REAL RANK ZERO ALGEBRAS HAVE THE CORONA FACTORIZATION PROPERTY

DAN KUCEROVSKY AND FRANCESC PERERA

Abstract. The purpose of this short note is to prove that a stable separable C^*-algebra with real rank zero has the so-called corona factorization property, that is, all the full multiplier projections are properly infinite. Enroute to our result, we consider conditions under which a real rank zero C^*-algebra admits an injection of the compact operators (a question already considered in [21]).

Introduction and Preliminaries

The corona factorization property has connections with extension theory and the internal structure of C^*-algebras. Its origins can be traced back to [6], and it has been subsequently studied in other papers, such as [9, 8, 10, 12, 13, 15, 16, 18].

Known examples of C^*-algebras satisfying this property include stabilizations of purely infinite simple C^*-algebras; stabilizations of exact C^*-algebras with minimal ranks and weakly unperforated K_0 group; and stable type I C^*-algebras with finite decomposition rank (see, e.g. [17]).

There are various equivalent formulations of the corona factorization property (see, e.g. [8, 13, 10]). We take as our definition the one below, which has a more K-theoretical flavour.

Recall that a projection p in a C^*-algebra A is *infinite* if there is a (non-zero) projection q such that $q < p$, yet $q \sim p$ in the Murray-von Neumann sense. If p is not infinite, then we say that p is *finite*. The (non-zero) projection p is said to be *properly infinite* provided that $p \oplus p \lesssim p$ (that is,

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the projection matrix $\left(\begin{smallmatrix} p & 0 \\ 0 & p \end{smallmatrix}\right)$ is equivalent to a subprojection of p). As usual, we say that an element c in a C^*-algebra A is *full* if the closed two-sided ideal $\overline{AcA}$ is all of A.

Definition 0.1. *Let A be a stable C^*-algebra. We say that A has the* corona factorization property *if all full projections in $\mathcal{M}(A)$ are properly infinite.*

Note that, because of the stability assumption, the above definition is in fact equivalent to the requirement that all full projections in $\mathcal{M}(A)$ are in fact Murray-von Neumann equivalent to $1_{\mathcal{M}(A)}$.

Recall that a C^*-algebra A has *real rank zero* provided the self-adjoint, invertible elements form a dense subset of the self-adjoint elements. Equivalently, A has real rank zero precisely when every self-adjoint element can be approximated arbitrarily well by a self-adjoint element with finite spectrum ([3]). The most natural question about the corona factorization property is perhaps whether a non-unital C^*-algebra with real rank zero has this property. This basic question has been mentioned in a number of papers and conference talks, see for example Question 4.17 in [17] which asks whether simple, separable, stable and nuclear real rank zero algebras have the corona factorization property.

A partial result in the positive direction was given in [14, Theorem 1.4], but there the authors require that the given real rank zero algebra admits an embedding of an AF-algebra with finite dimensional trace space.

We answer this question in general and affirmatively, without needing simplicity nor nuclearity assumptions. Our proof reduces to the case where the algebra has compact primitive ideal space, for which we give a characterization in Section 1. We proceed in Section 2 to establish an intermediate result that allows us to draw a generarization of a result in [19]. Our main result is proved in Section 3, where we also prove that if A is a separable C^*-algebra with real rank zero and compact primitive ideal space such that some matrix algebra is stable, then A contains a copy of the compact operators that trivially intersects every proper ideal of A (see [21], where this question was also considered). The corona factorization property has deep consequences in KK-theory, see for example Corollary 3.4.

1. A characterization of algebras with compact structure space

Proposition 1.1. *Let A be a separable C^*-algebra. Consider the following conditions:*

(i) *The primitive ideal space* Prim$A$ *is compact;*
(ii) *The full elements of A_+ in A form an open set.*

Then (i) implies (ii)*, and the converse also holds if A has moreover real rank zero.*

Proof. We first prove the implication (i) $\Rightarrow$ (ii). The Dauns-Hoffmann theorem [20, 4.4.4] shows that for each positive $x \in A_+$, the map $\check{x}\colon \mathrm{Prim}A \to \mathbb{R}$ given by $t \mapsto \|x + \ker t\|$ is lower semicontinuous. If the full elements of A_+ are not an open set, then there exists a full element $x \in A_+$ and a sequence $(y_n) \subset A_+$ that converges to x in norm, with none of the y_n full. By properties of quotient norms, the $\check{y}_n$ converge to $\check{x}$ uniformly. Each $\check{y}_n$ has a zero. By a triangle inequality argument, the infimum $\inf \check{x}$ is therefore zero. By one of the Weierstrass theorems, a lower semicontinuous function on a compact set attains its miminum, so that $\check{x}$ has a zero. But this contradicts the assumption that x is full in A_+.

We now prove the converse, so hence assume A has real rank zero. If the full positive elements form an open set, then by assumption we can approximate some full positive element by an element with finite spectrum, that is moreover full. By the functional calculus, we see that there exists a full projection, say $P \in A$. By Brown's theorem [2], we now have that the unital algebra PAP is Morita equivalent to A, and thus these algebras have the same primitive ideal spaces [22]. But it is well-known (see, e.g. [4]) that unital separable C^*-algebras have compact primitive ideal space. □

2. An intermediate result

In this section we prove a result that establishes the corona factorization property under somewhat strong hypothesis. As a corollary, we obtain a generalization of a result by E. Pardo ([19]), which says that under strong conditions on the algebra, multiplier projections that are not in the algebra are either properly infinite or finite.

Theorem 2.1. *Let B be a stable separable C^*-algebra. If, for every full multiplier projection P, the hereditary algebra PBP contains a full stable subalgebra, then all full multiplier projections are properly infinite.*

Before proving the above, we recall two of our earlier results. These are basically KK-theoretical results, stated here mostly without the language of KK-theory. The first of the quoted results [9, Lemma 10] is[1] :

Lemma 2.2. *Let A be stable and σ-unital. A hereditary subalgebra $\overline{cAc}$ generated by a positive element c of the multipliers of A is isomorphic to some hereditary subalgebra PAP generated by a multiplier projection P. If c is full in the multiplier algebra, then P is also full in the multiplier algebra.*

The next result is from [8, Theorem 3.2] (see also [6, Theorem 6]).

Lemma 2.3. *Let B be a stable and separable C^*-algebra. The following are equivalent:*

(i) *Every full multiplier projection is properly infinite;*
(ii) *Every weakly nuclear full extension of B by a separable C^*-algebra A is absorbing;*
(iii) *Every hereditary subalgebra $\overline{cBc}$ generated by a full positive element c of the multiplier algebra of B contains a stable full subalgebra.*

We now obtain the proof of Theorem 2.1 as a corollary of these lemmas:

Proof. Let c be a full multiplier element in $\mathcal{M}(B)$. By Lemma 2.2, there is a full multiplier projection P such that $\overline{cBc} \cong PBP$. Our hypothesis then implies that $\overline{cBc}$ contains a full stable subalgebra.

By Lemma 2.3, all full multiplier projections are therefore properly infinite. □

[1]The statement on fullness follows directly from a calculation with unitary equivalence in a Hilbert module, see the proof of [8, Theorem 4.4].

We have the following further corollary that more closely resembles Pardo's result.

Corollary 2.4. *Let B be a stable simple C^*-algebra. If no full multiplier projections are finite, then in fact all full multiplier projections are properly infinite.*

Proof. To deduce this corollary from the above theorem, it is sufficient to show that if a full multiplier projection P is infinite, then PBP contains a stable subalgebra. To do this, let v be a partial isometry with $v^*v = P$ and $vv^* \leq P$, with of course $vv^* \neq P$. Now, $C^*(v)$ is an abstract Toeplitz algebra, and thus contains a copy $\mathcal{K}$ of the compact operators. We notice that $\mathcal{K}$ is contained in the hereditary corner generated by P in the multipliers, and therefore the same is true of $\overline{\mathcal{K}B\mathcal{K}}$. Thus we have a stable subalgebra within PBP. □

Of course, it is not quite fair to call the above a generalization of Pardo's result, since his result is phrased in terms of individual projections, whereas ours has a stronger hypothesis involving all full multiplier projections. On the other hand, this stronger hypothesis allows us to use much weaker conditions on the algebra.

3. The corona factorization property for real rank zero algebras

Lemma 3.1. *Let A be a separable C^*-algebra with real rank zero such that has a full projection. If $M_n(A)$ is stable for some integer n, then there is a copy of $\mathcal{K}$ in A that does not nontrivially intersect any proper ideal of A.*

Proof. Let us first consider the special case where A is of type I. The Kirchberg-Winter decomposition rank of A is, in this case [23], equal to the topological dimension of the primitive ideal space of A, but on the other hand by the Bratteli–Elliott theorem [1] the real rank zero property implies that the primitive ideal space has a basis of compact open sets. In other words, the topological dimension of the spectrum is zero. In [5], it is shown that the decomposition rank of an algebra is zero if and only if the algebra is an AF algebra. Thus, A is an AF algebra and thus is stable if and only if it has no bounded traces. Since a bounded trace on A would extend to a bounded trace on the (stable) algebra $M_n(A)$, we see that the algebra A is stable. We notice that $p \otimes \mathcal{K} \subseteq A \otimes \mathcal{K}$, where p is a full projection in A, is a full copy of $\mathcal{K}$ inside $A \otimes \mathcal{K} \cong A$.

Having disposed of this quite nontrivial special case, we now consider the case where A is not of type I. We shall first show that there exists a copy of

$$C := \bigoplus_{n=1}^{\infty} M_n(\mathbb{C})$$

inside A, and that this copy does not nontrivially intersect any ideal of A.

Let P_N be a full projection in A. The hereditary subalgebra $P_N A P_N$ has no finite-dimensional representations. Thus, by the weak divisibility result from [21, Proposition 5.7], there is a unital $*$-homomorphism $\iota_N \colon \mathcal{F} \to P_N A P_N$ where $\mathcal{F}$ is a finite-dimensional C^*-algebra whose factors can be assumed to be matrix algebras of dimension greater than $2N$. Let us denote these matrix algebras by $F_1, F_2, \cdots, F_k$. As pointed out in [21] (see the remark after Proposition 5.7), we can suppose that one of the $\iota_N(F_i)$ is full in $P_N A P_N$. We may as well take $i = 1$.

Now, let us define a map $\jmath_N \colon M_N(\mathbb{C}) \to A$ by embedding $M_N(\mathbb{C})$ in F_1 as the top left corner. Let us also define $P_{N+1} := P_N - \jmath_n(1_{M_N})$, and notice that this is a full projection in A because it majorizes, for example, $\iota_N(e_{2N,2N})$.

We now have, by induction, defined a sequence of embeddings of $M_n(\mathbb{C})$ into A as orthogonal full subalgebras. (The base case of the induction begins with simply choosing any full projection of A as P_1.) We thus have the intertwining

$$
\begin{array}{ccc}
M_1(\mathbb{C}) & \longrightarrow & A \\
\downarrow & & \downarrow \\
M_1(\mathbb{C}) \oplus M_2(\mathbb{C}) & \longrightarrow & A \\
\downarrow & & \downarrow \\
M_1(\mathbb{C}) \oplus M_2(\mathbb{C}) \oplus M_3(\mathbb{C}) & \longrightarrow & A \\
\dots\dots\dots\dots & &
\end{array} \tag{3.1}
$$

which gives a copy of $C := \bigoplus_{n=1}^{\infty} M_n(\mathbb{C})$ inside A as claimed.

Now, writing the compact operators $\mathcal{K}$ as the closure of the algebraic direct limit $D := \mathrm{alglim}_{\vec{n}}(M_n(\mathbb{C}), x \mapsto x \oplus 0)$, we notice that there is a copy of D inside C. Since of course A is closed, we thus have a copy of $\mathcal{K}$ inside A as claimed. □

The next theorem shows that under minor technical hypotheses, real rank zero C^*-algebras have the corona factorization property. We note that stability is clearly a necessary condition if all full multiplier projections are to be properly infinite (the unit of the multipliers of a nonunital separable C^*-algebra is properly infinite if and only if the given algebra is stable[2]).

Theorem 3.2. *Let A be a stable separable real rank zero C^*-algebra with compact structure space. Then every full multiplier projection is properly infinite.*

Proof. Let P be a full multiplier projection. Then, for some positive integer N, we have that 1 in $\mathcal{M}(M_N(A))$ is subequivalent to $\bigoplus_1^N P$ (that is, 1 is equivalent to a subprojection of $\bigoplus_1^N P$). Since by stability 1 is properly infinite, $\bigoplus_1^N P$ is therefore also properly infinite. But the unit of the multipliers of $M_N(PAP)$ is therefore properly infinite, which implies that $M_N(PAP)$ is stable. Furthermore, we claim that PAP contains a full projection. Indeed, by the results in [2] and [22], the structure space of

[2]To see this, notice that if the unit of the multipliers is properly infinite, we have a unital embedding of $\mathcal{O}_2$ into the multipliers, hence a unital embedding of $\mathcal{O}_\infty$ into the multipliers, whereupon the main result of [7] gives stability

PAP is compact. By Proposition 1.1, the full positive elements of PAP are therefore an open set, and it is then easy to use the real rank zero property of PAP to find a full projection $p \in PAP$. By Lemma 3.1, there is now a full copy of the compact operators $\mathcal{K}$ in PAP. Theorem 2.1 finishes the proof. □

Our Lemma 2.3 gives a KK-theoretical corollary:

Corollary 3.3. *Let B be a stable separable real rank zero C^*-algebra with compact structure space. Every full weakly nuclear extension of B by a separable C^*-algebra A is absorbing.*

Further corollaries can be obtained, for example:

Corollary 3.4. *If B is a unital real rank zero C^*-algebra and A is a separable C^*-algebra, then all elements of $KK^1(A, B)$ can be taken to be full weakly nuclear extensions $\tau\colon A \to \mathcal{M}(B \otimes \mathcal{K})/B \otimes \mathcal{K}$.*

We may drop the condition on the structure space, but it appears that then we can only prove the existence of stable full subalgebras rather than full copies of the compact operators, $\mathcal{K}$. This is nevertheless sufficient:

Theorem 3.5. *Separable stable real rank zero C^*-algebras have the property that full multiplier projections are properly infinite.*

Proof. Under this hypothesis, the primitive ideal space is a second-countable zero-dimensional locally compact topological space. It may therefore be written as a countable union of disjoint compact open sets. By the Dauns-Hoffmann theorem, the support functions of these sets can be regarded as central projections in the multiplier algebra $\mathcal{M}(A)$. Let us denote these projections by q_i.

By Lemma 2.3, it is sufficient to prove that, given a positive full element c of the multiplier algebra $\mathcal{M}(A)$, the hereditary subalgebra $\overline{cAc}$ contains a stable full subalgebra. Since $\mathrm{Prim}(q_i A)$ is compact, we may use Theorem 3.2 followed by Lemma 2.3 again, and assume that $q_i\overline{cAc} \subseteq A$ contains a stable full subalgebra, say S_i. We may moreover suppose that the S_i are separable (since A is).

Define $S \subseteq \overline{cAc}$ to be

$$S := C_0(S_i) = \{(s_1, s_2, s_3, \cdots) : s_i \in S_i,\ \|s_i\| \to 0 \text{ as } i \to \infty\}.$$

We claim first that S is full. Let π be an arbitrary nonzero representation of A. We may extend it to a representation of $\mathcal{M}(A)$ by taking closures in the strict topology. Since the q_i converge strictly to $1_{\mathcal{M}(A)}$, at least one of the q_i is not in the kernel of π. But then since the corresponding S_i is full in q_iA, it follows that $\pi(S_i) \neq \{0\}$. This implies that S is not properly contained in the kernel of any nonzero representation.

Lastly, since S can be written as an inductive limit of stable separable subalgebras $\bigoplus_1^N S_i$, it is stable [7]. □

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Department of Mathematics, University of New Brunswick at Fredericton, NB, Canada E3B 5A3

E-mail address: dkucerov@unb.ca

Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Spain

E-mail address: perera@mat.uab.cat