MIXING INVARIANTS OF HYPERBOLIC 3-MANIFOLDS.

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Abstract : Let M be a compact hyperbolic 3-manifold with incompressible boundary. Consider a complete hyperbolic metric on $int(M)$. To each geometrically finite end of $int(M)$ are traditionnaly associated 3 different invariants : the hyperbolic metric associated to the conformal structure at infinity, the hyperbolic metric on the boundary of the convex core and the bending measured lamination of the convex core. In this note we show how invariants of different types can be realised in the different ends.

Introduction

A representation $\rho : \pi_1(M) \to Isom(\mathbb{H}^3)$ *uniformises* M if $\mathbb{H}^3/\rho(\pi_1(M))$ is homeomorphic to $int(M)$ in the homotopy class defined by ρ, if $\rho(\pi_1(M))$ is geometrically finite and minimally parabolic. We denote by $U(M)$ the set of representations uniformising M up to conjugacy. Consider a representation $\rho \in U(M)$ and denote by $M_\rho = \mathbb{H}^3/\rho(\pi_1(M))$ the quotient manifold. To each end of M_ρ is associated a component S of ∂M. If S is not a torus or a Klein bottle, the end associated to S has well defined invariants : the hyperbolic metric associated to the conformal structure at infinity, the hyperbolic metric on the boundary of the convex core (both element of the Teichmüller space $\mathcal{T}(S)$ of S) and the bending measured lamination of the convex core (element of $\mathcal{ML}(S)$).

Given an invariant for each component of ∂M, we want to ask wether or not there is a representation with these ends invariants. When all the ends have the same type, the problem have already been solved in the following ways. It follows from the theory of Ahlfors-Bers that the hyperbolic metric associated to the conformal structure at infinity produces a homeomorphism $U(M) \to \mathcal{T}(\partial_{\chi<0}M)$. By results of [KeS] and [EpM], it follows that the map $U(M) \to \mathcal{T}(\partial_{\chi<0}M)$ given by the metric on the boundary of the convex core is onto. The measured geodesic laminations that are the bending measured geodesic lamination of some $\rho \in U(M)$ have been described in [BoO].

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In the first theorem, we mix conformal structure on the boundary at infinity and bending measured geodesic lamination.

Theorem A. *Let M be a compact hyperbolic 3-manifold with incompressible boundary. Let S be an union of components of $\partial_{\chi<0}M$ and let $\lambda \in \mathcal{M}L(S)$ be a measured geodesic lamination. Denote by $S' = \partial_{\chi<0}M - S$ the complementary of S and let σ be a point in the Teichmüller space of S'. Assume that λ satisfies the following :*

a) the weigth of any closed leaf of λ is less than π;
b) $\exists\eta > 0$ such that, for any essential annulus or Möbius band E with $\partial E \subset S$, we have $i(\partial E, \lambda) \geq \eta$;

Then there is a representation $\rho \in U(M)$ uniformising M such that σ is the hyperbolic metric on the boundary at infinity of the ends corresponding to S' and λ is the union of the bending measured lamination of the ends of $\mathbb{H}^3/\rho(\pi_1(M))$ corresponding to the components of S.

As a special case consider a trivial I-bundle $M = S \times I$ over a closed surface. Given any $\lambda \in \mathcal{M}L(S)$ satisfying $a)$ and any $\sigma \in \mathcal{T}(S)$, it follows from Theorem A that there is $\rho \in U(M)$ such that the metric on the boundary at infinity of one end is σ and the bending measured lamination of the other end is λ.

Theorem B. *Let M be a compact hyperbolic 3-manifold with incompressible boundary. Let S be an union of components of $\partial_{\chi<0}M$ and let $\lambda \in \mathcal{M}L(S)$ be a measured geodesic lamination. Denote by $S' = \partial_{\chi<0}M - S$ the complementary of S and let σ be a point in the Teichmüller space of S'. Assume that λ satisfies the following :*

a) the weigth of any closed leaf of λ is less than π;
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Then there is a representation $\rho \in U(M)$ such that σ is the hyperbolic metric on the boundary of the convex core facing the end corresponding to $S' \subset \partial M$ and λ is the union of the bending measured lamination of the ends of $\mathbb{H}^3/\rho(\pi_1(M))$ corresponding to S.

We will deduce Theorems A and B from the study of local deformations of geometrically finite cone-manifolds done in [Bro] and compactness results, following the ideas of [BoO]. To have a simpler statement, we have considered manifolds with incompressible boundary and minimally parabolic representations. But the same proof would lead us to similar results for geometrically finite manifolds with compressible boundary.

1. Definitions

Since this paper is intended to be short, the definitions may appear to be a bit succinct. More detailed definitions can be found in [Th].

1.1. Geodesic Laminations.

Let S be a closed surface endowed with a complete hyperbolic metric. A *geodesic lamination* on S is a compact subset that is the disjoint union of complete embedded geodesics. Using the fact that two complete hyperbolic metrics on S are quasi-isometric, this definition can be made independent of the chosen metric on S. A geodesic lamination whose leaves are all closed is called a *multi-curve*.

A *measured geodesic lamination* λ is a transverse measure for some geodesic lamination $|\lambda|$. We will denote by $\mathcal{ML}(S)$ the space of measured geodesic laminations topologised with the topology of weak* convergence.

Let γ be a weighted simple closed geodesic with support $|\gamma|$ and weight w and let λ be a measured geodesic lamination, the intersection number $i(\gamma, \lambda)$ is w times the total mass of the measure induced on $|\gamma|$ by λ. The weighted simple closed curves are dense in $\mathcal{ML}(S)$ and this intersection number extends continuously to a function $i : \mathcal{ML}(S) \times \mathcal{ML}(S) \to \mathbb{R}$ (cf. [Bo]). A measured geodesic lamination λ is *arational* if for any simple closed curve c, we have $i(c, \lambda) > 0$.

1.2. 3-manifolds.

Let M be a compact 3-manifold. We will say that M is a *hyperbolic manifold* if its interior can be endowed with a complete hyperbolic metric. The manifold M has incompressible boundary if the map $i_* : \pi_1(S) \to \pi_1(M)$ induced by the inclusion map is injective for any component S of ∂M. Let $\rho : \pi_1(M) \to Isom(\mathbb{H}^3)$ be a faithful discrete representation such that $M_\rho = \mathbb{H}^3/\rho(\pi_1(M))$ is homeomorphic to the interior of M. The representation $\rho(\pi_1(M))$ is *minimally parabolic* if any parabolic isometry belongs to a rank 2 abelian subgroup. Let $L_\rho \subset \hat{\mathbb{C}} = \partial\overline{\mathbb{H}}^3$ be the limit set of $\rho(\pi_1(M))$ and let $\Omega_\rho = \hat{\mathbb{C}} - L_\rho$ be its domain of discontinuity. Let $H(\rho) \subset \mathbb{H}^3$ be the convex hull of L_ρ. The quotient $C(\rho)$ of $H(\rho)$ by $\rho(\pi_1(M))$ is the convex core of ρ and ρ is said to be *geometrically finite* if $C(\rho)$ has finite volume. A geometrically finite and minimally parabolic representation $\rho : \pi_1(M) \to Isom(\mathbb{H}^3)$ such that M_ρ is homeomorphic to the interior of M is said to *uniformise* M.

We will denote by $\partial_{\chi<0}M$ the union of the connected components of ∂M with negative Euler characteristic. Let $\rho \in U(M)$ be a representation that uniformises M. Its convex core $C(\rho)$ is naturally homeomorphic to $M - \partial_{\chi=0}M$. Each component of $M_\rho - C(\rho)$ is homeomorphic to some $S\times]0,\infty[$ where S is a component of ∂M. The components of $M_\rho - C(\rho)$ are in bijection with the non-parabolic ends of M_ρ. Thus to each end, that is not a rank 2 cusp, is associated a component of $\partial C(\rho)$ with its bending measured geodesic lamination and its induced hyperbolic metric (see [Th] or [CEG]). Furthermore to each such end is also associated a component of $\Omega_\rho/\rho(\pi_1(M))$ that is homeomorphic to S. This component has a natural structure of a Riemann surface and the corresponding hyperbolic metric is the *metric on the boundary at infinity* corresponding to $S \subset \partial M$.

2. Boundary at infinity and bending

We will start with the proof of Theorem A.

Proof of Theorem A. Consider a compact hyperbolic 3-manifolds with incompressible boundary and surfaces $S, S' \subset \partial_{\chi<0}M$ as in Theorem A. First we are going to show a result similar to the "Lemme de fermeture" in [BoO]. Consider a sequence of representations $\rho_n \in U(M)$. Denote by σ_n the hyperbolic metric on the boundary at infinity of the ends corresponding to the surface S' and by $\lambda_n \in \mathcal{ML}(S)$ the bending geodesic measured laminations of the ends facing S. We have :

Lemma 2.1. *Let $\rho_n \subset U(M)$ be a sequence of representations uniformising M such that $\sigma_n = \sigma$ for any n and that λ_n converges to a measured geodesic lamination $\lambda_\infty \in \mathcal{ML}(S)$ satisfying conditions a) and b) of Lemma A. Then, up to extracting a subsequence, $\{\rho_n\}$ converges to a representation ρ_∞ uniformising M such that σ is the hyperbolic metric on the components of the boundary at infinity corresponding to the surface S' and that λ_∞ is the bending measured geodesic lamination of the ends facing S.*

Proof. The existence of a subsequence that converges algebraically follows from the arguments of [Le2] (see also [BoO]) in the following way. Since the metric on the components of the conformal boundary corresponding to S' are constant, it follows from [EpM] that, up to extracting a subsequence, the representations $\rho_n(\pi_1(S'))$ converge algebraically. On S', we choose a measured geodesic lamination μ. Since λ satisfies condition b), we can choose μ so that $\exists \eta > 0$ such that $i(\partial E, \lambda_n) + i(\partial E, \mu) > \eta$ for any essential annulus or Möbius band E (for example if M is not an I-bundle, it is sufficient to take for μ an arational lamination). Since the representations $\rho_n(\pi_1(S'))$ converge algebraically, $l_{\rho_n}(\mu)$ is bounded, where $l_{\rho_n}(\mu)$ is the length of the geodesic representative of μ in $M_n = \mathbb{H}^3/\rho_n(\pi_1(M))$. It follows from [Bri] that $l_{\rho_n}(\lambda_n)$ is bounded as well. Now we can deduce from the proof of [Le2, Theorem 1] (see also [Le1, Theorem 6.5]) that the representations $\rho_n(\pi_1(M))$ converge algebraically, up to extracting a subsequence.

Since λ_n converges to λ, it follows from [Le2, Lemmas 4.3 and 4.6] that the sequence of convex pleated surfaces $f_n : S \to M_n$ corresponding to the boundary of the convex core converges to a convex pleated surface $f_\infty : S \to M_\infty$. Furthermore, it is proven in [Le3, Lemma 3.4] (see also [KeS]) that the bending measured geodesic lamination of f_∞ is λ_∞.

On the other hand, it follows from classical results on complex analysis that
$\Omega_{\rho_\infty}/\rho_\infty(\pi_1(M))$ contains a subsurface homeomorphic to S' and that the conformal structure on this subsurface corresponds to σ.

Finally, using the results of [Wa] as in the proof of [BoO, Lemme 21], we can conclude that ρ_∞ uniformises M. Now we have proven that ρ_∞ satisfies the conclusion of Lemma 2.1. □

Consider now a multi-curve $C \in S$ and let $U_C(M)$ be the set of representations
$\rho : \pi_1(M) \to Isom(\mathbb{H}^3)$ uniformising M such that if λ is the bending lamination of ρ, then $\lambda \cap S$ is supported by C. Define a map $b_C : U_C(M) \to]0,\pi[^k \times \mathcal{T}(S')$ that associates to a representation $\rho \in U_C(M)$ the bending angle along the k components of C (in $]0,\pi[^k$) and the point in $\mathcal{T}(S')$ given by the conformal structure of the components of $\Omega_\rho/\rho(\pi_1(M))$ corresponding to S'. Using the results of [Bro] and [BoO], we will show that b_C is a homeomorphism. Notice that the conditions on C for $U_C(M)$ to be empty are given by [BoO].

Lemma 2.2. *If $U_C(M)$ is not empty, the map b_C is a homeomorphism.*

Proof. Consider $\rho \in U_C(M)$ and denote by $D_S M$ the manifold obtained by gluing two copies of M along S. The multi-curve C defines a disjoint union of simple closed curves in $D_S M$ that we will denote by C as well. Let C_ρ be the convex core of ρ. The double $D_S C_\rho$ of $C_\rho - C$ along $S - C$ is naturally homeomorphic to $D_S M - C$. It induces an (generally) incomplete hyperbolic metric on the interior of $D_S M - C$. When we complete this metric we get a hyperbolic structure with cone singularities (see [Th] for definitions). A component of C with a weight $\theta < \pi$ corresponds to a component of the singular locus with cone angle equal to $2\pi - 2\theta$. To such a hyperbolic structure with cone singularities is associated a conformal structure on the boundary at infinity (see [Bro]). Let $b_{DC} : C(D_S M; C) \to]0, 2\pi[\times \mathcal{T}(S')$ be the map that to a geometrically finite hyperbolic structure in $D_S M$ with cone singularities along C associates its cone angles and the hyperbolic structure on the boundary at infinity. By [Bro], b_{DC} is a local homeomorphism.

Let us denote by $\Delta : U_C(M) \to C(D_S(M); C)$ the map given by the doubling construction described above. It follows from the arguments of [BoO, Lemme 23] that Δ is a homeomorphism into its image. By definition, we have $b_C = R^{-1} \circ b_{DC} \circ \Delta$ where $R :]0,\pi[\to]0, 2\pi[$ is the map defined by $\theta \mapsto 2\pi - 2\theta$. From that we deduce that b_C is a local homeomorphism.

It follows from Lemma 2.1 that b_C is a proper map. So b_C is a covering.

Let us adopt the convention that a singularity with cone angle 0 correspond to the complete hyperbolic metric on the complementary of the singularity. Let us extend b_C so that it is defined on geometrically finite metrics with some parabolics corresponding to components of C. Since the results of [Bro] and [BoO] extend to this case, this extension is a covering as well, following the same proof. Considering the complete hyperbolic metric on $D_S M - C$ with a given conformal structure at infinity, we get that b_C is a one sheeted covering, i.e. a homeomorphism. □

Notice that Proposition 2.2 proves Theorem A in the case where λ is a weighted multi-curve.

The proof of Theorem A is now very short. By the density of weighted simple closed curves in $\mathcal{ML}(\partial M)$ (see [CEG]), there is a sequence of weighted simple closed curves $\gamma_n \in \mathcal{ML}(S)$ converging to λ. From the proof of [Le1, Lemma 4.1] we can deduce that for n large enough, γ_n satisfies conditions $a)$ and $b)$ of Theorem A. So by Lemma 2.2, there is a sequence of representations ρ_n with $\sigma_n = \sigma$ and $\lambda_n = \gamma_n$ (using the notations of Lemma 2.1). By Lemma 2.1, a subsequence of ρ_n converges algebraically to a representation $\rho \in U(M)$ that satisfies the conclusion of Theorem A. □

3. The metric on the boundary of the convex core

The proof of Theorem B will follow the same strategy as the proof of Theorem A. We will first deduce from Lemma 2.2 the proof in the case where λ is a weighted multi-curve and then use a compactness argument to get the general case.

Proof of Theorem B. Consider a weighted multi-curve $\lambda \in \mathcal{ML}(S)$ and denote by $U_\lambda(M)$ the set of minimally parabolic representations $\rho : \pi_1(M) \to Isom(\mathbb{H}^3)$ uniformising M such that λ is the bending lamination of ρ on S. Let us denote by $b_\lambda : U_\lambda \to \mathcal{T}(S')$ the map that, to a representation $\rho \in U_\lambda$, associates the hyperbolic metric on the component of the boundary at infinity corresponding to S'. It follows from Lemma 2.2 that b_λ is a homeomorphism. On the other hand, we have a map $c_\lambda : U_\lambda \to \mathcal{T}(S')$ that to a representation $\rho \in U_\lambda$, associates the hyperbolic metric induced on the components of the boundary of the convex core corresponding to S'. The following Lemma, whose proof is well known, will allow us to conclude that c_λ is surjective.

Lemma 3.1. *The map $c_\lambda \circ b_\lambda^{-1} : \mathcal{T}(S') \to \mathcal{T}(S')$ is surjective.*

Proof. By [KeS], the map c_λ is continuous, so $c_\lambda \circ b_\lambda^{-1}$ is continuous. It can then be deduced from [EpM], that $c_\lambda \circ b_\lambda^{-1}$ is onto. □

So c_λ is surjective and we have proven Theorem B in the case where λ is a weighted multi-curve. It remains to show an anologous of Lemma 2.1 to conclude the proof. So consider a sequence of representations $\rho_n \in U(M)$ with bending laminations λ_n satisfying $\lambda_n \cap S \longrightarrow \lambda$ and induced metrics on the convex core σ on S'. The existence of such a sequence follows from the proof of Theorem B for weighted multi-curves and the density of weighted simple closed curves. Using the same arguments as in the proof of Lemma 2.1, we deduce that up to extracting a subsequence, ρ_n converge to ρ_∞.

Let $f_n : \partial_{\chi<0}(M) \to M_n$ be the convex pleated surface corresponding to the boundary of the convex core. Since λ_n converges to λ, it follows from [Le2] and [Le3] that $f_{n|S}$ converge to a convex pleated surface with bending lamination λ. On the other hand since the metric induced on S' by f_n does not depend on n, the pleated surfaces $f_{n|S'}$ converge to a convex pleated surface with induced metric σ. As in the proof of Theorem A, we can conclude, using [Wa], that ρ_∞ uniformises M. Thus ρ_∞ satisfies the conclusion of Theorem B. This conludes the proof.

□

References

[Bo] F. Bonahon, *Bouts des variétés hyperboliques de dimension* 3, Ann. of Math. (2) 124 (1986), 71-158.

[BoO] F. Bonahon, J.-P. Otal, *Lamination mesurées de plissage des variétés hyperboliques de dimension 3*, Ann. of Math. 160 (2004), 1013-1055.

[Bri] M. Bridgeman, *Average bending of convex pleated planes in hyperbolic three-space*, Invent. Math. 132 (1998), 381-391.

[Bro] K. Bromberg, *Rigidity of geometrically finite hyperbolic cone-manifolds*, Geom. Dedicata 105 (2004), 143-170.

[CEG] R.D. Canary, D.B.A. Epstein, P. Green, *Notes on notes of Thurston*, Analytical and Geometrical Aspects of hyperbolic Space (1987), 3-92.

[EpM] D.B.A. Epstein, A.Marden, *Convex hulls in hyperbolic space, a theorem of Sullivan, and measured pleated surfaces*, Analytical and Geometric Aspects of Hyperbolic Space (1987), 113-253.

[KeS] L. Keen, C. Series, *Continuity of convex hull boundaries*, Pac. J. Math. 127 (1988), 457-519.

[Le1] C. Lecuire, *An extension of Masur domain*, Proceedings of the Workshop "Spaces of Kleinian groups and hyperbolic 3-manifolds", London Math. Soc., Lecture Note Series 329 (2006).

[Le2] C. Lecuire, *Bending map and strong convergence*, Preprint.

[Le3] C. Lecuire, *Continuity of the bending map*, Preprint.

[MoS] J.W. Morgan, P.B. Shalen, *Degenerations of Hyperbolic structures III: actions of 3-manifolds groups on trees and Thurston's compactness theorem*, Ann. of Math. 127 (1988), 457-519.

[Th] W.P. Thurston, *The topology and geometry of 3-manifolds*, Notes de cours, Université de Princeton, 1976-79.

[Wa] F. Waldhausen, *On irreductible 3-manifolds which are sufficiently large*, Ann. Math. 87 (1968), 56-88.