb{R}^2$, deducing

$$\int_{\mathbb{R}^2} x_i n(x,t)\, dx = \int_{\mathbb{R}^2} x_i n_0(x)\, dx$$

for almost every $t > 0$ and $i \in \{1, 2\}$.

The case of the second moment is done analogously. Taking $\varphi_\varepsilon(x) = |x|^2\, \psi_\varepsilon(x)$ as test function in (1.1), we obtain (14). Proceeding as above, we can pass to the limit using Lebesgue's dominated convergence theorem deducing

$$\int_{\mathbb{R}^2} |x|^2\, n(x,t)\, dx = \int_{\mathbb{R}^2} |x|^2\, n_0(x)\, dx + \int_0^t 4M \left(1 - \frac{\chi M}{8\pi}\right) dt$$

and obtaining the desired result. □

The previous result allows us to restrict our analysis to the case where the center of mass is zero by the translation invariance of the PKS system (1).

3. Asymptotic behaviour in the case $\chi M = 8\pi$

This section is devoted to the proof of the main theorem (Theorem 1.3 for the critical mass $\chi M = 8\pi$). In Section 3.1, we characterize the possible blow-up profile of the Patlak-Keller-Segel system (1) with critical mass. More precisely, if the solution of the PKS system (1) blows-up, then it does it like a delta dirac of mass $8\pi/\chi$ at the center of mass of the solution.

In Section 3.2 we prove that the solution to the Patlak-Keller-Segel (1) cannot blow-up in finite time. As a consequence, we show the global in time existence of free-energy solutions to the PKS system (1) for the critical mass $\chi M = 8\pi$. Finally, Section 3.3 is devoted to demonstrate that the solution to the PKS system (1) concentrates as a delta dirac of mass $8\pi/\chi$ at the center of mass of the solution in infinite time.

3.1. **How does it blow-up?** Let us first recall the proof of the upper bound on the entropy functional

$$\mathcal{S}[n] = \int_{\mathbb{R}^2} n(x) \log n(x)\, dx,$$

in the sub-critical case, namely the mass $M < 8\pi/\chi$ as done in [13]. By the monotonicity of the free energy (4) the quantity

$$\mathcal{F}[n](t) = (1-\theta)\, \mathcal{S}[n](t) + \theta \left[\mathcal{S}[n](t) + \frac{\chi}{4\pi\theta} \iint_{\mathbb{R}^2\times\mathbb{R}^2} n(x)\, n(y) \log|x-y|\, dx\, dy \right]$$

is bounded from above by $\mathcal{F}[n_0]$. We choose $\theta = \frac{\chi M}{8\pi}$ and apply the Logarithmic Hardy-Littlewood-Sobolev inequality (5) to get:

$$(1-\theta)\, \mathcal{S}[n](t) - \theta\, C(M) \le \mathcal{F}[n_0]\,.$$

If $\chi M < 8\pi$, then $\theta < 1$ and

$$\mathcal{S}[n](t) \le \frac{\mathcal{F}[n_0] + \theta\, C(M)}{1-\theta}\,.$$

However, in the case when $\chi M = 8\pi$, the above argument does not imply that an upper bound on $\mathcal{S}[n](t)$ although $\mathcal{F}[n](t)$ is bounded from above and from below. Nevertheless, a localization argument of the free energy and a smart use of the Logarithmic Hardy-Littlewood-Sobolev inequality (5) allows to show:

Lemma 3.1 (Characterization of Blowing-up Profile)**.** *Under hypotheses* (2) *on the initial data, assume that* T^* *the maximum time of existence of the free-energy solution* n *to the Patlak-Keller-Segel system* (1) *with initial data* n_0 *of critical mass* $M = 8\,\pi/\chi$ *is finite. If* $\{t_p\}_{p\in\mathbb{N}} \nearrow T^*$ *when* $p \to \infty$*, then* $t_p \mapsto n(x,t_p)$ *converges to a delta dirac of mass* $8\,\pi/\chi$ *concentrated at the center of mass in the measure sense as* $p \to \infty$*.*

The main ideas of the proof of Lemma 3.1 reads as follows: we assume by contradiction that the weak-* limit of $t_p \mapsto n(x,t_p)$, namely $dn^*(x)$ is not a delta dirac. Hence, there exists a ball B_{r_1} in which the mass of dn^* is some α such that $0 < \alpha < M = 8\,\pi/\chi$. We apply the Logarithmic Hardy-Littlewood-Sobolev inequality (5) to balls and annulii (see Figure 1). By adding the corresponding terms, we can prove that a variation of the Logarithmic Hardy-Littlewood-Sobolev inequality (5) holds for the solution $n(x,t_p)$ in $\mathbb{R}^2$, from which we obtain a uniform bound on the entropy $\mathcal{S}[n](t_p)$. This contradicts the choice of the maximal time of existence of the free-energy solution due to the characterization in Proposition 1.2.

Proof. We first remark that the second moment is preserved in time due to (13) for the critical mass $\chi\,M = 8\,\pi$. This together with the conservation of mass shows that the sequence of positive integrable functions $n(x,t_p)$ is tight for the weak-* convergence as measures preserving its mass by Prokhorov's theorem [12]. Thus, there exists a weakly-* converging subsequence towards a limiting measure $dn^* \in \mathcal{M}(\mathbb{R}^2)$ with mass M. It is obvious that the argument can be reduced to weakly-* converging subsequences, so we will do so and we will keep the same index for the time sequence.

Let us now assume by contradiction that there is no formation of a delta dirac of mass $M = 8\,\pi/\chi$ as $t \nearrow T^*$. For any t_p and $r > 0$, we introduce the mass density on balls

$$\alpha_p(r) := \int_{B_r} n(x,t_p)\,dx$$

which is a non decreasing function on $\mathbb{R}^+$ and let us define analogously the mass density $\alpha^*(r)$ for dn^* as:

$$\alpha^*(r) := \int_{B_r} dn^*(x).$$

Saying that the positive measure dn^* of mass M is not $M\delta_0$ is equivalent to assert that

$$\lim_{r\to 0^+} \alpha^*(r) < \frac{8\,\pi}{\chi}$$

since the function α^* is non-decreasing and $\lim_{r\to\infty}\alpha^*(r) = 8\,\pi/\chi$. Let us remark that α^* is continuous for almost every $r > 0$.

By changing the origin if necessary, using again the translation invariance of the system, we can assume without loss of generality that α^* is not identically 0 in the neighbourhood of 0. Then, there exists $r > 0$ for which $0 < \alpha^*(r) < M$ and r is a point of continuity of α^*. Then, for small enough $\eta > 0$, there exist r_1 and r_2 with $r_1 < r < r_2$ such that

$$0 < \alpha^*(r) - \eta < \alpha^*(r_1) \le \alpha^*(r) \le \alpha^*(r_2) < \alpha^*(r) + \eta < \frac{8\,\pi}{\chi}\,.$$

Since the time sequence $n(x,t_p) \rightharpoonup dn^*$ as $p \to \infty$, we can assume without loss of generality that

$$0 < \alpha^*(r) - \eta < \alpha_p(r_1) \le \alpha_p(r) \le \alpha_p(r_2) < \alpha^*(r) + \eta < \frac{8\,\pi}{\chi}$$

for p large enough. As a consequence, for all p large enough, we have

$$\alpha^*(r) - \eta \le \int_{B_{r_1}} n(x,t_p)\,dx\,, \quad \frac{8\,\pi}{\chi} - \alpha^*(r) - \eta \le \int_{B^c_{r_2}} n(x,t_p)\,dx \quad \text{and}$$

$$\int_{B_{r_2}\setminus B_{r_1}} n(x,t_p)\,dx \le 2\,\eta\,.$$

Let $\rho := (1/3)(r_2 - r_1)$. We introduce the annulii $S_1 := B_{r_1+\rho} \setminus B_{r_1}$, $S_2 := B^c_{r_2-\rho} \setminus B^c_{r_2} = B_{r_2} \setminus B_{r_2-\rho}$ and $S_3 := (B_{r_2} \setminus B_{r_1}) \setminus (S_1 \cup S_2)$ (see Figure 1 below).

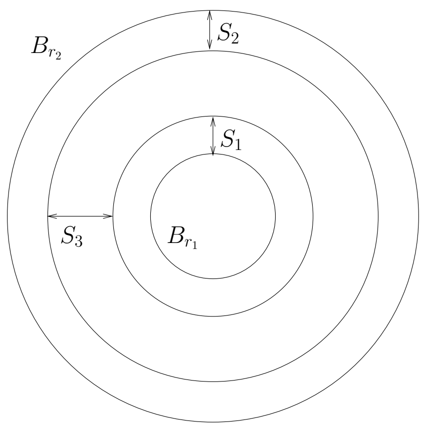

FIGURE 1. Definition of the different sets

In the rest C will denote several constants, not necessarily positive, depending only on the mass M, the value of the initial second moment and η. We apply the Logarithmic Hardy-Littlewood-Sobolev inequality (5) to $n_p(x) := n(x, t_p)$ on the sets $B_{r_1+\rho}$, $B^c_{r_2-\rho}$ and $B_{r_2} \setminus B_{r_1}$ for p large enough to obtain

$$\int_{B_{r_1+\rho}} n_p(x)\,dx \int_{B_{r_1+\rho}} n_p(x)\ \log n_p(x)\,dx + 2 \iint_{B_{r_1+\rho}\times B_{r_1+\rho}} n_p(x)\,n_p(y)\ \log|x-y|\,dx\,dy \geq C_{r_1,\rho}$$

$$\int_{B^c_{r_2-\rho}} n_p(x)\,dx \int_{B^c_{r_2-\rho}} n_p(x)\ \log n_p(x)\,dx + 2 \iint_{B^c_{r_2-\rho}\times B^c_{r_2-\rho}} n_p(x)\,n_p(y)\ \log|x-y|\,dx\,dy \geq C_{r_2,\rho}$$

$$\int_{B_{r_2}\setminus B_{r_1}} n_p(x)dx \int_{B_{r_2}\setminus B_{r_1}} n_p(x)\ \log n_p(x)dx + 2 \iint_{B_{r_2}\setminus B_{r_1}\times B_{r_2}\setminus B_{r_1}} n_p(x)\,n_p(y)\ \log|x-y|dxdy \geq C_{r_1,r_2}$$

We expand $n\log n = n\log^+ n - n\log^- n$ in the first terms and disregard the negative part contribution. Using

$$B_{r_1+\rho} = B_{r_1} \cup S_1, \quad B^c_{r_2-\rho} = B^c_{r_2} \cup S_2, \qquad \text{and} \quad B_{r_2} \setminus B_{r_1} = S_1 \cup S_2 \cup S_3$$

adding the terms and collecting terms to reconstruct the integral in the whole $\mathbb{R}^2$ of the positive contribution of the entropy, we deduce

$$\begin{aligned} &I_1 + I_2 - I_3 + I_4 := \\ &\mathcal{K}_p \int_{\mathbb{R}^2} n_p(x)\ \log^+ n_p(x)\,dx\ +\ 2 \iint_{\mathbb{R}^2\times\mathbb{R}^2} n_p(x)\,n_p(y)\ \log|x-y|\,dx\,dy \\ &\qquad -\ 4 \iint_{[B_{r_1}\times B^c_{r_1+\rho}]\cup[(S_1\cup S_3)\times B^c_{r_2}]} n_p(x)\ n_p(y)\ \log|x-y|\,dx\,dy \\ &\qquad +\ 2 \iint_{[S_1\times S_1]\cup[S_2\times S_2]} n_p(x)\ n_p(y)\ \log|x-y|\,dx\,dy \geq C \end{aligned}$$

with

$$\mathcal{K}_p := \max\{a_1, a_1+a_2, a_2, a_2+a_3, a_3\} = \max\{a_1+a_2, a_2+a_3, \}\ ,$$

and

$$a_1 := \int_{B_{r_1+\rho}} n_p(x)\,dx, \quad a_2 := \int_{(B_{r_2}\setminus B_{r_1})} n_p(x)\,dx, \quad \text{and } a_3 := \int_{B^c_{r_2-\rho}} n_p(x)\,dx,$$

where the second and the fourth terms in the first expression of $\mathcal{K}_p$ are due to the fact that S_1 and S_2 are counted twice for the positive contribution of the entropy. We estimate the third term I_3 by using $\log|x-y| \geq \log\rho$ in the sets where the integral is defined to obtain

$$I_3 \geq 4M^2 \log\rho.$$

Since the second moment of the solutions is constant in time by (13), we deduce

$$I_4 \le 2 \iint_{[S_1\times S_1]\cup[S_2\times S_2]} n_p(x)\, n_p(y)\ \log^+|x-y|\,dx\,dy$$
$$\le C \iint_{[S_1\times S_1]\cup[S_2\times S_2]} n_p(x)\, n_p(y)\ (1+|x|^2+|y|^2)\,dx\,dy \le C$$

Moreover, we can estimate the factor $\mathcal{K}_p$ as

$$\mathcal{K}_p \le \mathcal{K} := \max\left\{\alpha^*(r)+3\,\eta, \frac{8\,\pi}{\chi} - \alpha^*(r)+3\,\eta\right\}$$

which is positive and strictly smaller than $\frac{8\,\pi}{\chi}$ for $\eta < (1/6)\min\{\alpha^*(r); \frac{8\,\pi}{\chi} - \alpha^*(r)\}$. Summarizing, we have obtained

$$\mathcal{K}\int_{\mathbb{R}^2} n_p(x)\ \log^+ n_p(x)\,dx\ +\ 2\iint_{\mathbb{R}^2\times\mathbb{R}^2} n_p(x)\, n_p(y)\ \log|x-y|\,dx\,dy \ge C$$

for all p big enough with $0<\mathcal{K}<\frac{8\,\pi}{\chi}$. Now, taking into account Lemma 2.2 and the conservation of the second moment in (13), we obtain

$$\int_{\mathbb{R}^2} n_p \log^- n_p\,dx \le \frac{1}{2}\int_{\mathbb{R}^2} |x|^2 n_p\,dx + \log(2\pi)\int_{\mathbb{R}^2} n_p\,dx + \frac{1}{e}\ \le C$$

and thus

$$\mathcal{K}\int_{\mathbb{R}^2} n_p(x)\ \log n_p(x)\,dx\ +\ 2\iint_{\mathbb{R}^2\times\mathbb{R}^2} n_p(x)\, n_p(y)\ \log|x-y|\,dx\,dy \ge C$$

for p big enough with $0<\mathcal{K}<\frac{8\,\pi}{\chi}$.

Repeating the same arguments as in the introduction to this subsection, see [13] for the subcritical case, and using the estimate on the free energy, we deduce

$$\mathcal{S}[n_p] \le \frac{\mathcal{F}[n_0]+\theta\, C(M)}{1-\theta}\ .$$

with $\theta = \frac{\chi\,\mathcal{K}}{8\pi}$, for all p big enough.

This fact contradicts the choice of T^* as the maximal time of existence of a free-energy solution n since

$$\lim_{p\to\infty} \mathcal{S}[n_p] = +\infty.$$

This contradiction comes from the assumption

$$\lim_{r\to 0^+} \alpha^*(r) < \frac{8\,\pi}{\chi},$$

and thus, $\alpha^*(r) = M$ or equivalently $dn^* = M\delta_0$ for all converging subsequences, completing the desired result. □

It is very easy to verify that the previous proof also works assuming that the entropy diverges if $T^* = \infty$. More precisely, we have:

Corollary 3.2 (Infinite Time Aggregation)**.** *Under hypotheses* (2) *on the initial data, assume that the free-energy solution n to the Patlak-Keller-Segel system* (1) *with initial data n_0 of critical mass $\chi M = 8\pi$ is global in time and that*

$$\lim_{t \nearrow \infty} \int_{\mathbb{R}^2} n(x,t) \log n(x,t)\, dx = +\infty.$$

Then $t \mapsto n(x,t)$ converges to a delta dirac of mass $8\pi/\chi$ concentrated at the center of mass in the measure sense as $p \to \infty$.

3.2. **When does it blow-up?**

Proposition 3.3 (Existence of global in time solution)**.** *Under assumptions* (2) *on the initial data n_0, there exists a nonnegative free-energy solution n to the Patlak-Keller-Segel system* (1) *on $[0,\infty)$.*

Proof. We start by reminding the reader about a "by-now standard" gain of integrability result [24, 34, 33].

Lemma 3.4 (De la Vallèe-Poussin theorem)**.** [33] *Let μ be a non-negative measure and $\mathcal{F} \subset L^1_\mu(\mathbb{R}^2)$. The set $\mathcal{F}$ is uniformly integrable in $L^1_\mu(\mathbb{R}^2)$ if and only if $\mathcal{F}$ is uniformly bounded in $L^1_\mu(\mathbb{R}^2)$ and there exists a convex function $\Phi_0 \in \mathcal{C}^\infty([0,\infty))$ such that $\Phi_0(0) = \Phi_0'(0) = 0$, Φ_0' is concave, $\Phi_0(r) \le r\,\Phi_0'(r) \le 2\,\Phi_0(r)$ if $r > 0$, $r^{-1}\Phi_0(r)$ is concave, $\lim_{r\to\infty} \frac{\Phi_0(r)}{r} = \lim_{r\to\infty} \Phi_0'(r) = \infty$ and*

$$\sup_{f\in\mathcal{F}} \int_{\mathbb{R}^2} \Phi_0(|f|)\, d\mu < \infty\ .$$

We apply De la Vallèe-Poussin theorem (Lemma 3.4) to $f(x) = |x|^2$ with $d\mu = n_0(x)\,dx$. Hence, there exists Φ_0 satisfying the properties enumerated in De la Vallèe-Poussin theorem (Lemma 3.4) such that $\Phi_0(|x|^2)\, n_0(x)$ is in $L^1(\mathbb{R}^2)$. It is straightforward, based on the properties of Φ_0 in Lemma 3.4, to show for all $r > 0$

$$r\,\Phi_0''(r) \le 2\frac{\Phi_0(r)}{r}. \tag{15}$$

Now, let us look for the evolution of

$$\int_{\mathbb{R}^2} \Phi_0(|x|^2)\, n(x,t)\, dx.$$

Lemma 3.5. *If $\chi M = 8\pi$, and n is a maximal free-energy solution of the Patlak-Keller-Segel system (1) with initial data n_0, then for any smooth convex function $\Phi : \mathbb{R} \longrightarrow [0,\infty)$ such that $\Phi(0) = 0$*

$$\frac{d}{dt}\int_{\mathbb{R}^2} \Phi\left(|x|^2\right)\, n(x,t)\, dx \leq 4\int_{\mathbb{R}^2} |x|^2\, \Phi''\left(|x|^2\right)\, n(x,t)\, dx$$

Proof. We compute for all $t \in (0,T^*)$

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^2} \Phi\left(|x|^2\right)\, n(x,t)\, dx &= 4\int_{\mathbb{R}^2} \left[\Phi'\left(|x|^2\right) + |x|^2\Phi''\left(|x|^2\right)\right] n(x,t)\, dx \\ &\quad - \frac{1}{\pi}\int_{\mathbb{R}^2} x\,\Phi'\left(|x|^2\right)\frac{x-y}{|x-y|^2}\, n(x,t)\, n(y,t)\, dx\, dy \\ &= 4\int_{\mathbb{R}^2} \left[\Phi'\left(|x|^2\right) + |x|^2\Phi''\left(|x|^2\right)\right] n(x,t)\, dx \\ &\quad - \frac{1}{2\pi}\int_{\mathbb{R}^2}\left[x\,\Phi'\left(|x|^2\right) - y\,\Phi'\left(|y|^2\right)\right]\frac{x-y}{|x-y|^2}\, n(x,t)\, n(y,t)\, dx\, dy.\end{aligned}$$

On the other hand, we have for any $(x,y,s,t) \in \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}$

$$\begin{aligned}&(x\,\Phi'(s) - y\,\Phi'(t))\,(x-y) = \\ &\quad = |x|^2\Phi'(s) - (x\cdot y)\ \Phi'(s) - (x\cdot y)\ \Phi'(t) + |y|^2\Phi'(t) \\ &\quad = \frac{1}{2}\,|x-y|^2\,[\Phi'(s) + \Phi'(t)] + \frac{1}{2}\left(|x|^2 - |y|^2\right)[\Phi'(s) - \Phi'(t)]\ .\end{aligned}$$

As a consequence, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^2}\left(x\,\Phi'\left(|x|^2\right) - y\,\Phi'\left(|y|^2\right)\right)\frac{x-y}{|x-y|^2}\, n(x,t)\, n(y,t)\, dx\, dy = \\ &\quad = \frac{1}{2}\int_{\mathbb{R}^2}\left\{\left[\Phi'\left(|x|^2\right) + \Phi'\left(|y|^2\right)\right] + \frac{|x|^2 - |y|^2}{|x-y|^2}\left[\Phi'\left(|x|^2\right) - \Phi'\left(|y|^2\right)\right]\right\}\cdot \\ &\quad\quad \cdot n(x,t)\, n(y,t)\, dx\, dy\end{aligned}$$

In particular, as Φ is convex $\left(|x|^2 - |y|^2\right)\left[\Phi'\left(|x|^2\right) - \Phi'\left(|y|^2\right)\right]$ is non-negative and as $\chi M = 8\,\pi$

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^2} \Phi\left(|x|^2\right)\, n(x,t)\, dx &\leq 4\int_{\mathbb{R}^2} \left[\Phi'\left(|x|^2\right) + |x|^2\Phi''\left(|x|^2\right)\right] n(x,t)\, dx \\ &\quad - 4\int_{\mathbb{R}^2} \Phi'\left(|x|^2\right)\, n(x,t)\, dx\end{aligned}$$

which ends the proof. □

Applying Lemma 3.5 to Φ_0, we have

$$\frac{d}{dt}\int_{\mathbb{R}^2} \Phi_0\left(|x|^2\right)\, n(x,t)\, dx \leq 4\int_{\mathbb{R}^2} |x|^2\Phi_0''\left(|x|^2\right)\, n(x,t)\, dx\ .$$

Using the Remark 15 and the properties of Φ_0 in De la Vallèe-Poussin theorem (Lemma 3.4), we conclude there exist $c_1, c_2 > 0$ such that

$$|x|^2 \Phi_0''\left(|x|^2\right) \leq 2\frac{\Phi_0\left(|x|^2\right)}{|x|^2} \leq c_1 \Phi_0\left(|x|^2\right) + c_2,$$

and thus,

$$\frac{d}{dt}\int_{\mathbb{R}^2} \Phi_0\left(|x|^2\right)\, n(x,t)\, dx \leq 4\, c_1 \int_{\mathbb{R}^2} \Phi_0\left(|x|^2\right)\, n(x,t)\, dx + 4c_2 M\ . \tag{16}$$

Gronwall's lemma implies that

$$\int_{\mathbb{R}^2} \Phi_0\left(|x|^2\right)\, n(x,t)\, dx \leq e^{c_1 t}\left[\int_{\mathbb{R}^2} \Phi_0\left(|x|^2\right)\, n_0(x)\, dx + \frac{4c_2 M}{c_1}\right] \tag{17}$$

for all $t \in (0, T^*)$.

Now, let us prove that the maximal time of existence cannot be finite. Assume by contradiction that $T^* < \infty$. We first observe that in the case $\chi\, M = 8\,\pi$, the second-momentum of a free-energy solution to the Patlak-Keller-Segel system (1) is conserved due to (13):

$$\int_{\mathbb{R}^2} |x|^2\, n_0(x)\, dx = \int_{\mathbb{R}^2} |x|^2\, n(x,t)\, dx > 0\ . \tag{18}$$

Let us take $\{t_p\}_{p\in\mathbb{N}} \nearrow T^*$. Equation (18) together with the tail-control of the set of densities $\{|x|^2 n(x,t_p)\}_{p\in\mathbb{N}}$ due to (17), implies the tightness of the densities $\{|x|^2 n(x,t_p)\}_{p\in\mathbb{N}}$ in $\mathcal{M}(\mathbb{R}^2)$ by Prokhorov's theorem.

As a conclusion, the sequence of densities $\{n(x,t_p)\}_{p\in\mathbb{N}}$ converges weakly-$*$ as measures towards $dn^* \in \mathcal{M}(\mathbb{R}^2)$ with

$$\int_{\mathbb{R}^2} |x|^2\, dn^*(x) = \int_{\mathbb{R}^2} |x|^2\, n_0(x)\, dx > 0, \tag{19}$$

contradicting the fact that dn^* should coincide with $M\delta_0$ due to Lemma 3.1. □

3.3. **Does it blow-up?** We proved in Lemma 3.3 that if n is a non-negative free-energy solution to the Patlak-Keller-Segel system (1) with initial data n_0 on $[0,T]$ for any $T > 0$, then n satisfies

$$\frac{d}{dt}\mathcal{F}[n](t) \leq -\int_{\mathbb{R}^2} u(x,t)\, \left|\nabla \log u(x,t) - \chi \nabla v(x,t)\right|^2\, dx\ . \tag{20}$$

Let us see that the blow-up of the cell density does happen at $t \nearrow \infty$.

Lemma 3.6 (Blow-up in infinite time)**.** *Under assumptions* (2) *on the initial data n_0, given any free-energy solution n of* (1)*, we have*

$$\lim_{t\to\infty} n(t) = \frac{8\,\pi}{\chi}\delta_{M_1} \quad \textit{weakly-* as measures.}$$

The central idea for this lemma is that if we can extend (20) to the limit when t goes to infinity, we could prove the convergence of the solution to a L^1-stationary solution with a finite second moment. However, all integrable stationary solutions of the PKS system (1) with critical mass have infinite second momentum.

Proof. Consider, without loss of generality, that $M_1 = 0$. Assume by contradiction the existence of an increasing sequence of times $\{t_p\}_{p\in\mathbb{N}} \nearrow \infty$ for which

$$\mathcal{S}[n_p] = \int_{\mathbb{R}^2} n(x,t_p)\, \log n(x,t_p)\, dx$$

is bounded. This together with the conservation of the second moment for the solutions due to (13) shows that there exists a subsequence, denoted with the same index for simplicity, converging weakly in $L^1(\mathbb{R}^2)$ towards a density $n_\infty^* \in L^1(\mathbb{R}^2)$ by Dunford-Pettis theorem. Moreover, the second moment of the limiting density satisfies

$$0 < \int_{\mathbb{R}^2} |x|^2\, n_\infty^*(x)\, dx \le \int_{\mathbb{R}^2} |x|^2\, n_0(x)\, dx < \infty \tag{21}$$

since the concentration towards a delta dirac at 0 is ruled out by the uniform estimate on the entropy $\mathcal{S}[n_p]$.

On the other hand, as previously noticed, a direct consequence of the Logarithmic Hardy-Littlewood-Sobolev inequality (5) is that the free energy $\mathcal{F}[n]$ is bounded from below:

$$\mathcal{F}[n_0] - \liminf_{t\to\infty} \mathcal{F}[n](t) = \lim_{t\to\infty} \int_0^t \left(\int_{\mathbb{R}^2} n(x,s)\, |\nabla \log n(x,s) - \chi \nabla c(x,s)|^2\, dx \right) ds.$$

As a consequence the Fisher information is integrable and,

$$\lim_{t\to\infty} \int_t^\infty \left(\int_{\mathbb{R}^2} n(x,s)\, |\nabla \log n(x,s) - \chi \nabla c(x,s)|^2\, dx \right) ds = 0\ ,$$

which shows that, up to the extraction of sub-sequences, the limit $n_\infty(s,x)$ of $(s,x) \mapsto n(x,t+s)$ when t goes to infinity satisfies

$$\nabla \log n_\infty - \chi \nabla c_\infty = 0\ , \quad c_\infty = -\frac{1}{2\pi} \log|\cdot| * n_\infty\ ,$$

where the first equation holds in the distribution sense at least almost everywhere in the support of n_∞. We have skipped most of the details of this passing to the limit argument since it follows the same steps as in the proof of convergence towards self-similar behavior in the subcritical case done in [13] and in the existence theorem in Section 2. Let us point out, that this is equivalent to the fact that (n_∞, c_∞) solves the nonlocal nonlinear elliptic equation

$$u_\infty = M\,\frac{e^{\chi\, v_\infty}}{\int_{\mathbb{R}^2} e^{\chi\, v_\infty}\,dx} = -\Delta v_\infty\ , \quad \text{with} \quad v_\infty = -\frac{1}{2\pi}\log|\cdot| * u_\infty\ . \tag{22}$$

Moreover, by [18, Theorem 1], the solutions to (22) are radially symmetric. In the case $\chi\,M = 8\,\pi$, [11] and [51] characterized explicitly the family of radial stationary solutions to (22) as being n_b defined in (3). For all b, the stationary solutions n_b have infinite second momentum contradicting (21).

As a conclusion, we have shown that the global free-energy solutions of the PKS system (1) satisfies

$$\lim_{t\nearrow\infty}\int_{\mathbb{R}^2} n(x,t)\log n(x,t)\;dx = +\infty.$$

Now, a direct application of Corollary 3.2 gives the desired result. □

Remark 3.7 (Stationary states and Minimizers of Free Energy)**.** *The free energy $\mathcal{F}[n]$ has an interesting scaling property: introduce $n_\lambda(x) := \lambda^{-2}\,n(x/\lambda)$ then*

$$\mathcal{F}[n_\lambda] = \mathcal{F}[n] + 2\,M\,\log\epsilon\left(\frac{\chi\,M}{8\,\pi} - 1\right)\ .$$

In the case $\chi\,M = 8\,\pi$, $\mathcal{F}[n]$ is thus invariant by this scaling.

On the other hand, the extremal functions of the Logarithmic Hardy-Littlewood-Sobolev inequality (5) *are given up to a conformal automorphism by $\overline{n} := A\left(1+|x|^2\right)^{-2}$. Combining these two remarks we obtain for any $\lambda > 0$,*

$$-C(M) = \mathcal{F}[\overline{n}] = \mathcal{F}[\overline{n}_\lambda]\ .$$

Hence, in the case $\chi\,M = 8\,\pi$, the free energy $\mathcal{F}[n]$ achieves its minimum $C(8\,\pi/\chi) = (8\,\pi/\chi)(1+\log\pi - \log(8\,\pi/\chi))$ in its "stationary solution" the delta dirac. Actually, one can give a sense to $\frac{8\pi}{\chi}\delta_0$ as stationary solution to system (1) *as being the weak-* limit of the suitably scaled as above stationary solutions in* (22)*.*

Note also that when $\chi\,M < 8\,\pi$ there are no stationary solutions for the Patlak-Keller-Segel system (1)*. Indeed, in the case $\chi\,M < 8\,\pi$ there is uniqueness of the stationary self-similar solution as proved in* [13, Theorem 2.1] *and* [11]*.*

Let us finally point out that if moments larger than 2 are initially bounded, then they must diverge as $t \to \infty$. This completes the picture of the long time asymptotics of global in time solutions of the PKS sytem (1) with critical mass: solutions exists globally and concentrate toward the delta dirac at the center of mass as time diverges, while their second moment is preserved and larger moments become unbounded, i.e., the time sequence of measures with equal mass given by the second moments is not tight. This fact means that the "small" mass escaping at infinity in space is "large enough" to have this effect on moments.

Lemma 3.8 (Blow-up of moments in infinite time)**.** *Assume the initial data n_0 verifies* (2) *and $|x|^{2k}n_0 \in L^1(\mathbb{R}^2)$, with $k > 1$, then any free-energy solution n of* (1) *with initial data n_0 satisfies*

$$\lim_{t \nearrow \infty} \int_{\mathbb{R}^2} |x|^{2k} \, n(x,t) \, dx = +\infty.$$

Proof. Consider, without loss of generality, that $M_1 = 0$. Let us start by pointing out the propagation of moments of order $2k$ for all times due to (16), *i.e.*, for any $T > 0$, there exists a constant C_T such that

$$\int_{\mathbb{R}^2} |x|^{2k} \, n(x,t) \, dx \le C_T$$

for almost every $t \in (0,T)$.

Now, assume by contradiction the existence of an increasing sequence of times $\{t_p\}_{p\in\mathbb{N}} \nearrow \infty$ for which $|x|^{2k} \, n(x,t_p)$ is in $L^1(\mathbb{R}^2)$. This together with the conservation of the second moment for the solutions due to (13) shows that there exists a subsequence, denoted with the same index for simplicity, converging weakly-* as measures in $\mathcal{M}(\mathbb{R}^2)$ towards a density $dn^* \in \mathcal{M}(\mathbb{R}^2)$ satisfying

$$\int_{\mathbb{R}^2} |x|^{2k} \, dn^* = \int_{\mathbb{R}^2} |x|^{2k} \, n_0(x) \, dx > 0$$

by Prokhorov's theorem. This fact contradicts that the limiting density is a delta dirac concentrated at 0 due to Lemma 3.6. □

Acknowledgements.- The authors are grateful to Vincent Calvez and Philippe Laurençot for stimulating discussions and fruitful comments. AB acknowledges the support of ESF Network "Global" and bourse Lavoisier. JAC acknowledges the support from DGI-MEC (Spain) project MTM2005-08024. AB and JAC acknowledge partial support of the Acc. Integ./Picasso program HF2006-0198. NM is partially supported by National Science Foundation grant DMS-0403983. We thank the Centre de Recerca Matemàtica (Barcelona) for providing an excellent atmosphere for research.

References

[1] A. Arnold, J. A. Carrillo, L. Desvillettes, J. Dolbeault, A. Jüngel, C. Lederman, P. A. Markowich, G. Toscani and C. Villani, *Entropies and equilibria*

of many-particle systems: an essay on recent research, Monatsh. Math., 142 (2004), pp. 35–43.

[2] J.-P. Aubin, *Un théorème de compacité*, C. R. Acad. Sci. Paris, 256 (1963), pp. 5042–5044.

[3] F. Bavaud, *Equilibrium properties of the Vlasov functional: the generalized Poisson-Boltzmann-Emden equation*, Rev. Modern Phys., 63 (1991), pp. 129–148.

[4] W. Beckner, *Sharp Sobolev inequalities on the sphere and the Moser-Trudinger inequality*, Ann. of Math. (2), 138 (1993), pp. 213–242.

[5] R. Benguria, *PhD Thesis*, PhD thesis, Princeton University, 1979.

[6] R. Benguria, H. Brézis, and E. H. Lieb, *The Thomas-Fermi-von Weizsäcker theory of atoms and molecules*, Comm. Math. Phys., 79 (1981), pp. 167–180.

[7] P. Biler, *Local and global solvability of some parabolic systems modelling chemotaxis*, Adv. Math. Sci. Appl., 8 (1998), pp. 715–743.

[8] P. Biler, G. Karch, P. Laurençot and T. Nadzieja, *The 8π-problem for radially symmetric solutions of a chemotaxis model in a disc*, Top. Meth. Nonlin. Anal., 27 (2006), pp. 133-147.

[9] ———, *The 8π-problem for radially symmetric solutions of a chemotaxis model in the plane*, Math. Meth. Appl. Sci., 29 (2006), pp. 1563-1583.

[10] P. Biler and T. Nadzieja, *A class of nonlocal parabolic problems occurring in statistical mechanics*, Colloq. Math., 66 (1993), pp. 131–145.

[11] ———, *Global and exploding solutions in a model of self-gravitating systems*, Rep. Math. Phys., 52 (2003), pp. 205–225.

[12] P. Billingsley, *Convergence of Probability Measures*, John Wiley & Sons, Inc., New York, 1999.

[13] A. Blanchet, J. Dolbeault and B. Perthame, *Two-dimensional Keller-Segel model: optimal critical mass and qualitative properties of the solutions*. Electronic Journal of Differential Equations, 44 (2006), pp. 1-33.

[14] V. Calvez and J. A. Carrillo, *Volume effects in the Keller-Segel model: energy estimates preventing blow-up*, Journal Mathématiques Pures et Appliquées, 86 (2006), pp. 155-175.

[15] V. Calvez, B. Perthame and M. Sharifi tabar, *Modified Keller-Segel system and critical mass for the log interaction kernel*. preprint 2006.

[16] E. Carlen and M. Loss, *Competing symmetries, the logarithmic HLS inequality and Onofri's inequality on S^n*, Geom. Funct. Anal., 2 (1992), pp. 90–104.

[17] F. A. C. C. Chalub, P. A. Markowich, B. Perthame, and C. Schmeiser, *Kinetic models for chemotaxis and their drift-diffusion limits*, Monatsh. Math., 142 (2004), pp. 123–141.

[18] W. X. Chen, and C. Li, *Classification of solutions of some nonlinear elliptic equations*, Duke Math. J., 63 (1991), pp. 615–622.

[19] S. Childress and J. K. Percus, *Nonlinear aspects of chemotaxis*, Math. Biosci., 56 (1981), pp. 217–237.

[20] L. Corrias, B. Perthame and H. Zaag, *A chemotaxis model motivated by angiogenesis*, C. R. Math. Acad. Sci. Paris, 336 (2003), pp. 141–146.

[21] ———, *Global solutions of some chemotaxis and angiogenesis systems in high space dimensions*, Milan J. Math., 72 (2004), pp. 1–29.

[22] J. Dolbeault and B. Perthame, *Optimal critical mass in the two-dimensional Keller-Segel model in $\mathbb{R}^2$*, C. R. Math. Acad. Sci. Paris, 339 (2004), pp. 611–616.

[23] C. J. van Duijn, I. A. Guerra and M. A. Peletier, *Global existence conditions for a nonlocal problem arising in statistical mechanics*, Adv. Differential Equations, 9 (2004), pp. 133–158.

[24] E. Gabetta, G. Toscani, W. Wennberg, *Metrics for Probability Distributions and the Trend to Equilibrium for Solutions of the Boltzmann Equation*, J. Statist. Phys., 81 (1995), pp. 901–934.

[25] H. Gajewski and K. Zacharias, *Global behaviour of a reaction-diffusion system modelling chemotaxis*, Math. Nachr., 195 (1998), pp. 77–114.

[26] B. Gidas, W. M. Ni, and L. Nirenberg, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys., 68 (1979), pp. 209–243.

[27] M. A. Herrero and J. J. L. Velázquez, *Singularity patterns in a chemotaxis model*, Math. Ann., 306 (1996), pp. 583–623.

[28] D. Horstmann, *From 1970 until present: the Keller-Segel model in chemotaxis and its consequences. I*, Jahresber. Deutsch. Math.-Verein., 105 (2003), pp. 103–165.

[29] ———, *From 1970 until present: the Keller-Segel model in chemotaxis and its consequences. II*, Jahresber. Deutsch. Math.-Verein., 106 (2004), pp. 51–69.

[30] W. Jäger and S. Luckhaus, *On explosions of solutions to a system of partial differential equations modelling chemotaxis*, Trans. Amer. Math. Soc., 329 (1992), pp. 819–824.

[31] E. F. Keller and L. A. Segel, *Initiation of slide mold aggregation viewed as an instability*, J. Theor. Biol., 26 (1970).

[32] R. Kowalczyk, *Preventing blow-up in a chemotaxis model*, J. Math. Anal. Appl., 305 (2005), pp. 566–588.

[33] Ph. Laurençot, *A "de la Vallée-Poussin" theorem*, personal communication.

[34] Ph. Laurençot and S. Mischler, *The continuous coagulation-fragmentation equation with diffusion*, Arch. Rational Mech. Anal., 162 (2002), pp. 45-99.

[35] E. H. Lieb, *Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities*, Ann. of Math. (2), 118 (1983), pp. 349–374.

[36] J.-L. Lions, *Équations différentielles opérationnelles et problèmes aux limites*, Die Grundlehren der mathematischen Wissenschaften, Bd. 111, Springer-Verlag, Berlin, 1961.

[37] A. Marrocco, *Numerical simulation of chemotactic bacteria aggregation via mixed finite elements*, M2AN Math. Model. Numer. Anal., 37 (2003), pp. 617–630.

[38] T. Nagai, T. Senba and K. Yoshida, *Application of the Trudinger-Moser inequality to a parabolic system of chemotaxis*, Funkcial. Ekvac., 40 (1997), pp. 411–433.

[39] Y. Naito, *Symmetry results for semilinear elliptic equations in $\mathbf{R}^2$*, in Proceedings of the Third World Congress of Nonlinear Analysts, Part 6 (Catania, 2000), vol. 47, 2001, pp. 3661–3670.

[40] Y. Naito and T. Suzuki, *Self-similar solutions to a nonlinear parabolic-elliptic system*, Taiwanese J. Math, 8 (2004), pp. 43–55.

[41] V. Nanjundiah, *Chemotaxis, signal relaying and aggregation morphology*, Journal of Theoretical Biology, 42 (1973), pp. 63–105.

[42] H. G. Othmer and A. Stevens, *Aggregation, blowup, and collapse: the ABCs of taxis in reinforced random walks*, SIAM J. Appl. Math., 57 (1997), pp. 1044–1081.

[43] T. Padmanabhan, *Statistical mechanics of gravitating systems*, Phys. Rep., 188 (1990), pp. 285–362.

[44] C. S. Patlak, *Random walk with persistence and external bias*, Bull. Math. Biophys., 15 (1953), pp. 311–338.

[45] B. Perthame, *PDE models for chemotactic movements: parabolic, hyperbolic and kinetic*, Appl. Math., 49 (2004), pp. 539–564.
[46] T. Senba and T. Suzuki, *Weak solutions to a parabolic-elliptic system of chemotaxis*, J. Funct. Anal., 191 (2002), pp. 17–51.
[47] ———, *Applied analysis*, Imperial College Press, London, 2004. Mathematical methods in natural science.
[48] J. Simon, *Compact sets in the space $L^p(0,T;B)$*, Ann. Mat. Pura Appl. (4), 146 (1987), pp. 65–96.
[49] A. Stevens, *The derivation of chemotaxis equations as limit dynamics of moderately interacting stochastic many-particle systems*, SIAM J. Appl. Math., 61 (2000), pp. 183–212 (electronic).
[50] T. Suzuki, *Free energy and self-interacting particles*, Progress in Nonlinear Differential Equations and their Applications, 62, Birkhäuser Boston Inc., Boston, MA, 2005.
[51] J. J. L. Velázquez, *Stability of some mechanisms of chemotactic aggregation*, SIAM J. Appl. Math., 62 (2002), pp. 1581–1633 (electronic).
[52] ———, *Point dynamics in a singular limit of the Keller-Segel model. I. Motion of the concentration regions*, SIAM J. Appl. Math., 64 (2004), pp. 1198–1223 (electronic).
[53] ———, *Point dynamics in a singular limit of the Keller-Segel model. II. Formation of the concentration regions*, SIAM J. Appl. Math., 64 (2004), pp. 1224–1248 (electronic).
[54] G. Wolansky, *Comparison between two models of self-gravitating clusters: conditions for gravitational collapse*, Nonlinear Anal., 24 (1995), pp. 1119–1129.

CRM (Centre de Recerca Matemàtica), Universitat Autònoma de Barcelona, E-08193 Bellaterra, Spain. http://www.ceremade.dauphine.fr/∼blanchet/
E-mail address: blanchet@ceremade.dauphine.fr

ICREA (Institució Catalana de Recerca i Estudis Avançats) and Departament de Matemàtiques, Universitat Autònoma de Barcelona, E-08193 Bellaterra, Spain. http://kinetic.mat.uab.es/∼carrillo/
E-mail address: carrillo@mat.uab.es

Department of Mathematics, Courant Institute of Mathematical Sciences, New York University, NY 10012, U.S.A. http://www.math.nyu.edu/faculty/masmoudi/
E-mail address: masmoudi@cims.nyu.edu