Global exponential stability of nonautonomous neural network models with unbounded delays

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Abstract

For a nonautonomous class of n-dimensional differential system with infinite delays, we give sufficient conditions for its global exponential stability, without showing the existence of an equilibrium point, or a periodic solution, or an almost periodic solution. We apply our main result to several concrete neural network models, studied in the literature, and a comparison of results is given. Contrary to usual in the literature about neural networks, the assumption of bounded coefficients is not need to obtain the global exponential stability. Finally, we present numerical examples to illustrate the effectiveness of our results.

Keywords: Cohen-Grossberg neural networks, Infinite distributed delays, Infinite discrete delays, Global exponential stability, Unbounded coefficients.

Mathematics Subject Classification: 34K20, 34K25, 39A30, 92B20.

1 Introduction

In 1983, Cohen and Grossberg [8] presented and studied the well-known neural network model described by the following system of ordinary differential equations

$$x_i'(t) = -a_i(x_i(t))\left[b_i(x_i(t)) - \sum_{j=1}^{n} c_{ij} f_j(x_j(t)) + I_i\right], \quad t \geq 0,\ i = 1, \dots, n, \tag{1.1}$$

where $a_i(u)$ are the amplification functions, $b_i(u)$ are the self-signal functions, $f_j(u)$ are the activation functions, c_{ij} represent the connection weights, and I_i denote the inputs from outside of the system. As particular situation of (1.1), we have the well-known Hopfield neural network model

$$x_i'(t) = -b_i(x_i(t)) + \sum_{j=1}^{n} c_{ij} f_j(x_j(t)) + I_i, \quad t \geq 0,\ i = 1, \dots, n, \tag{1.2}$$

studied by Hopfield [16, 17] in 1982 and 1984 respectively.

Due to the finite switching speed of the amplifiers and the communication time between neurons, differential equations describing neural networks should incorporate time delays. In 1989, Marcus and Westervelt [22] introduced for the first time a discrete delay in the Hopfield model (1.2), and they observed that the delay can destabilize the system. In fact, the delays can affect the dynamic behavior of neural network models [3, 22] and, for this reason, stability of delayed neural network models has been investigated extensively (see [1, 2, 4, 6, 7, 9, 11, 18, 19, 20, 23, 25, 26, 27, 28, 29, 30], and the references therein). Another relevant fact to take into account is that the neuron charging time, the interconnection weights, and the external inputs often change as time proceeds. Thus, neural network models with temporal structure of neural activities should be introduced and investigated (see [7, 25]).

For neural network models with time-varying coefficients, many authors derive sufficient conditions ensuring that all solutions converge exponentially to zero or to an equilibrium point [9, 19, 30]. Other authors assume periodic, or almost periodic, coefficient functions and derive sufficient conditions ensuring the existence of a periodic, or almost periodic, solution and its global exponential stability

[7, 18, 20, 26, 27]. We should say that, in the significative recent research papers [1, 2], the authors consider neural network models with weighted pseudo-almost automorphic coefficients and general conditions are assumed to prove the existence and global exponential stability of a weighted pseudo-almost automorphic solution. Studies about global exponential stability of nonautonomous neural network models without an equilibrium point, or a periodic solution, or an almost periodic solution are few and the authors always assume bounded coefficient functions [28, 29].

In this paper we study a general nonautonomous n-dimensional differential equation with infinite delays, which includes most of the neural network models as particular situations. In spite of our main motivation was to apply the obtained result to neural network models, the studied system is general enough to include, as subclass, models from others research areas such as Lotka-Volterra [5, 10]. Moreover, we should remark that the global exponential stability is obtained without assuming, or showing, the existence of an equilibrium point, or a periodic solution, or an almost periodic solution, or even an another solution and, when we apply this stability criterion to neural network models, the boundedness of coefficients are not required. To the best of our knowledge, a few authors have studied the stability of neural network models without assuming bounded coefficient functions [21].

After the introduction, the present paper is divided into four sections. Section 2 is a preliminary section, where some notations and definitions are introduced, a generalized nonautonomous n-dimensional differential equation with infinite delays is formulated, and the hypotheses are given. In section 3, we prove the main results about global exponential stability of the model. The section 4 is dedicated to apply our results to neural network models. First we obtain a stability criterion for a generalized Cohen-Grossberg neural network model with infinite distributed and discrete delays, then we apply it to several examples, showing that the studied model includes, as subclass, Hopfield, Cohen-Grossberg, and Bidirectional Associative Memory (BAM) models. Also in this section, a brief comparison of our stability criteria with the literature is given. Finally, in section 5, we present two numerical simulations to illustrate the effectiveness of our results.

2 Notations and model description

In this paper, we always consider the vectorial space $\mathbb{R}^n$, for $n \in \mathbb{N}$, equipped with the maximum norm, i.e. $|x| = \max\{|x_i| : i = 1, \ldots, n\}$ for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, and we say that a vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ is positive, denoting by $x > 0$, if $x_i > 0$ for all $i = 1, \ldots, n$.

We consider the Banach space

$$UC_\varepsilon^n = \left\{\varphi \in C((-\infty, 0]; \mathbb{R}^n) : \sup_{s\leq 0} |\varphi(s)|\, \mathrm{e}^{\varepsilon s} < \infty,\ \varphi(s)\, \mathrm{e}^{\varepsilon s} \text{ is uniformly continuous on } (-\infty, 0]\right\},$$

for a convenient $\varepsilon > 0$, equipped with the norm $||\varphi||_\varepsilon = \sup_{s\leq 0} |\varphi(s)|\, \mathrm{e}^{\varepsilon s}$. In what follows, we fix $\varepsilon > 0$. In applications, section 4, we will see how to choose the positive constant ε.

We also consider the Banach space $BC = BC((-\infty, 0]; \mathbb{R}^n)$ of bounded and continuous functions, $\varphi : (-\infty, 0] \longrightarrow \mathbb{R}^n$, equipped with the norm $||\varphi|| = \sup_{s\leq 0} |\varphi(s)|$. We note that $BC \subseteq UC_\varepsilon^n$ and $\|\varphi\|_\varepsilon \leq \|\varphi\|$, for $\varphi \in BC$.

For an open set $D \subseteq BC$ and $f : [0, +\infty) \times D \longrightarrow \mathbb{R}^n$ a continuous function, we consider, in the phase space UC_ε^n, the functional differential equation (FDE) given in general setting by

$$x'(t) = f(t, x_t), \qquad t \geq 0, \tag{2.1}$$

where, as usual, x_t denotes the function $x_t : (-\infty, 0] \longrightarrow \mathbb{R}^n$ defined by $x_t(s) = x(t + s)$ for $s \leq 0$.

It is known that UC_ε^n is an admissible phase space for (2.1) in the sense of [12, 13] (see [13, Theorem 1.2] and [13, Remark 2.3]), consequently the standard existence, uniqueness, continuous dependence type results are available (see [15]). Here, we assume that f has enough smooth properties to ensure the existence and uniqueness of solution for the initial value problem, denoting by $x(t, t_0, \varphi)$ the solution of (2.1) with initial condition $x_{t_0} = \varphi$, for $t_0 \geq 0$ and $\varphi \in UC_\varepsilon^n$. From [15], if f maps closed bounded subsets of its domain into bounded sets of $\mathbb{R}^n$, then the solution $x(t, t_0, \varphi)$ of (2.1) is extensible to $(-\infty, a]$, with $a > t_0$, whenever it is bounded.

In the phase space UC_ε^n, we consider the following nonautonomous functional differential system with infinite delays,

$$x_i'(t) = -a_i(t, x_i(t)) \left[b_i(t, x_i(t)) + \sum_{k=1}^{K} \sum_{j=1}^{n} f_{ijk}(t, x_{j_t}) \right], \qquad t \geq 0,\ i = 1, \dots, n, \tag{2.2}$$

where $n, K \in \mathbb{N}$ and $a_i : [0, +\infty) \times \mathbb{R} \longrightarrow (0, \infty)$, $b_i : [0, +\infty) \times \mathbb{R} \longrightarrow \mathbb{R}$, and $f_{ijk} : [0, +\infty) \times UC_\varepsilon^1 \longrightarrow \mathbb{R}$ are continuous functions. We should say that model (2.2) is general enough to include several well known biological models, such as Lotka-Volterra [5, 10], as well as neural network models, such as Hopfield, Cohen-Grossberg, and BAM. As, in this paper, we intend to apply the results to neural network models, we restrict our attention to initial bounded conditions, i.e.,

$$x_{t_0} = \varphi, \qquad \text{with} \qquad \varphi \in BC, \tag{2.3}$$

for some $t_0 \geq 0$.

By one hand, taking $g(s) = \mathrm{e}^{-\varepsilon s}$ in [13, Theorem 2.1] (see also [13, Remark 3.2]), we know that UC_ε^n is an admissible phase space, as we said above. By another hand, the model (2.2) has the form (2.1) with $f : [0, +\infty) \times UC_\varepsilon^n \to \mathbb{R}^n$ defined by

$$f(t, \varphi) = (f_1(t, \varphi), \dots, f_n(t, \varphi)), \qquad t \geq 0,\ \varphi = (\varphi_1, \dots, \varphi_n) \in UC_\varepsilon^n, \tag{2.4}$$

where

$$f_i(t, \varphi) = -a_i(t, \varphi_i(0)) \left[b_i(t, \varphi_i(0)) + \sum_{k=1}^{K} \sum_{j=1}^{n} f_{ijk}(t, \varphi_j) \right]. \tag{2.5}$$

Consequently, as f is a continuous function, from [15, Theorem 2.1] we conclude that the initial value problem (2.2)-(2.3) has a solution.

In the sequel, for (2.2) the following hypotheses will be considered:

(A1) for each $i \in \{1, \dots, n\}$, there exists $\overline{a}_i, \underline{a}_i > 0$ such that

$$\underline{a}_i \leq a_i(t, u) \leq \overline{a}_i, \qquad \forall t \geq 0,\ \forall u \in \mathbb{R};$$

(A2) for each $i \in \{1, \dots, n\}$, there exists $D_i : [0, +\infty) \longrightarrow \mathbb{R}$ such that

$$D_i(t) a_i^2(t, u) \leq \frac{\partial}{\partial t} a_i(t, u), \qquad \forall t > 0,\ \forall u \in \mathbb{R};$$

(A3) for each $i \in \{1, \dots, n\}$, there exists $\beta_i : [0, +\infty) \longrightarrow (0, +\infty)$ such that

$$(b_i(t, u) - b_i(t, v))/(u - v) \geq \beta_i(t), \qquad \forall t \geq 0,\ \forall u, v \in \mathbb{R},\ u \neq v;$$

(A4) for each $i, j \in \{1, \dots, n\}$ and $k \in \{1, \dots, K\}$, $f_{ijk} : [0, +\infty) \times UC_\varepsilon^1 \longrightarrow \mathbb{R}$ is a Lipschitz function on the second variable i.e., there exists a continuous function $F_{ijk} : [0, +\infty) \longrightarrow [0, +\infty)$ such that

$$|f_{ijk}(t, \varphi) - f_{ijk}(t, \psi)| \leq F_{ijk}(t) \|\varphi - \psi\|_\varepsilon, \qquad \forall t \geq 0, \forall \varphi, \psi \in UC_\varepsilon^1;$$

(A5) there exists a continuous function $\lambda : [0, +\infty) \to (0, +\infty)$ such that, for each $i \in \{1, \dots, n\}$ and $t_0 \geq 0$,

$$\underline{a}_i \big(\beta_i(t) + D_i(t) \big) - \sum_{k=1}^{K} \sum_{j=1}^{n} \overline{a}_j F_{ijk}(t)\, \mathrm{e}^{\int_{t_0}^{t} \lambda(u) - \varepsilon\, du} > \lambda(t) \;\; \text{and} \;\; \int_{t_0}^{t} [\lambda(u) - \varepsilon]\, du \geq 0, \tag{2.6}$$

for all $t \geq t_0$.

First, we remark that, if all functions, $a_i(t,u)$, do not depend explicitly on time i.e., $a_i(t,u) = a_i(u)$, $i = 1,\dots,n$, then $D_i(t) = 0$ and assumption (A2) will be relaxed. We also remark that the hypothesis (A3) is trivially satisfied if $b_i(t,u) = \beta_i(t)u$ for all $t \geq 0$ and $u \in \mathbb{R}$. Finally, as we can see by the last example in section 5, the hypotheses set (A1)-(A5) does not imply the boundedness of solutions of (2.2).

Without assuming the existence of an equilibrium point, or a periodic solution, our main purpose is to establish sufficient conditions for the global exponential stability of (2.2), whose definition we recall here.

Definition 2.1. *The system* (2.2) *is said to be* globally exponentially stable *if there are* $\delta > 0$ *and* $C \geq 1$ *such that*

$$|x(t,t_0,\varphi) - x(t,t_0,\psi)| \leq C\,\mathrm{e}^{-\delta(t-t_0)}\,\|\varphi-\psi\|, \qquad \forall t_0 \geq 0,\ \forall t \geq t_0,\ \forall \varphi,\psi \in BC.$$

It should be mentioned that the above definition of global exponential stability is the usual in the literature on neural networks with unbounded delay [25, 28, 29], but it does not even imply the stability of (2.2) in the phase space UC_ε^n, i.e., relative to the norm $\|\cdot\|_\varepsilon$.

3 Global exponential stability

First we show that the solutions of (2.2), with bounded initial conditions, are defined on $\mathbb{R}$.

Lemma 3.1. *Assume (A1), (A3), and (A4). For* $t_0 \geq 0$ *and* $\varphi \in BC$*, the solution* $x(t) = x(t,t_0,\varphi)$ *of* (2.2) *is defined on* $\mathbb{R}$.

Proof. By simplicity, we denote

$$g_i(t,\varphi) := \sum_{k=1}^{K}\sum_{j=1}^{n} f_{ijk}(t,\varphi_j),$$

for each $i \in \{1,\dots,n\}$, $t \geq 0$, and $\varphi = (\varphi_1,\dots,\varphi_n) \in UC_\varepsilon^n$. As f_{ijk} are Lipschitz functions on the second variable, then g_i are also Lipschitz function on the second variable, i.e.,

$$|g_i(t,\varphi) - g_i(t,\psi)| \leq G_i(t)\|\varphi-\psi\|_\varepsilon, \qquad \forall t \geq 0, \forall \varphi,\psi \in UC_\varepsilon^n,$$

with $G_i(t) := \displaystyle\sum_{k=1}^{K}\sum_{j=1}^{n} F_{ijk}(t)$.

Let $x(t) = (x_1(t),\dots,x_n(t))$ be the maximal solution of initial value problem (2.2)-(2.3), with $t \in (-\infty,a)$ for some $a \in (t_0,+\infty]$, and define $z(t) = (z_1(t),\dots,z_n(t)) := (|x_1(t)|,\dots,|x_n(t)|)$. For each $i \in \{1,\dots,n\}$, from (2.2) and (A1) we have

$$\begin{aligned}
z_i'(t) &= \mathrm{sign}(x_i(t))x_i'(t)\\
&= -\mathrm{sign}(x_i(t))a_i(t,x_i(t))\left[b_i(t,x_i(t)) + g_i(t,x_t)\right]\\
&\leq -\mathrm{sign}(x_i(t))a_i(t,x_i(t))\big[b_i(t,x_i(t)) - b_i(t,0)\big] + \overline{a}_i|g_i(t,x_t) - g_i(t,0)| + \overline{a}_i|b_i(t,0) + g_i(t,0)|,
\end{aligned}$$

and, from (A3), (A4), and integrating over $[t_0, t]$, we obtain

$$\begin{aligned} z_i(t) &\le z_i(t_0) - \int_{t_0}^{t} \operatorname{sign}(x_i(u)) a_i(u, x_i(u)) \big[b_i(u, x_i(u)) - b_i(u, 0)\big] du + \int_{t_0}^{t} \overline{a}_i |g_i(u, x_u) - g_i(u, 0)| du + \\ &\quad + \int_{t_0}^{t} \overline{a}_i |b_i(u, 0) + g_i(u, 0)| du \\ &\le \|\varphi\| - \int_{t_0}^{t} a_i(u, x_i(u)) \operatorname{sign}(x_i(u)) x_i(u) \beta_i(u) du + \int_{t_0}^{t} \overline{a}_i G_i(u) \|x_u\|_\varepsilon du + \\ &\quad + \int_{t_0}^{t} \overline{a}_i |b_i(u, 0) + g_i(u, 0)| du \\ &\le \|\varphi\| + \int_{t_0}^{t} \overline{a}_i G_i(u) \|x_u\| du + \int_{t_0}^{t} \overline{a}_i |b_i(u, 0) + g_i(u, 0)| du \\ &\le \|\varphi\| + \int_{t_0}^{t} [\overline{b}(u, 0) + \overline{g}(u, 0)] du + \int_{t_0}^{t} \overline{G}(u) \|z_u\| du \end{aligned} \tag{3.1}$$

where $\overline{G}(u) = \max_i |\overline{a}_i G_i(u)|$, $\overline{b}(u, 0) = \max_i |\overline{a}_i b_i(u, 0)|$, and $\overline{g}(u, 0) = \max_i |\overline{a}_i g_i(u, 0)|$. Defining the continuous function $\phi : [t_0, +\infty) \longrightarrow [0, +\infty)$ by $\phi(t) = \|\varphi\| + \int_{t_0}^{t} \overline{b}(u, 0) + \overline{g}(u, 0) du$, from (3.1) we obtain, for $t \ge t_0$,

$$\|z_t\| \le \phi(t) + \int_{t_0}^{t} \overline{G}(u) \|z_u\| du$$

and, by the generalized Gronwall's inequality (see [14]) we have

$$\|z_t\| \le \phi(t) + \int_{t_0}^{t} \overline{G}(u) \phi(u) e^{\int_u^t \overline{G}(v) dv} du. \tag{3.2}$$

By one hand, as f, defined by (2.4)-(2.5), takes bounded subsets of $[0, +\infty) \times UC_\varepsilon^n$ into bounded sets of $\mathbb{R}^n$, from Continuation Theorem [15, Theorem 2.4], we have

$$\lim_{t \to a} \|x_t\| = \lim_{t \to a} \|z_t\| = +\infty. \tag{3.3}$$

By another hand, the functions $\bar{G} : \mathbb{R} \to [0, +\infty)$ and $\phi : [t_0, +\infty) \to [0, +\infty)$ are continuous and, from (3.2) and (3.3) we conclude that $a = +\infty$. □

Remark 3.1 It is easy to see that Lemma 3.1 is also true if we have

(A4') for each $i, j \in \{1, \dots, n\}$ and $k \in \{1, \dots, K\}$, there exists a continuous function $F_{ijk} : [0, +\infty) \longrightarrow (0, +\infty)$ such that

$$|f_{ijk}(t, \varphi) - f_{ijk}(t, \psi)| \le F_{ijk}(t) \|\varphi - \psi\|, \qquad \forall t \ge 0, \forall \varphi, \psi \in UC_\varepsilon^1$$

instead of (A4).

Now, we state our main result on the global exponential stability of (2.2).

Theorem 3.2. *Assume (A1)-(A5). Then the model* (2.2) *is globally exponentially stable.*

Proof. Following the notation in Definition 2.1, fix $\delta = \varepsilon$ and

$$C := \frac{\max_j \{\overline{a}_j\}}{\min_j \{\underline{a}_j\}} = \frac{\max_j \{\underline{a}_j^{-1}\}}{\min_j \{\overline{a}_j^{-1}\}} \ge 1. \tag{3.4}$$

Let $t_0 > 0$, $\varphi = (\varphi_1, \ldots, \varphi_n) \in BC$, and $\psi = (\psi_1, \ldots, \psi_n) \in BC$, and consider the solutions $x(t) = x(t, t_0, \varphi)$ and $y(t) = x(t, t_0, \psi)$ of (2.2) defined on $\mathbb{R}$. Define, for $t \geq t_0$, $V(t) = V(t, t_0, \varphi, \psi) = (V_1(t), \ldots, V_n(t))$ by

$$V_i(t) := \mathrm{e}^{\int_{t_0}^{t} \lambda(u) du} \operatorname{sign}(x_i(t) - y_i(t)) \int_{y_i(t)}^{x_i(t)} \frac{1}{a_i(t, u)} du, \qquad i = 1, \ldots, n. \tag{3.5}$$

First we prove that

$$|V(t)| \leq \max_j \left\{ \underline{a}_j^{-1} \right\} \|\varphi - \psi\|, \qquad \forall t \geq t_0. \tag{3.6}$$

Clearly, from (A1) and (3.5), we have

$$V_i(t_0) \leq \underline{a}_i^{-1} |x_i(t_0) - y_i(t_0)| \leq \max_j \left\{ \underline{a}_j^{-1} \right\} \|\varphi - \psi\|.$$

Now, to get a contradiction, we assume that the inequality (3.6) is false. Consequently, as $V(t)$ is a positive continuous vector function, there is $t_1 > t_0$ such that

$$|V(t_1)| > \max_j \left\{ \underline{a}_j^{-1} \right\} \|\varphi - \psi\|.$$

Defining

$$T := \min \left\{ t \in [t_0, t_1] : |V(t)| = \max_{s \in [t_0, t_1]} |V(s)| \right\}$$

and choosing $i \in \{1, \ldots, n\}$ such that $V_i(T) = |V(T)|$, we have $V_i(T) > 0$, $V_i'(T) \geq 0$ and $V_i(T) > |V(t)|$ for all $t < T$.

On the other hand, from (2.2), (A2), (A3), and (A4) we have

$$\begin{aligned}
V_i'(T) &= \lambda(T) V_i(T) + \mathrm{e}^{\int_{t_0}^{T} \lambda(u)\, du} \operatorname{sign}(x_i(T) - y_i(T)) \left(\frac{1}{a_i(T, x_i(T))} x_i'(T) - \frac{1}{a_i(T, y_i(T))} y_i'(T) + \right. \\
&\quad \left. + \int_{y_i(T)}^{x_i(T)} - \frac{\partial_t a_i(T, u)}{a_i^2(T, u)}\, du \right) \\
&= \lambda(T) V_i(T) + \mathrm{e}^{\int_{t_0}^{T} \lambda(u)\, du} \operatorname{sign}(x_i(T) - y_i(T)) \Bigg(b_i(T, y_i(T)) - b_i(T, x_i(T)) + \\
&\quad + \sum_{k=1}^{K} \sum_{j=1}^{n} \left(f_{ijk}(T, y_{j_T}) - f_{ijk}(T, x_{j_T}) \right) + \int_{y_i(T)}^{x_i(T)} - \frac{\partial_t a_i(T, u)}{a_i^2(T, u)}\, du \Bigg) \\
&\leq \lambda(T) V_i(T) + \mathrm{e}^{\int_{t_0}^{T} \lambda(u)\, du} \Bigg(- \beta_i(T) |x_i(T) - y_i(T)| + \sum_{k=1}^{K} \sum_{j=1}^{n} F_{ijk}(T) \|x_{j_T} - y_{j_T}\|_\varepsilon \\
&\quad - D_i(T) |x_i(T) - y_i(T)| \Bigg)
\end{aligned}$$

The definition (3.5) and the hypothesis (A1) imply that $\mathrm{e}^{\int_{t_0}^{T} \lambda(u)\, du} |x_i(T) - y_i(T)| \geq \underline{a}_i V_i(T)$ and the hypothesis (A5) implies that $\beta_i(T) + D_i(T) > 0$. Consequently

$$\begin{aligned} V_i'(T) &\leq \lambda(T)V_i(T) - \underline{a}_i(\beta_i(T) + D_i(T))V_i(T) + \mathrm{e}^{\int_{t_0}^{T} \lambda(u)\,du} \sum_{k=1}^{K} \sum_{j=1}^{n} F_{ijk}(T) \cdot \\ &\quad \cdot \max\left\{ \sup_{s \leq t_0 - T} |x_j(T+s) - y_j(T+s)|\, \mathrm{e}^{\varepsilon s}, \sup_{t_0 - T < s \leq 0} |x_j(T+s) - y_j(T+s)|\, \mathrm{e}^{\varepsilon s} \right\} \\ &\leq \lambda(T)V_i(T) - \underline{a}_i(\beta_i(T) + D_i(T))V_i(T) + \mathrm{e}^{\int_{t_0}^{T} \lambda(u)\,du} \sum_{k=1}^{K} \sum_{j=1}^{n} F_{ijk}(T) \cdot \\ &\quad \cdot \max\left\{ \|\varphi_j - \psi_j\|\, \mathrm{e}^{\varepsilon(t_0 - T)}, \sup_{t_0 - T < s \leq 0} |x_j(T+s) - y_j(T+s)|\, \mathrm{e}^{\varepsilon s} \right\}. \end{aligned} \tag{3.7}$$

Again from (3.5) and (A1), we have

$$|x_i(T+s) - y_i(T+s)| \leq \mathrm{e}^{-\int_{t_0}^{T+s} \lambda(u)\,du}\, \overline{a}_i V_i(T+s)$$

for all $i \in \{1, \ldots, n\}$ and $s \in [t_0 - T, 0]$, which implies that

$$\begin{aligned} V_i'(T) &\leq \lambda(T)V_i(T) - \underline{a}_i(\beta_i(T) + D_i(T))V_i(T) + \mathrm{e}^{\int_{t_0}^{T} \lambda(u)\,du} \sum_{k=1}^{K} \sum_{j=1}^{n} F_{ijk}(T) \cdot \\ &\quad \cdot \max\left\{ \|\varphi_j - \psi_j\|\, \mathrm{e}^{\varepsilon(t_0 - T)}, \sup_{t_0 - T < s \leq 0} \mathrm{e}^{-\int_{t_0}^{T+s} \lambda(u)\,du + \varepsilon s}\, \overline{a}_j V_j(T+s) \right\}. \end{aligned}$$

Since (A5) holds, $V_i(T) > |V(t)|$ for all $t < T$, and $V_i(T) > \max_j \left\{ \overline{a}_j^{-1} \right\} \|\varphi - \psi\|$, we conclude that

$$\begin{aligned} V_i'(T) &\leq \lambda(T)V_i(T) - \underline{a}_i(\beta_i(T) + D_i(T))V_i(T) + \mathrm{e}^{\int_{t_0}^{T} \lambda(u) - \varepsilon\,du} \sum_{k=1}^{K} \sum_{j=1}^{n} F_{ijk}(T) \cdot \\ &\quad \cdot \overline{a}_j \max\left\{ \frac{\|\varphi_j - \psi_j\|}{\overline{a}_j}, \sup_{t_0 - T < s \leq 0} \frac{\mathrm{e}^{-\int_{t_0}^{T+s} \lambda(u)\,du + \varepsilon s}}{\mathrm{e}^{\varepsilon(t_0 - T)}} V_j(T+s) \right\} \\ &\leq \lambda(T)V_i(T) - \underline{a}_i(\beta_i(T) + D_i(T))V_i(T) + \mathrm{e}^{\int_{t_0}^{T} \lambda(u) - \varepsilon\,du} \sum_{k=1}^{K} \sum_{j=1}^{n} F_{ijk}(T) \cdot \\ &\quad \cdot \overline{a}_j \max\left\{ V_i(T), \sup_{t_0 - T < s \leq 0} \mathrm{e}^{-\int_{t_0}^{T+s} \lambda(u) - \varepsilon\,du}\, V_j(T+s) \right\} \\ &\leq \left(\lambda(T) - \underline{a}_i(\beta_i(T) + D_i(T)) + \sum_{k=1}^{K} \sum_{j=1}^{n} \overline{a}_j\, \mathrm{e}^{\int_{t_0}^{T} \lambda(u) - \varepsilon\,du} F_{ijk}(T) \right) V_i(T) < 0, \end{aligned}$$

which is a contradiction and (3.6) holds.

From (A1) we obtain, for all $t \geq t_0$,

$$|V(t)| \geq \mathrm{e}^{\int_{t_0}^{t} \lambda(u)\,du} |x_i(t) - y_i(t)| \min_j \left\{ \overline{a}_j^{-1} \right\}, \qquad \forall i \in \{1, \ldots, n\},$$

and from (3.6) we have

$$|x(t) - y(t)|\, \mathrm{e}^{\int_{t_0}^{t} \lambda(u)\,du} \min_j \left\{ \overline{a}_j^{-1} \right\} \leq |V(t)| \leq \max_j \left\{ \underline{a}_j^{-1} \right\} \|\varphi - \psi\|.$$

Consequently, by (A5),

$$|x(t) - y(t)| \le \frac{\max_j \{\underline{a}_j^{-1}\}}{\min_j \{\overline{a}_j^{-1}\}} \mathrm{e}^{-\int_{t_0}^t \lambda(u)\,du} \|\varphi - \psi\| \le C\,\mathrm{e}^{-\varepsilon(t-t_0)} \|\varphi - \psi\|, \qquad \forall t \ge t_0,$$

with C defined by (3.4), which means that (2.2) is globally exponentially stable. □

The next result shows that the same conclusion can be obtained if we assume

(A5$_d$) there exist a continuous function $\lambda : [0, +\infty) \longrightarrow (0, +\infty)$ and $d = (d_1, \dots, d_n) > 0$ such that, for each $i \in \{1, \dots, n\}$ and $t_0 \ge 0$,

$$\underline{a}_i\big(\beta_i(t) + D_i(t)\big) - \sum_{k=1}^{K}\sum_{j=1}^{n} \overline{a}_j d_i^{-1} d_j F_{ijk}(t)\,\mathrm{e}^{\int_{t_0}^t \lambda(u)-\varepsilon\,du} > \lambda(t) \ \text{ and } \ \int_{t_0}^t [\lambda(u) - \varepsilon]\,du \ge 0,$$

for all $t \ge t_0$.

instead of (A5). We remark that (A5$_d$) is slightly weaker than (A5).

Corollary 3.3. *Assume (A1)-(A4) and (A5$_d$). Then the model* (2.2) *is globally exponentially stable.*

Proof. With the change $y_i(t) = d_i^{-1} x_i(t)$, the system (2.2) is transformed into

$$y_i'(t) = -a_i(t, d_i y_i(t)) d_i^{-1} \left[b_i(t, d_i y_i(t)) + \sum_{k=1}^{K}\sum_{j=1}^{n} f_{ijk}(t, d_j y_{j_t}) \right], \qquad t \ge 0,\, i = 1, \dots, n. \tag{3.8}$$

Defining, for each $i, j = 1, \dots, n$ and $k = 1, \dots, K$, $\tilde{a}_i(t, u) := a_i(t, d_i u)$, $\tilde{b}_i(t, u) := d_i^{-1} b_i(t, d_i u)$, and $\tilde{f}_{ijk}(t, \varphi) := d_i^{-1} f_{ijk}(t, d_j \varphi)$, for all $t \ge 0$, $u \in \mathbb{R}$, $\varphi \in UC_\varepsilon^1$, the model (3.8) has the form

$$y_i'(t) = -\tilde{a}_i(t, y_i(t)) \left[\tilde{b}_i(t, y_i(t)) + \sum_{k=1}^{K}\sum_{i=1}^{n} \tilde{f}_{ijk}(t, y_{j_t}) \right], \qquad t \ge 0,\, i = 1, \dots, n \tag{3.9}$$

and the hypotheses (A1), (A2), and (A3) hold with $\underline{\tilde{a}}_i = \underline{a}_i$, $\overline{\tilde{a}}_i = \overline{a}_i$, $\tilde{D}_i(t) = D_i(t)$, and $\tilde{\beta}_i(t) = \beta_i(t)$. From (A4), for $\varphi, \psi \in UC_\varepsilon^1$ we have

$$|\tilde{f}_{ijk}(t, \varphi) - \tilde{f}_{ijk}(t, \psi)| = d_i^{-1} |f_{ijk}(t, d_j \varphi) - f_{ijk}(d_j \psi)| \le \frac{d_j}{d_i} F_{ijk}(t) \|\varphi - \psi\|_\varepsilon,$$

which implies that $\tilde{f}_{ijk}$ are Lipschitz function on the second variable with $\tilde{F}_{ijk}(t) := d_i^{-1} d_j F_{ijk}(t)$, for all $i, j \in \{1, \dots, n\}$ and $k \in \{1, \dots, K\}$, which means that hypothesis (A4) is satisfied. Finally, from (A5$_d$) the hypothesis (A5) holds for (3.9) and the conclusion follows from Theorem 3.2. □

Using the same ideas presented in the proof of Theorem 3.2 and Corollary 3.3, we also obtain the next result.

Theorem 3.4. *Assume (A1), (A2), (A3), (A4') and*

(A5') *there exist a continuous function $\lambda : [0, +\infty) \longrightarrow (0, +\infty)$ and $d = (d_1, \dots, d_n) > 0$ such that, for each $i \in \{1, \dots, n\}$ and $t_0 \ge 0$,*

$$\underline{a}_i\big(\beta_i(t) + D_i(t)\big) - \sum_{k=1}^{K}\sum_{j=1}^{n} \overline{a}_j d_i^{-1} d_j F_{ijk}(t)\,\mathrm{e}^{\int_{t_0}^t \lambda(u)\,du} > \lambda(t) \ \text{ and } \ \int_{t_0}^t [\lambda(u) - \varepsilon]\,du \ge 0,$$

for all $t \ge t_0$.

Then the model (2.2) *is globally exponentially stable.*

Considering in (2.2) $a_i(t,u) = 1$ for all $i = 1, \dots, n$, $t \geq 0$, and $u \in \mathbb{R}$, we get the following generalized Hopfield neural network model

$$x_i'(t) = -b_i(t, x_i(t)) + \sum_{k=1}^{K} \sum_{j=1}^{n} f_{ijk}(t, x_{j_t}), \qquad t \geq 0, \ i = 1, \dots, n, \tag{3.10}$$

for which its global exponential stability was studied in [9]. Trivially, hypotheses (A1) and (A2) hold with $\underline{a}_i = \overline{a}_i = 1$ and $D_i(t) = 0$ respectively, and we have the next result.

Corollary 3.5. *Assume (A3) and (A4').*

If there exist a continuous function $\lambda : [0, +\infty) \longrightarrow (0, +\infty)$ and $d = (d_1, \dots, d_n) > 0$ such that, for each $i \in \{1, \dots, n\}$ and $t_0 \geq 0$,

$$\beta_i(t) - \sum_{k=1}^{K} \sum_{j=1}^{n} d_i^{-1} d_j F_{ijk}(t) \, e^{\int_{t_0}^{t} \lambda(u)\, du} > \lambda(t) \quad \text{and} \quad \int_{t_0}^{t} [\lambda(u) - \varepsilon] \, du \geq 0, \qquad \forall t \geq t_0,$$

then the model (3.10) *is globally exponentially stable.*

Remark 3.2. We note that Corollary 3.5 improves the exponential stability criterion in [9], because here we have a model with unbounded delays.

4 Neural network models

In this section, we shall apply the stability criteria in section 3 to the following generalized nonautonomous Cohen-Grossberg neural network model with unbounded delays

$$\begin{aligned} x_i'(t) = -a_i(t, x_i(t)) \Bigg[b_i(t, x_i(t)) + \sum_{k=1}^{K} \sum_{j=1}^{n} \Bigg(p_{ijk}(t) h_{ijk}(x_j(t - \tau_{ijk}(t)) \\ + q_{ijk}(t) l_{ijk} \left(\int_{-\infty}^{0} g_{ijk}(x_j(t+s)) \, d\eta_{ijk}(s) \right) \Bigg) + I_i(t) \Bigg], \ t \geq 0, \, i = 1, \dots, n, \end{aligned} \tag{4.1}$$

where $a_i : [0, +\infty) \times \mathbb{R} \to (0, +\infty)$, $b_i : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$, $p_{ijk}, q_{ijk}, I_i : [0, +\infty) \to \mathbb{R}$, $\tau_{ijk} : [0, +\infty) \to [0, +\infty)$, and $h_{ijk}, l_{ijk}, g_{ijk} : \mathbb{R} \to \mathbb{R}$ are continuous functions, and $\eta_{ijk} : (-\infty, 0] \to \mathbb{R}$ are non-decreasing, bounded, and normalized i.e. $\eta_{ijk}(0) - \eta_{ijk}(-\infty) = 1$, for all $i, j \in \{1, \dots, n\}$, $k \in \{1, \dots, K\}$. Assume that there exists $\xi > 0$ such that

$$\int_{-\infty}^{0} e^{-\xi s} \, d\eta_{ijk}(s) < \infty, \qquad i, j = 1, \dots, n, \ k = 1, \dots, K. \tag{4.2}$$

As we are going to illustrate with some examples in this section, the model (4.1) is general enough to include several types of neural network models present in the literature. However, we should remark that the model (4.1) is not a neutral-type model and it does not have leakage delays. Thus, our following stability criterion can not be applied to the neural network models introduced in the recent research papers [1, 2].

Theorem 4.1. *Consider* (4.1)*, where a_i, b_i, p_{ijk}, q_{ijk}, and I_i are continuous functions such that (A1), (A2), and (A3) hold, h_{ijk}, l_{ijk}, and g_{ijk} are Lipschitz functions with Lipschitz constants H_{ijk}, L_{ijk}, and G_{ijk} respectively, and η_{ijk} are non-decreasing, bounded, and normalized functions such that* (4.2) *holds for some $\xi > 0$.*

If there are $d = (d_1, \dots, d_n) > 0$, $\alpha > 0$ and $\theta_{ijk} \geq 0$ such that, for each $i \in \{1, \dots, n\}$ and $t_0 \geq 0$,

$$\underline{a}_i \big(\beta_i(t) + D_i(t) \big) - \sum_{k=1}^{K} \sum_{j=1}^{n} \overline{a}_j \frac{d_j}{d_i} \Big(|p_{ijk}(t)| H_{ijk} \, e^{\theta_{ijk} \tau_{ijk}(t)} + |q_{ijk}(t)| L_{ijk} G_{ijk} \Big) > \alpha, \qquad \forall t \geq t_0, \tag{4.3}$$

and

$$\theta_{ijk} > 0 \text{ when } \tau_{ijk}(t) \text{ is an unbounded function,}$$

then the model (4.1) *is globally exponentially stable.*

Proof. For simplicity, consider $t_0 = 0$ and, as in the proof of Corollary 3.3, after a change of variable, we may assume that $d_i = 1$ for all $i \in \{1, \ldots, n\}$.

From (4.3), it is easy to conclude that there is $\nu > 0$ such that, for all $i = 1, \ldots, n$ and $t \geq t_0$,

$$\underline{a}_i\big(\beta_i(t) + D_i(t)\big) - \sum_{k=1}^{K}\sum_{j=1}^{n} \overline{a}_j \left(|p_{ijk}(t)|H_{ijk}\nu_{ijk}(t) + |q_{ijk}(t)|L_{ijk}G_{ijk}(1+\nu)\right) > \nu, \ \ \forall t \geq 0, \quad (4.4)$$

where

$$\nu_{ijk}(t) = \begin{cases} 1+\nu, & \text{if } \theta_{ijk} = 0 \\ \mathrm{e}^{\theta_{ijk}\tau_{ijk}(t)}, & \text{if } \theta_{ijk} > 0 \end{cases}. \quad (4.5)$$

Define the positive numbers

$$\tau := \max_{i,j,k}\left(\sup_{t\geq 0}\left\{\tau_{ijk}(t) : \tau_{ijk} \text{ is a bounded function}\right\}\right)$$

and $\underline{\theta} := \min_{i,j,k}\{\theta_{ijk} : \theta_{ijk} > 0\}$. As in the proof of [11, Theorem 4.3], from (4.2), we conclude that there is $\varsigma \in (0, \xi)$ such that

$$\int_{-\infty}^{0} e^{-\varsigma s} d\eta_{ijk}(s) < 1+\nu, \qquad i,j = 1, \ldots, n, \ k = 1, \ldots, K. \quad (4.6)$$

Let $\varepsilon := \min\left\{\underline{\theta}, \nu, \varsigma, \dfrac{\log(1+\nu)}{\tau}\right\}$ and consider system (4.1) in the phase space UC_ε^n. We remark that model (4.1) has the form (2.2) with

$$f_{ijk}(t,\varphi) = p_{ijk}(t)h_{ijk}(\varphi(-\tau_{ijk}(t))) + q_{ijk}(t)l_{ijk}\left(\int_{-\infty}^{0} g_{ijk}(\varphi(s))\, d\eta_{ijk}(s)\right) + \frac{I_i(t)}{nK},$$

for all $t \geq 0$, $\varphi \in UC_\varepsilon^1$, $i,j \in \{1, \ldots, n\}$, $k \in \{1, \ldots, K\}$.

As we shall see below, $\varepsilon \leq \nu$ and $\varepsilon \leq \varsigma$ are needed to deal with the distributed delays, and $\varepsilon \leq \underline{\theta}$ and $\varepsilon \leq \frac{\log(1+\nu)}{\tau}$ are needed to deal with the discrete delays in the model (4.1). We claim that $\mathrm{e}^{\varepsilon\tau_{ijk}(t)} \leq \nu_{ijk}(t)$ for all $t \geq 0$, $i,j = 1, \ldots, n$, and $k = 1, \ldots, K$. In fact, if $\theta_{ijk} > 0$ ($\tau_{ijk}(t)$ unbounded $\Longrightarrow \theta_{ijk} > 0$), then we have $\theta_{ijk} \geq \underline{\theta} \geq \varepsilon > 0$ and consequently

$$\mathrm{e}^{\varepsilon\tau_{ijk}(t)} \leq \mathrm{e}^{\theta_{ijk}\tau_{ijk}(t)} = \nu_{ijk}(t),$$

where $\nu_{ijk}(t)$ is defined by (4.5). If $\theta_{ijk} = 0$, then $\tau_{ijk}(t)$ is a bounded function, thus $\tau_{ijk}(t) \leq \tau$ and

$$\mathrm{e}^{\varepsilon\tau_{ijk}(t)} \leq \mathrm{e}^{\varepsilon\tau} \leq \mathrm{e}^{\frac{\log(1+\nu)}{\tau}\tau} = \mathrm{e}^{\log(1+\nu)} = 1+\nu = \nu_{ijk}(t).$$

For $\varphi, \psi \in UC_\varepsilon^1$ and $t \geq 0$, since h_{ijk}, l_{ijk}, and g_{ijk} are Lipschitz functions and η_{ijk} are non-decreasing, we have

$$
\begin{aligned}
|f_{ijk}(t,\varphi)-f_{ijk}(t,\psi)| &\le \big|p_{ijk}(t)\big|\cdot\big|h_{ijk}(\varphi(-\tau_{ijk}(t)))-h_{ijk}(\psi(-\tau_{ijk}(t)))\big|+\\
&+\big|q_{ijk}(t)\big|\cdot\left|l_{ijk}\bigg(\int_{-\infty}^{0} g_{ijk}(\varphi(s))\,d\eta_{ijk}(s)\bigg)-l_{ijk}\bigg(\int_{-\infty}^{0} g_{ijk}(\psi(s))\,d\eta_{ijk}(s)\bigg)\right|\\
&\le \big|p_{ijk}(t)\big|H_{ijk}\big|\varphi(-\tau_{ijk}(t))-\psi(-\tau_{ijk}(t))\big|+\\
&+\big|q_{ijk}(t)\big|L_{ijk}\left|\int_{-\infty}^{0}\Big[g_{ijk}(\varphi(s))-g_{ijk}(\psi(s))\Big]\,d\eta_{ijk}(s)\right|\\
&\le \big|p_{ijk}(t)\big|H_{ijk}\frac{\big|(\varphi-\psi)(-\tau_{ijk}(t))\big|}{\mathrm{e}^{\varepsilon\tau_{ijk}(t)}}\,\mathrm{e}^{\varepsilon\tau_{ijk}(t)}+\\
&+\big|q_{ijk}(t)\big|L_{ijk}G_{ijk}\int_{-\infty}^{0}\frac{\big|(\varphi-\psi)(s)\big|}{\mathrm{e}^{-\varepsilon s}}\,\mathrm{e}^{-\varepsilon s}\,d\eta_{ijk}(s)\\
&\le \big|p_{ijk}(t)\big|H_{ijk}\|\varphi-\psi\|_\varepsilon\,\mathrm{e}^{\varepsilon\tau_{ijk}(t)}+\big|q_{ijk}(t)\big|L_{ijk}G_{ijk}\int_{-\infty}^{0}\|\varphi-\psi\|_\varepsilon\,\mathrm{e}^{-\varepsilon s}\,d\eta_{ijk}(s)\\
&\le \bigg(\big|p_{ijk}(t)\big|H_{ijk}\,\mathrm{e}^{\varepsilon\tau_{ijk}(t)}+\big|q_{ijk}(t)\big|L_{ijk}G_{ijk}\int_{-\infty}^{0}\mathrm{e}^{-\varsigma s}\,d\eta_{ijk}(s)\bigg)\|\varphi-\psi\|_\varepsilon\\
&\le \bigg(\big|p_{ijk}(t)\big|H_{ijk}\nu_{ijk}(t)+\big|q_{ijk}(t)\big|L_{ijk}G_{ijk}(1+\nu)\bigg)\|\varphi-\psi\|_\varepsilon.
\end{aligned}
$$

This means that, for each $i,j=1,\dots,n$, and $k=1,\dots,K$,

$$|f_{ijk}(t,\varphi)-f_{ijk}(t,\psi)|\le F_{ijk}(t)\|\varphi-\psi\|_\varepsilon,\qquad \forall t\ge 0,\ \forall\varphi,\psi\in UC^1_\varepsilon,$$

with $F_{ijk}(t):=\big|p_{ijk}(t)\big|H_{ijk}\nu_{ijk}(t)+\big|q_{ijk}(t)\big|L_{ijk}G_{ijk}(1+\nu)$, and from (4.4) we conclude that (A5) holds with $\lambda(t)=\varepsilon$. Now, the conclusion follows from Theorem 3.2. □

Example 4.1. If we take $K=2$, $a_i(t,u)=1$, $b_i(t,u)=b_i(t)u$, $p_{ij1}(t)=c_{ij}(t)$, $p_{ij2}(t)=d_{ij}(t)$, $h_{ij1}(u)=h_{ij2}(u)=g_{ij1}(u)=-f_j(u)$, $\tau_{ij1}(t)=0$, $\tau_{ij2}(t)=\tau_{ij}(t)$, $q_{ij1}(t)=e_{ij}(t)$, $q_{ij2}(t)=0$, $l_{ij1}(u)=u$, with $b_i:[0,+\infty)\to(0,+\infty)$, $c_{ij},d_{ij},e_{ij}:[0,+\infty)\to\mathbb{R}$, $\tau_{ij}:[0,+\infty)\to[0,+\infty)$, $f_j:\mathbb{R}\to\mathbb{R}$ continuous functions, and

$$\eta_{ij1}(s)=\int_{-\infty}^{s}k_{ij}(-v)\,dv,\qquad s\in(-\infty,0],\ i,j=1,\dots,n,$$

where $k_{ij}:[0,+\infty)\longrightarrow[0,+\infty)$ are piecewise continuous functions, then the model (4.1) becomes the following Hopfield neural network model

$$
\begin{aligned}
x_i'(t)=-b_i(t)x_i(t)+\sum_{j=1}^{n}c_{ij}(t)f_j(x_j(t))+\sum_{j=1}^{n}d_{ij}(t)f_j(x_j(t-\tau_{ij}(t)))\\
+\sum_{j=1}^{n}e_{ij}(t)\int_{-\infty}^{0}k_{ij}(-s)f_j(x_j(t+s))\,ds+I_i(t),
\end{aligned}
\tag{4.7}
$$

for $t\ge 0$ and $i=1,\dots,n$. Applying Theorem 4.1 to model (4.7), we have the following result.

Corollary 4.2. *Consider* (4.7)*, where* $b_i : [0, +\infty) \longrightarrow (0, +\infty)$*,* $c_{ij}, d_{ij}, e_{ij}, I_i : [0, +\infty) \longrightarrow \mathbb{R}$*, and* $\tau_{ij} : [0, +\infty) \to [0, +\infty)$ *are continuous,* $f_j : \mathbb{R} \longrightarrow \mathbb{R}$ *are Lipschitz functions with Lipschitz constants* F_j*, and* $k_{ij} : [0, +\infty) \to [0, +\infty)$ *are piecewise continuous functions such that*

$$\int_0^{+\infty} k_{ij}(t)\, dt = 1, \quad \text{and} \quad \int_0^{+\infty} k_{ij}(t)\, \mathrm{e}^{\xi t}\, dt < +\infty, \quad i, j = 1, \ldots, n, \tag{4.8}$$

for some $\xi > 0$.

If there are $d = (d_1, \ldots, d_n) > 0$*,* $\alpha > 0$*, and* $\theta_{ij} \geq 0$ *such that, for each* $i \in \{1, \ldots, n\}$ *and* $t_0 \geq 0$*,*

$$b_i(t) - \sum_{j=1}^{n} \frac{d_j}{d_i} F_j \left(|c_{ij}(t)| + |d_{ij}(t)|\, \mathrm{e}^{\theta_{ij}\tau_{ij}(t)} + |e_{ij}(t)| \right) > \alpha, \quad \forall t \geq t_0, \tag{4.9}$$

and

$$\theta_{ij} > 0 \text{ when } \tau_{ij}(t) \text{ is an unbounded function,}$$

then the model (4.7) *is globally exponentially stable.*

Remark 4.1. The exponential stability of (4.7) was recently studied in [28], with the p-norm in $\mathbb{R}^n$, $p \geq 1$. (The case $p = \infty$ was not treated in [28].) The Corollary 4.2 gives a new stability criterion with the ∞-norm, which complements the result in [28]. Moreover, it is relevant to observe that Q. Zhang et al. [28] assumed in addition that, for each $i, j = 1, \ldots, n$, $b_i(t)$, $c_{ij}(t)$, $d_{ij}(t)$, $e_{ij}(t)$ $I_i(t)$, and $\tau_{ij}(t)$ are bounded functions and $\tau_{ij}(t)$ is differentiable satisfying $\sup_{t \geq 0} \tau'_{ij}(t) < 1$.

Example 4.2. In [19], the following Cohen-Grossberg neural network model was considered

$$x'_i(t) = -a_i(t, x_i(t)) \left[b_i(t, x_i(t)) + \sum_{j=1}^{n} c_{ij}(t) f_j(x_j(t - \tau_{ij}(t))) + \sum_{j=1}^{n} d_{ij}(t) \int_{-\infty}^{0} k_{ij}(-s) g_j(x_j(t+s))\, ds + I_i(t) \right], \tag{4.10}$$

for $t \geq 0$ and $i = 1, \ldots, n$, where $a_i, b_i : [0, +\infty) \times \mathbb{R} \longrightarrow (0, +\infty)$, $c_{ij}, d_{ij}, I_i : [0, +\infty) \longrightarrow \mathbb{R}$ are continuous functions and $f_j, g_j : \mathbb{R} \longrightarrow \mathbb{R}$ are Lipschitz functions with Lipschitz constants F_j, G_j, respectively, and $k_{ij} : [0, +\infty) \longrightarrow [0, +\infty)$ are piecewise continuous functions such that (4.8) holds, $i, j = 1, \ldots, n$.

Clearly, model (4.10) is also a particular case of (4.1) and from Theorem 4.1 we obtain the next result.

Corollary 4.3. *Consider* (4.10) *under the hypotheses above and (A1), (A2), and (A3) hold.*

If there are $d = (d_1, \ldots, d_n) > 0$*,* $\alpha > 0$*, and* $\theta_{ij} \geq 0$ *such that, for each* $i \in \{1, \ldots, n\}$ *and* $t_0 \geq 0$*,*

$$\underline{a}_i\big(\beta_i(t) + D_i(t)\big) - \sum_{j=1}^{n} \overline{a}_j \frac{d_j}{d_i} \left(|c_{ij}(t)| F_j\, \mathrm{e}^{\theta_{ij}\tau_{ij}(t)} + |d_{ij}(t)| G_j \right) > \alpha, \ \forall t \geq t_0, \tag{4.11}$$

and

$$\theta_{ij} > 0 \text{ when } \tau_{ij}(t) \text{ is an unbounded function,}$$

then the model (4.10) *is globally exponentially stable.*

Remark 4.2. In [19], B. Liu assumed a different set of hypotheses to prove that all solutions of (4.10) converge exponentially to zero. However, we remark that, in [19], the author assumed that coefficient functions $c_{ij}(t)$ and $d_{ij}(t)$ are bounded.

Example 4.3. In [29], the following Cohen-Grossberg neural network model was considered

$$x_i'(t) = -a_i(x_i(t)) \left[b_i(t, x_i(t)) + \sum_{j=1}^{n} c_{ij}(t) f_j(x_j(t)) + \right.$$

$$\left. + \sum_{j=1}^{n} d_{ij}(t) f_j(x_j(t - \tau_{ij}(t))) + I_i(t) \right], \ t \geq 0, \tag{4.12}$$

$i = 1, \dots, n$, where $a_i : \mathbb{R} \longrightarrow (0, +\infty)$, $b_i : [0, +\infty) \times \mathbb{R} \longrightarrow (0, +\infty)$, $c_{ij}, d_{ij}, I_i : [0, +\infty) \longrightarrow \mathbb{R}$ are continuous functions and $f_j : \mathbb{R} \longrightarrow \mathbb{R}$ are Lipschitz functions with Lipschitz constants F_j, $i, j = 1, \dots, n$.

Clearly, model (4.12) is still a particular case of (4.1) and from Theorem 4.1 we obtain the following stability criterion.

Corollary 4.4. *Consider* (4.12) *under the hypotheses above and (A1) and (A3) hold.*

If there are $d = (d_1, \dots, d_n) > 0$, $\alpha > 0$, and $\theta_{ij} \geq 0$ such that, for each $i \in \{1, \dots, n\}$ and $t_0 \geq 0$,

$$\underline{a}_i \beta_i(t) - \sum_{j=1}^{n} \overline{a}_j F_j \frac{d_j}{d_i} \left(|c_{ij}(t)| + |d_{ij}(t)| \, \mathrm{e}^{\theta_{ij} \tau_{ij}(t)} \right) > \alpha, \qquad \forall t \geq t_0, \tag{4.13}$$

and

$$\theta_{ij} > 0 \text{ when } \tau_{ij}(t) \text{ is an unbounded function,}$$

then the model (4.12) *is globally exponentially stable.*

Remark 4.3. For the particular model (4.12), W. Zhao [29] obtained its global exponential stability assuming the conditions above and the following additional hypotheses:

1. The functions $a_i(u)$ are locally Lipschitz;
2. For each $i, j = 1, \dots, n$, $b_i(t, 0)$, $c_{ij}(t)$, $d_{ij}(t)$, and $\tau_{ij}(t)$ are bounded functions.

Hence, it is clear that our Corollary 4.4 strongly improves the main result in [29].

Example 4.4. Consider the following BAM neural network model

$$\begin{cases} x_i'(t) = -\widetilde{c}_i(t) \widetilde{b}_i(x_i(t)) + \displaystyle\sum_{j=1}^{m} a_{ij}(t) f_j(y_j(t)) \\ \qquad + \displaystyle\sum_{j=1}^{m} e_{ij}(t) \int_{-\infty}^{0} k_{ij}(-s) h_j(y_j(t - \tau_{ij} + s)) ds + \widetilde{I}_i(t), \ i = 1, \dots, k, \\ y_j'(t) = -c_j(t) b_j(y_j(t)) + \displaystyle\sum_{i=1}^{k} \widetilde{a}_{ji}(t) \widetilde{f}_i(x_i(t)) \\ \qquad + \displaystyle\sum_{i=1}^{k} \widetilde{e}_{ji}(t) \int_{-\infty}^{0} \widetilde{k}_{ji}(-s) \widetilde{h}_i(x_i(t - \sigma_{ji} + s)) ds + I_j(t), \ j = 1, \dots, m, \end{cases} \tag{4.14}$$

where $k, m \in \mathbb{N}$, $\tau_{ij}, \sigma_{ji} \in [0, +\infty)$, $a_{ij}, \widetilde{a}_{ji}, e_{ij}, \widetilde{e}_{ji}, \widetilde{I}_i, I_j : [0, +\infty) \longrightarrow \mathbb{R}$, $\widetilde{c}_i, c_j : [0, +\infty) \longrightarrow (0, +\infty)$, and $\widetilde{b}_i, b_j, f_j, \widetilde{f}_i, h_j, \widetilde{h}_i : \mathbb{R} \to \mathbb{R}$ are continuous functions, and $k_{ij}, \widetilde{k}_{ji} : [0, +\infty) \longrightarrow [0, +\infty)$ are piecewise continuous functions such that

$$\int_0^{+\infty} k_{ij}(t) \, dt = 1, \qquad \text{and} \qquad \int_0^{+\infty} k_{ij}(t) \, \mathrm{e}^{\xi t} \, dt < +\infty, \tag{4.15}$$

$$\int_0^{+\infty} \widetilde{k}_{ji}(t) \, dt = 1, \qquad \text{and} \qquad \int_0^{+\infty} \widetilde{k}_{ji}(t) \, \mathrm{e}^{\xi t} \, dt < +\infty, \tag{4.16}$$

for some $\xi > 0$, $i = 1, \ldots, k$, $j = 1, \ldots, m$. The model (4.14) was previously studied in [9, 25] and, as we can see in [9], it is also a particular case of (4.1). Consequently, from Theorem 4.1, we have the following result.

Corollary 4.5. *Consider* (4.14) *where $f_j, \widetilde{f}_i, h_j, \widetilde{h}_i$ are Lipschitz functions with Lipschitz constant, $F_j, \widetilde{F}_i, H_j, \widetilde{H}_i$ respectively, $k_{ij}, \widetilde{k}_{ji}$ are piecewise continuous functions such that* (4.15) *and* (4.16) *hold, and there exist positive numbers $\beta_j, \widetilde{\beta}_i$ such that*

$$(\widetilde{b}_i(u) - \widetilde{b}_i(v))/(u - v) \geq \widetilde{\beta}_i,$$

and

$$(b_j(u) - b_j(v))/(u - v) \geq \beta_j,$$

for all $u, v \in \mathbb{R}$, $u \neq v$, $i = 1, \ldots, k$, $j = 1, \ldots, m$.

If there exist $\widetilde{d} = (\widetilde{d}_1, \ldots, \widetilde{d}_k) > 0$, $d = (d_1, \ldots, d_m) > 0$, and $\alpha > 0$ such that, for all $i = 1, \ldots, k$, $j = 1, \ldots, m$, and $t \geq t_0$,

$$\widetilde{d}_i\left(\widetilde{\beta}_i \widetilde{c}_i(t) - \alpha\right) - \sum_{j=1}^{m} d_j\left(|a_{ij}(t)|F_j + |e_{ij}(t)|H_j\right) > 0, \ \ \forall t \geq t_0, \tag{4.17}$$

$$d_j\left(\beta_j c_j(t) - \alpha\right) - \sum_{i=1}^{k} \widetilde{d}_i\left(|\widetilde{a}_{ji}(t)|\widetilde{F}_i + |\widetilde{e}_{ji}(t)|\widetilde{H}_i\right) > 0, \ \ \forall t \geq t_0, \tag{4.18}$$

then system (4.14) *is globally exponentially stable.*

Remark 4.4. System (4.14) was studied in [25] assuming bounded coefficient functions $a_{ij}(t)$, $\widetilde{a}_{ji}(t)$, $\widetilde{c}_i(t)$, $c_j(t)$, $e_{ij}(t)$, $\widetilde{e}_{ji}(t)$, $\widetilde{I}_i(t)$, and $I_j(t)$. Here, it is possible to have unbounded coefficient functions and system (4.14) is only a particular case of (4.1), hence our Theorem 4.1 is more general than the main stability result in [25].

5 Numerical simulations

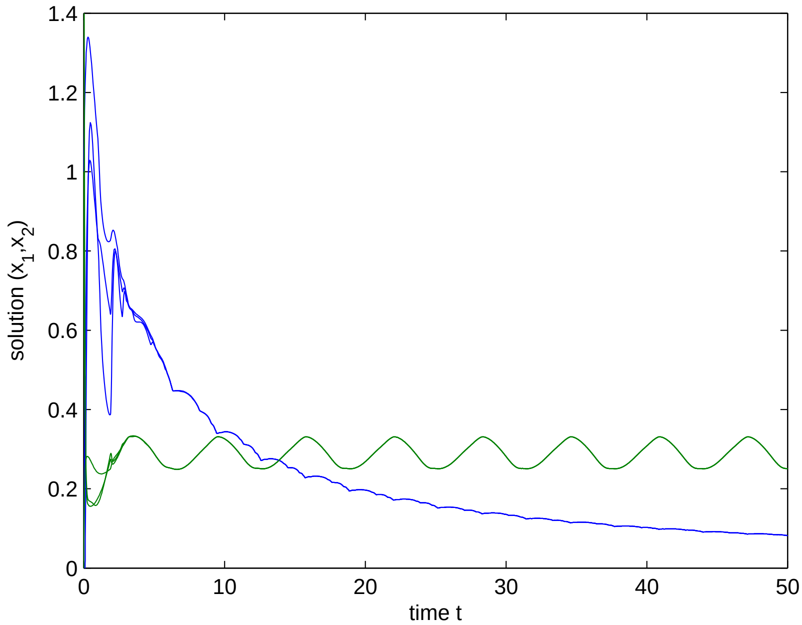
In this section, we give numerical examples to illustrate the effectiveness of the new results presented in this paper. We always use the Matlab software, [24], to plot the numerical simulations of solutions.

Example 5.1. The model

$$\begin{cases} x_1'(t) = -\min\left\{\sqrt{|x_1(t)|} + 1, 2\right\}\Big[(2t+3)x_1(t) - t\arctan(x_1(t - 2|\sin(t)|)) \\ \qquad\qquad\qquad\qquad\qquad\qquad - \tanh(x_2(t-1)) - 4\Big] \\ x_2'(t) = -\left(\cos(x_2(t)) + 2\right)\Big[7x_2(t) + \dfrac{\sin(t)}{t+1}\arctan\left(x_1(t - \log(t+1))\right) \\ \qquad\qquad\qquad\qquad\qquad\qquad + \cos(t)\tanh(x_2(t - 2|\sin(t)|)) - 2\Big] \end{cases} \tag{5.1}$$

is a particular situation of the system (4.12). The conditions of Corollary 4.4 are satisfied with $d = (1, 1)$, $\underline{a}_1 = 1$, $\overline{a}_1 = 2$, $\underline{a}_2 = 1$, $\overline{a}_2 = 3$, $\alpha = 1$, $\theta_{11} = \theta_{12} = \theta_{22} = 0$, and $\theta_{21} = 1$, hence the system is globally exponentially stable, which means that all solutions converge to each other exponentially (see the numerical simulation of three solutions of (5.1) in Figure 1).

Remark 5.1. We remark that the amplification function $a_1(u) = \min\left\{\sqrt{|u|}+1, 2\right\}$ is not locally Lipschitz and $d_{11}(t) = t$ and $\tau_{21} = \log(t+1)$ are unbounded functions, thus the stability result of W. Zhao [29] can not be applied to this example. Moreover, the delay function $\tau(t) = \tau_{11}(t) = \tau_{21}(t) = 2|\sin(t)|$ is not differentiable and it does not satisfy $\sup_{t\geq 0,\, t\neq k\pi} \tau'(t) < 1$ as required in [28].



(a) Solutions $(x_1(t), x_2(t))$ of equation (5.1) with initial condition $\varphi(s) = (e^s, \cos(s))$, $\varphi(s) = (0,0)$, and $\varphi(s) = (2\sin(s)-1, 2)$, for $s \leq 0$, respectively.

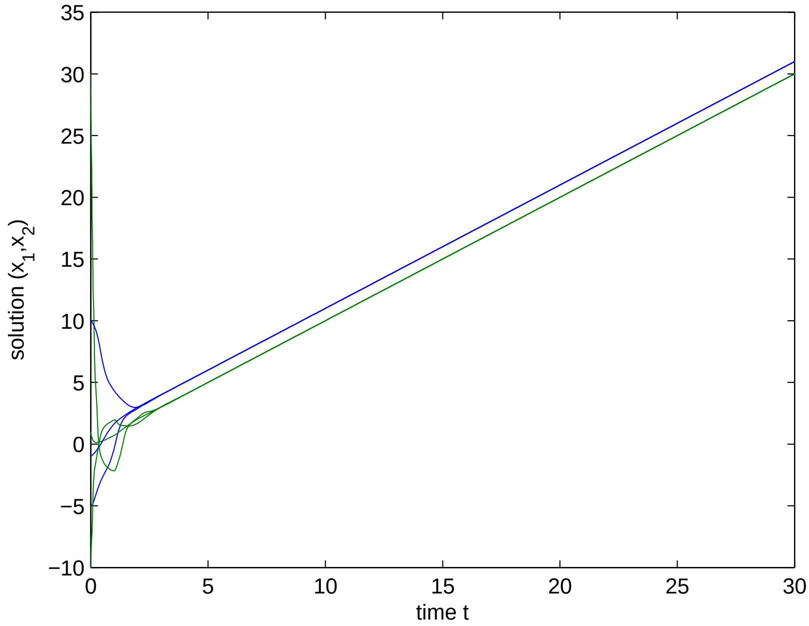
Figure 1: Behavior of three solutions of (5.1).

With the following last example, we illustrate that there exist models of the form (2.2), satisfying all hypotheses (A1)-(A5), with unbounded solutions.

Example 5.2. Finally, we consider the following example of a particular situation of model (4.12).

$$\begin{cases} x_1'(t) = -\big(\cos(x(t)-1)+2\big)\left[\dfrac{2t+1}{\cos t+2}x_1(t) - \dfrac{t}{(4+2\sin t)(\cos t+2)}\sin(x_1(t-1)) \right. \\ \qquad\qquad \left. + \dfrac{1}{12}x_2(t) - \dfrac{1}{\cos t+2}\left(\dfrac{t}{\sin t+2}+2t^2+\dfrac{5}{2}t+2\right) - \dfrac{1}{12}t\right] \\ x_2'(t) = -\left(\sin(x_2(t))+2\right)\left[\dfrac{9}{\sin t+2}x_2(t) - \dfrac{1}{\sin t+2}\sin(x_1(t-1)) + \dfrac{1}{\sin t+2}x_2(t-1) \right. \\ \qquad\qquad \left. + \dfrac{\sin t - 10t}{\sin t+2}\right] \end{cases} \tag{5.2}$$

The hypotheses of Corollary 4.4 are satisfied with $d = (1,1)$, $\underline{a}_1 = 1$, $\overline{a}_1 = 3$, $\underline{a}_2 = 1$, $\overline{a}_2 = 3$, $\theta_{11} = \theta_{21} = \theta_{22} = 0$, $F_1 = F_2 = 1$, and $\alpha = \frac{1}{13}$, hence the system is globally exponentially stable. It is easy to verify that $(x_1(t), x_2(t)) = (t+1, t)$ is a solution of (5.2), thus all solutions of (5.2) converge to it exponentially and they are unbounded (see the numerical simulation of three solutions of (5.2) in Figure 2).



(a) Solutions $(x_1(t), x_2(t))$ of equation (5.2) with initial condition $\varphi(s) = (-5\,\mathrm{e}^s, 30\cos s)$, $\varphi(s) = (10, -10)$, and $\varphi(s) = (-\cos s, \mathrm{e}^s)$, for $s \leq 0$, respectively.

Figure 2: Behavior of three solutions of (5.2).

6 Conclusion

We have presented a criterion for the global exponential stability of a nonautonomous differential equation with infinite delays (2.2). We have applied this criterion to a generalized nonautonomous Cohen-Grossberg neural network model (4.1) to obtain a global exponential stability result, which is simple to verify and directly applicable to several neural network models such as Hopfield model (4.7), Cohen-Grossberg models (4.10) and (4.12), and BAM model (4.14). The corollaries 4.2 and 4.3 present different stability criteria for the models studied in [19, 28] and Corollary 4.4 gives a improvement of the main result in [29].

We should remark that, contrary to the usual in the literature about neural networks, the stability results in this paper do not require the boundedness of coefficients.

As the model (2.2) is general enough to include biological models also, in a forthcoming work, we shall exploit our main result, Theorem 3.2, to get new global stability criteria for population models.

Another important research line possible to be followed is to consider the model (4.1) in time-space scales, as in the relevant papers [1, 2], and to study if a stability criterion like Theorem 4.1 holds.

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