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# On Vietoris' number sequence and combinatorial identities with quaternions

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**Abstract**

Ruscheweyh and Salinas showed in 2004 the relationship of a celebrated theorem of Vietoris (1958) about the positivity of certain sine and cosine sums with the function theoretic concept of stable holomorphic functions in the unit disc. The present paper reveals that the coefficient sequence in Vietoris' theorem is identical to a number sequence obtained by a new combinatorial identity which involves generators of quaternions. In this sense Vietoris' sequence of rational numbers combines seemingly disperse subjects in Real, Complex and Hypercomplex Analysis. Thereby we show that a non-standard application of Clifford algebra tools is able to reveal new insights in objects of combinatorial nature.

*Keywords:* Vietoris' number sequence, quaternions, combinatorial identities

## 1 Introduction

In the center of our attention lies the sequence of rational numbers

$$1,\ \frac{1}{2},\ \frac{1}{2},\ \frac{3}{8},\ \frac{3}{8},\ \frac{5}{16},\ \frac{5}{16},\frac{35}{128},\frac{35}{128},\frac{63}{256},\frac{63}{256},\frac{231}{1024},\frac{231}{1024},\ldots. \tag{1}$$

which by means of the *generalized central binomial coefficient* $\binom{k}{\lfloor \frac{k}{2} \rfloor}$ can be written in compact form (cf. [4]) as

$$\mathcal{S} = (c_k)_{k\geq 0}\,, \quad \text{where} \;\; c_k = \frac{1}{2^k}\binom{k}{\lfloor \frac{k}{2} \rfloor}, \quad k \geq 0. \tag{2}$$

Seemingly for the first time this sequence appeared in the context of positive trigonometric sums in a celebrated paper of L. Vietoris [17]. Askey's version ([2, p. 5]) of Vietoris' theorem is the following:

**Theorem 1 (L. Vietoris)**

$$\sum_{k=1}^{n} a_k \sin k\theta > 0, \quad 0 < \theta < \pi, \quad \textit{and} \quad \sum_{k=0}^{n} a_k \cos k\theta > 0, \quad 0 \leq \theta < \pi,$$

*where*

$$a_{2k} = a_{2k+1} = \frac{(\frac{1}{2})_k}{k!}, \qquad k = 0, 1, \ldots, \tag{3}$$

*with* $(\cdot)_k$ *as the raising factorial in the classical form of the Pochhammer symbol.*

We call attention to the fact that because of (3), the coefficients in the *sine* sum used in Askey's as well as in Vietoris' original version are exactly the elements of $\mathcal{S}$ in (2) or, explicitly, in (1). Obviously, demanding in (3) that $a_{2k}$ and $a_{2k+1}$ coincide, the sequence of coefficients in the *cosine* sum differs from (1) by the inclusion of $a_0 = 1$ and the shift of the indices by one to the left, i.e. $a_0 = 1 \quad \text{and} \quad a_{k+1} = c_k, \; k \geq 0$. Even though this small difference, we call $\mathcal{S}$ in the sequel simply *Vietoris' number sequence*. Compared with the traditional way of defining the coefficient sequence by (3), the use of the properties of the *generalized central binomial coefficient* allowed at least a unique representation (2) with consecutively running index $k$.

Before continuing with the specific task of the present paper, it seems worthwhile to mention the other areas in which Vietoris' theorem played an important role. Using the arsenal of real analysis methods in positivity theory, Askey and Steinig showed in [3] the embedding of Vietoris' results in general problems for Jacobi polynomials, including their relation to other subjects in Harmonic Analysis. Later on, Ruscheweyh and Salinas showed in [15] an interesting relationship of Vietoris' theorem with the function theoretic concept of stable holomorphic functions in the unit disc. The common origin of the present paper with others like, for example, [9, 5] where the sequence $\mathcal{S}$ was already mentioned in different contexts, is the field of Hypercomplex Analysis, particularly the study of monogenic (or Clifford-holomorphic) Appell polynomials [1, 8, 10]. Recently in [6], the authors obtained even some number theoretic results for a related to $\mathcal{S}$ integer number sequence (sequence A283208 in The On-Line Encyclopedia of Integer Sequences, published electronically at https://oeis.org).

The goal of the present paper concerns the surprising appearance of $\mathcal{S}$ in a relation between the generators of Hamilton's well known non-commutative algebra $\mathbb{H}$ of quaternions (see e.g. [10]), relying only on elementary properties of a adequately generalized binomial formula for the quaternions. Taking into account that $\mathbb{H}$ can be considered as a Clifford algebra $\mathcal{C}\ell_{0,n}$, for $n = 2$, the generalization of our results for an arbitrary $n \geq 2$ will be treated in an extended version of this paper by applying intrinsic properties of monogenic Appell polynomials in terms of several hypercomplex variables.

## 2 Hamilton's quaternions come into the play

Consider a quaternion $q \in \mathbb{H}$ written as

$$q = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}, \;\; \text{where} \;\; \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1.$$

Due to non-commutativity the formal expansion of a binomial with two imaginary units (quaternion generators) $(\mathbf{i} + \mathbf{j})^k, \; k \geq 0$, will not directly lead to Pascal's triangle, as the case $k = 3$ shows:

$$(\mathbf{i}+\mathbf{j})^3 = \mathbf{i}^3 + (\mathbf{iij}+\mathbf{iji}+\mathbf{jii}) + (\mathbf{ijj}+\mathbf{jij}+\mathbf{jji}) + \mathbf{j}^3. \tag{4}$$

But that will happen if we try to embed the non-commutative multiplication into the concept of a $k-nary$ symmetric (or permutative) operation. Therefore let $a_i$ stay for one of the generators $\mathbf{i}$ or $\mathbf{j}$ and write the quaternionic $k$-fold product of $k-s$ generators $\mathbf{i}$ and $s$ generators $\mathbf{j}$, respectively, in the general form of a symmetric " $\times$ " product ([13]), i.e.

$$\mathbf{i}^{k-s} \times \mathbf{j}^s := \frac{1}{k!} \sum_{\pi(i_1,\ldots,i_n)} a_{i_1} a_{i_2} \cdots a_{i_k} \tag{5}$$

where the sum runs over ***all*** permutations of all $(i_1, \ldots, i_n)$. Then, by taking into account the repeated use of $\mathbf{i}$ and $\mathbf{j}$ on the right hand side of (5), we can write

$$\mathbf{i}^{k-s} \times \mathbf{j}^s = \frac{(k-s)!s!}{k!} \sum_{\pi(i_1,\ldots,i_n)} a_{i_1} a_{i_2} \cdots a_{i_k} = \left[\binom{k}{s}\right]^{-1} \sum_{\pi(i_1,\ldots,i_n)} a_{i_1} a_{i_2} \cdots a_{i_k}$$

where now the sum runs only over ***all distinguished*** permutations of all $(i_1, \ldots, i_n)$. Applying, for example, the convention (5) to (4) we obtain now, for $k = 3$ the expansion written with binomial coefficients in the form

$$(\mathbf{i}+\mathbf{j})^3 = \binom{3}{0}\mathbf{i}^3 + \binom{3}{1}\mathbf{i}^2 \times \mathbf{j} + \binom{3}{2}\mathbf{i} \times \mathbf{j}^2 + \binom{3}{3}\mathbf{j}^3. \tag{6}$$

Analogously, the expansion of $(\mathbf{i}+\mathbf{j})^k$ for any $k \geq 0$ follows now the rules of the ordinary binomial expansion in an evident way and leads to[1]

$$(\mathbf{i}+\mathbf{j})^k = \left[\sum_{s=0}^{k} \binom{k}{s} (\mathbf{i}^{k-s} \times \mathbf{j}^s)\right], \quad k \geq 0. \tag{7}$$

Needless to say that the generalized binomial formula (7) is a key for studying combinatorial relations with quaternions in the following sections.

We will show now that another step towards our goal is the evaluation of expressions of the form $(\mathbf{i}^{k-s} \times \mathbf{j}^s)\ \ k \geq 0,\ s = 0, 1, \ldots k$.

# 3 Evaluating symmetric products of quaternion generators

Notice that for $k \geq 2$ the influence of the non-commutativity of the ordinary quaternionic product is evident. This can be illustrated by the following exam-

[1] An obvious generalization of (5) to the case of more than two generators used in the general case of $\mathcal{C}\ell_{0,n}$ for $n \geq 2$ leads to a polynomial formula.

ples:

$$\mathbf{i} \times \mathbf{j} = \frac{1!1!}{2!}(\mathbf{ij} + \mathbf{ij}) = 0,$$

$$\mathbf{i}^2 \times \mathbf{j} = \binom{3}{1}^{-1}(\mathbf{iij} + \mathbf{iji} + \mathbf{jii}) = -\frac{1}{3}\mathbf{j}, \tag{8}$$

$$\mathbf{i} \times \mathbf{j}^2 = \binom{3}{1}^{-1}(\mathbf{ijj} + \mathbf{jij} + \mathbf{jji}) = -\frac{1}{3}\mathbf{i}. \tag{9}$$

To obtain a general rule for those products we refer to an early version of the famous *Faá di Bruno formula* for the derivative of a composed function (see [11] and [12]) as it was used in [7].

**T. Abadie's formula**
*If $f$ and $g$ are real functions of $\lambda$, with a sufficient number of derivatives, then*

$$(g \circ f)^{(s)}(\lambda) = \sum_{l=0}^{s} \binom{s}{l} g^{(l)}\big(f(\lambda)\big) \Big\{ \frac{d^{s-l}}{dh^{s-l}} \big(\Delta_h f(\lambda)\big)^l \Big\}_{h=0},$$

*where $\Delta_h f(\lambda) := \frac{f(\lambda+h)-f(\lambda)}{h}$ is the difference quotient of $f$.*

Consider now the polynomial $F_k(\lambda)$ of degree $k$ in the real parameter $\lambda$,

$$F_k(\lambda) = (\mathbf{i} + \lambda\mathbf{j})^k = \sum_{s=0}^{k} \binom{k}{s} \lambda^s \mathbf{i}^{k-s} \times \mathbf{j}^s$$

and note that

$$\mathbf{i}^{k-s} \times \mathbf{j}^s = \frac{F_k^{(s)}(0)}{s!\binom{k}{s}}. \tag{10}$$

Since

$$F_k(\lambda) = \begin{cases} (-1-\lambda^2)^{\frac{k}{2}}, & \text{if} \quad k \text{ even;} \\ (-1-\lambda^2)^{\frac{k-1}{2}}(\mathbf{i} + \lambda\mathbf{j}), & \text{if} \quad k \text{ odd,} \end{cases} \tag{11}$$

it can be composed, for even $k$, in the form $F_k(\lambda) = (g \circ f)(\lambda)$ with suitably chosen functions

$$g(\lambda) = (-1-\lambda)^{\frac{k}{2}} \text{ and } f(\lambda) = \lambda^2.$$

whereas the case of an odd $k$ can be reduced to the previous case by the relation

$$F_k(\lambda) = F_{k-1}(\lambda)(\mathbf{i} + \lambda\mathbf{j}).$$

Applying *T. Abadie's formula* and following the proof of *Proposition 1* in [7, p. 1730], about generalized powers of hypercomplex variables, one gets finally the values of (10) in the form:

$$\mathbf{i}^{k-s} \times \mathbf{j}^{s} = \begin{cases} (-1)^{\frac{k}{2}} \binom{\frac{k}{2}}{\frac{s}{2}} \binom{k}{s}^{-1}, & k \text{ even and } s \text{ even;} \\ 0, & k \text{ even and } s \text{ odd;} \\ (-1)^{\frac{k-1}{2}} \binom{\frac{k-1}{2}}{\frac{s}{2}} \binom{k}{s}^{-1} \mathbf{i}, & k \text{ odd and } s \text{ even;} \\ (-1)^{\frac{k-1}{2}} \binom{\frac{k-1}{2}}{\frac{s-1}{2}} \binom{k}{s}^{-1} \mathbf{j}, & k \text{ odd and } s \text{ odd.} \end{cases} \tag{12}$$

**Remark 1** The examples in the beginning of this section, in particular relations (8) and (9), confirm very well the last three equalities in (12). It is evident, that there are, depending on the relative parities of $k$ resp. $s$ only four types of values of $\mathbf{i}^{k-s} \times \mathbf{j}^{s}$, namely (i) real and different from zero, (ii) equal to zero, (iii) a real multiple of $\mathbf{i}$ and (iv) a real multiple of $\mathbf{j}$. The reason for this, at first glance, surprising result is based on the following facts. Obviously, an integer power of the type of a so-called *reduced purely imaginary quaternion* $\underline{q} = \mathbf{i} + \mathbf{j}$, is either a real number (if $k$ is even) or again a reduced purely imaginary quaternion (if $k$ is odd; cf. (11)). Besides this, the symmetric products in which such a power (7) is additively decomposed avoid the appearance of ordinary mixed products like, for example, $\mathbf{i} \cdot \mathbf{j}$ as mutually annihilating summands. This becomes directly plausible if we look for example to the binomial expansion for even $k = 2$ or $k = 4$ where only entries of type (i) and type (ii) are present. An example for an odd $k$ with $2 \times \frac{k+1}{2} = k + 1$ alternating entries of type (iii) and (iv) is (6). In both cases we can recognize the symmetric structure of the corresponding lines in a Pascal triangle with quaternionic entries.

# 4 A combinatorial identity for Vietoris' number sequence

Before coming to our main result, let us still remember a well known combinatorial identity (cf. [14, p. 130] or [16, p. 44]) that we need for its proof, namely

$$\sum_{t=0}^{m} \binom{2t}{t} \binom{2m-2t}{m-t} = 4^{m}, \tag{13}$$

which in turn can be proved by evaluating the square of the generating function of the central binomial coefficients and its derivatives at $x = 0$.

Now we can prove

**Theorem 2** *Let* $\mathbf{i}$ *and* $\mathbf{j}$ *be two generators of a reduced purely imaginary quaternion. Then the following combinatorial identity holds*

$$\binom{k}{\lfloor \frac{k}{2} \rfloor} \left[ \sum_{s=0}^{k} \binom{k}{s} (\mathbf{i}^{k-s} \times \mathbf{j}^{s})^{2} \right] = (-2)^{k}. \tag{14}$$

Taking into account the form of the elements of Vietoris' number sequence (2), formula (14) can be rewritten in order to obtain the representation of Vietoris' number sequence by symmetric products of the generators $\mathbf{i}$ and $\mathbf{j}$.

**Corollary** [*Representation of Vietoris' number sequence*]

$$c_k = (-1)^k \left[ \sum_{s=0}^{k} \binom{k}{s} (\mathbf{i}^{k-s} \times \mathbf{j}^s)^2 \right]^{-1} . \tag{15}$$

*Proof of Theorem 2*

As an auxiliary calculation we determine the square of the symmetric products in (12) multiplied by $\binom{k}{s}$ and distinguish between $k = 2m$ and $k = 2m+1$ resp. $s = 2t$ and $s = 2t+1$ for the different parities. We get immediately

$$\binom{k}{s} (\mathbf{i}^{k-s} \times \mathbf{j}^s)^2 = \begin{cases} \dfrac{\binom{m}{t}^2}{\binom{2m}{2t}}, & k = 2m \quad \text{and } s = 2t; \\ 0, & k = 2m \quad \text{and } s = 2t+1; \\ \dfrac{-\binom{m}{t}^2}{\binom{2m+1}{2t}}, & k = 2m+1 \quad \text{and } s = 2t; \\ \dfrac{-\binom{m}{t}^2}{\binom{2m+1}{2t+1}}, & k = 2m+1 \quad \text{and } s = 2t+1. \end{cases} \tag{16}$$

Now we consider two cases corresponding to the parity of $k$.

I. $k$ even

Denote by $A_k$ the left-hand side of (14). The use of (16) (note that the second case implies that the sum over odd values of $s$ completely vanishes) together with (13) allows to write

$$\begin{aligned} A_{2m} &= \tbinom{2m}{m} \Big[ \sum_{s=0}^{2m} \tbinom{2m}{s} (\mathbf{i}^{2m-s} \times \mathbf{j}^s)^2 \Big] = \tbinom{2m}{m} \Big[ \sum_{t=0}^{m} \tbinom{2m}{2t} (\mathbf{i}^{2m-2t} \times \mathbf{j}^{2t})^2 + 0 \Big] \\ &= \tbinom{2m}{m} \Bigg[ \sum_{t=0}^{m} \frac{\binom{m}{t}^2}{\binom{2m}{2t}} \Bigg] = \sum_{t=0}^{m} \tfrac{(2t)!}{t!t!} \cdot \tfrac{(2m-2t)!}{(m-t)!(m-t)!} \\ &= 4^m = (-2)^{2m} \end{aligned}$$

II. $k$ odd

In this case we apply the third and the fourth case of (16) and proceed analogously to the former case.

$$
\begin{aligned}
A_{2m+1} &= \tbinom{2m+1}{m}\Big[\sum_{s=0}^{2m+1}\tbinom{2m}{s}(\mathbf{i}^{2m+1-s}\times\mathbf{j}^{s})^{2}\Big] = \tbinom{2m+1}{m}\Bigg[\sum_{t=0}^{m}\frac{-\binom{m}{t}^{2}}{\binom{2m+1}{2t}} - \sum_{t=0}^{m}\frac{\binom{m}{t}^{2}}{\binom{2m+1}{2t+1}}\Bigg] \\
&= \tbinom{2m+1}{m}\sum_{t=0}^{m}\tbinom{m}{t}^{2}\Big[\tfrac{(2t)!(2m-2t+1)!}{(2m+1)!}+\tfrac{(2m-2t)!(2t+1)!}{(2m+1)!}\Big] \\
&= -2\sum_{t=0}^{m}\tfrac{(2t)!}{t!t!}\cdot\tfrac{(2m-2t)!}{(m-t)!(m-t)!} \\
&= -2\cdot 4^{m} = (-2)^{2m+1}.
\end{aligned}
$$

□

We finish with examples for the first values of $c_k$ in (15) resp. (1),

$$
\begin{aligned}
c_0 =&1 \\
c_1 =&(-1)^1\left[\mathbf{i}^2+\mathbf{j}^2\right]^{-1} = \frac{1}{2} \\
c_2 =&(-1)^2\left[(\mathbf{i}^2)^2+\binom{2}{1}(\mathbf{i}\times\mathbf{j})^2+(\mathbf{j}^2)^2\right]^{-1} = \frac{1}{2} \\
c_3 =&(-1)^3\left[(\mathbf{i}^3)^2+\binom{3}{1}(\mathbf{i}^2\times\mathbf{j})^2+\binom{3}{2}(\mathbf{i}\times\mathbf{j}^2)^2+(\mathbf{j}^3)^2\right]^{-1} = \frac{3}{8} \\
c_4 =&(-1)^4\left[(\mathbf{i}^4)^2+\binom{4}{1}(\mathbf{i}^3\times\mathbf{j})^2+\binom{4}{2}(\mathbf{i}^2\times\mathbf{j}^2)^2+\binom{4}{3}(\mathbf{i}\times\mathbf{j}^3)^2+(\mathbf{j}^4)^2\right]^{-1} = \frac{3}{8}.
\end{aligned}
$$


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