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Approaching an Overdamped System as a Quadratic Eigenvalue Problem

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Abstract: In viscous material systems, time and stress dependent instabilities often occur. The evolution of visco-elastic systems under external stress has already been modeled by applying a matricial dynamics equation comprehending elasticity and viscosity matrices. In this study we report a novel formulation for such kind of systems in an overdamped regime as a nonlinear quadratic eigenvalue problem. The results presented were obtained after solving the eigenvalue equation of several sets of discrete damped mass-spring systems.

Keywords: Quadratic eigenvalue problem, visco-elastic systems, damped mass-spring system.

1 Introduction

The harmonic oscillator is the paradigm to all condensed matter. In a structured material system, resonance occurs when the structure is excited by external forces whose frequencies mime the natural frequencies/modes of the system itself. Then, the system vibrations are amplified towards infinity and it becomes unstable. The natural modes of a structure can be seen as the solution of an eigenvalue problem, that is quadratic when damping effects are included in the model. When considering the evolution of a damped visco-elastic systems under external stress, it may be identified as a nonlinear quadratic eigenvalue problem which models the second order differential equation of the momentum balance of the system. This was confirmed by the modularization of an interconnected 2D damped mass-spring system for which the solution of the dynamics equation was successfully applied as herein presented.

After briefly addressing the problem of damped visco-elastic systems under external stress and introducing its physical dynamics equation, it is addressed as a quadratic eigenvalue problem. For this formulation new contributions were developed. A case study of four, nine and sixteen damped mass-spring system is analyzed as a QEP, calling upon MatLab.

2 Quadratic Eigenvalue Problem

Under an external applied F the dynamics of a system is governed by the momentum balance equation (Newton second law). Considering elasticity and viscosity, it follows:

$$M\ddot{u}(t)+B\dot{u}(t)+Ku(t)=F(t) \tag{1}$$

where M is the mass matrix (symmetric and positive definite), K is the elasticity matrix (positive definite), B is the viscosity matrix (symmetric), and $u(t)$ stands for the n point masses individual displacements ($M,K,B,\in R^{nxn}$, $u(t)\in R^n$). The (static) resistance to displacement is provided by a spring of elasticity K, while the (dynamic) energy loss mechanism is represented by a damper B and $F(t)$ represents the external force ($F(t)\in R^n$).

The general solution to the homogeneous equation has the form:

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$$u(t) = ve^{\lambda t}$$

where λ and v are a scalar and a vector of dimension n, respectively.

The solution of the dynamic equation (1) can be expressed as a nonlinear eigenvalue problem in terms of the eigensolution of the corresponding Quadratic Eigenvalue Problem (QEP).

The Quadratic Eigenvalue Problem has extensive applications in areas such as the dynamic analysis of structures with proportional damping models, electrical circuit simulation or linear stability of flows in fluid mechanics.The review paper by Tisseur and Meerbergen [1] describes many applications of the QEP.

Given $M, B, K \in \mathbb{C}^{r\times r}$, the QEP formal definition consists on finding scalars λ and nonzero vectors $v \in C^r$ and $w \in C^r$, such that:

$$(\lambda^2 M + \lambda B + K)v = 0 \quad \text{and} \quad w^*(\lambda^2 M + \lambda B + K) = 0,$$

where v and w^* are the left and the right eigenvectors corresponding to the eigenvalue λ (w^* denotes the conjugate transpose of w) [1]. In all, QEP has $2r$ eigenvalues with up to $2r$ right and $2r$ left eigenvectors, though no more than r eigenvectors linearly independent.

In the matrix polynomial of degree 2

$$Q(\lambda) = \lambda^2 M + \lambda B + K$$

the coefficients of the matrix are quadratic polynomials in the scalar λ.

Matrix $Q(\lambda)$ is *self-adjoint* if $Q(\lambda) = Q(\overline{\lambda})^*$ for all $\lambda \in \mathbb{C}$ or, equivalently, if M, B, and K are Hermitian matrices. Knowing that the eigenvalues of a self-adjoint matrix $Q(\lambda)$ are real or arise in complex conjugate pairs:

$$Q(\lambda)v = 0 \;\Leftrightarrow\; w^* Q(\overline{\lambda}) = 0,$$

where v is a right eigenvector of λ and w is a left eigenvector of $\overline{\lambda}$. When the matrices are real, then the sets of left and right eigenvectors coincide.

When the matrices M, B, and K are real and symmetric, $M, B > 0$, and $K \geq 0$, since v is an eigenvector, the roots of $v^* Q(\lambda)v = 0$ are

$$\lambda = \left(-(v^*Bv) \pm \sqrt{(v^*Bv)^2 - 4(v^*Mv)\,(v^*Kv)} \right) / (2v^*Mv),$$

and we say that the system is *overdamped* when it is satisfied the overdamping condition:

$$min_{||v||_2=1} \left[(v^*Bv)^2 - 4(v^*Mv)(v^*Kv) \right] > 0.$$

We observe that for $B > 0$ and $K > 0$, we have $Re(\lambda) < 0$, all the eigenvalues are real and nonpositive, lying in the left half-plane and the system is stable [1].

3 A 2-D model of interconnected masses

As an example of a QEP, we consider the interconnected 2D damped mass-spring system illustrated in Figure 1. Each mass has 2 degrees of freedom and is connected to all the others by a spring and a damper with constants κ (elasticity) and b (friction or viscous damping), respectively.

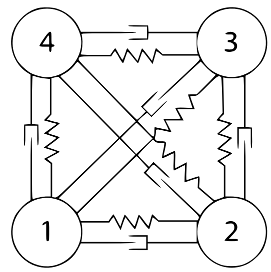

Fig. 1: An 8-degrees of freedom damped mass-spring system.

The second order differential equation (1) governs the behavior of the system where the mass matrix M is diagonal and the elasticity and damping matrices, K and B are written according to equation (2) [2].

Initially we have taken a 4-point square geometry, strictly regular first, and slightly distorted afterwards - either contracted or stretched by the displacement of one of its mass-points (see Figure 2). In further essays, the set has been expanded to a 9 and to a 16 point masses arrangement, equally distorted by the contraction or stretching of a single point.

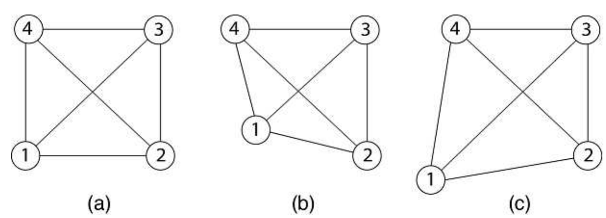

Fig. 2: Set of 4 point-mass in a regular (square) geometry (a) and in a contracted (b) and stretched (c) polygon.

Assuming that all masses m are equal and that the elastic κ and damping b constants are the same for every pair of interconnected masses, we can seek for solutions at the overdamping regime.

The associated mass matrix M is a $2n \times 2n$ diagonal, and the elasticity K and damping B matrices are $2n \times 2n$ symmetric and defined after [2] as,

$$K = \kappa \left(\begin{array}{cc|cc} P & O_\alpha & O_\gamma & O_\delta \\ O_\alpha & Q & O_\beta & O_\theta \\ \hline O_\gamma & O_\beta & R & O_\varphi \\ O_\delta & O_\theta & O_\varphi & S \end{array} \right), \quad B = b \left(\begin{array}{cc|cc} P & O_\alpha & O_\gamma & O_\delta \\ O_\alpha & Q & O_\beta & O_\theta \\ \hline O_\gamma & O_\beta & R & O_\varphi \\ O_\delta & O_\theta & O_\varphi & S \end{array} \right), \quad (2)$$

where the block sub-matrices O_i ($i = \alpha, \beta, \gamma$, θ, φ and δ), P, Q, R and S traduce the geometrical relations between each mass-point and its neighbors. Angles α, β, γ, θ, φ and δ result from the direction defined by every couple of mass-points relative to the X-axis (see details in [2]).

4 Results

The case study from now on reported is an overdamped visco-elastic system taking the mass-points to have one unit value and the ratio of elasticity to viscosity coefficients to be $1/10$, so that the over damped regime is guaranteed. In this frame, matrices K and B have been built and the eigenvalues of equation $Q(\lambda)v = 0$ obtained in MatLab, by calling the function $polyeig(K,B,M)$ [3].

The above mentioned 4, 9 and 16-point sets have been successively taken as shown in Figure 3.

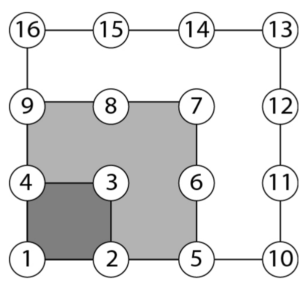

Fig. 3: The sets of 4, 9 and 16 mass-points.

The eigenvalues of the three sets of mass-points, either in regular geometry, or contracted or stretched, have been obtained and are plotted in Figure 4. Table 1 shows only the 4-point case to exemplify.

All the eigenvalues are real and non-positive, as expected for an overdamped regime [1]. The eigenvalues are degenerate for the regular geometries, but degeneracy is partially broken both for contracted and stretched geometries due to its lower degree of symmetry.

Increasing the set of mass-points from 4 to 9 to 16 leads to an increasing number of non-zero eigenvalues, though many of then very close to null.

Extending the system from 4 to 9 to 16 mass-points to an arbitrarily large number of points will eventually lead from discrete to continuous sets of eigenvalues.

5 Final remarks

The introduced QEP case can be generalized for an arbitrary large number of disordered point masses, belonging to a single material (characterized by a single set of m, κ and b), or to heterogeneous - blended or layered - materials with different sets of parameters.

Table 1: Eigenvalue of the QEP for 4 overdamped mass-spring systems regular, contracted and stretched.

4 mass-points geometry		
regular	contracted	stretched
0,0000	0 ,0000	0,0000
0,0000	0,0000	0,0000
0,0000	0,0000	0,0000
0,0000	0,0000	0,0000
-0,0100	-0,0100	-0,0100
-0,0100	-0,0100	-0,0100
-0,0100	-0,0100	-0,0100
-0,0100	-0,0100	-0,0100
-0,0100	-0,0100	-0,0100
-0,0100	-0,0100	-0,0100
-58,5700	-55,3000	-55,8951
-58,5700	-61,7600	-61,1943
-199,9900	-198,6300	-199,0939
-199,9900	-198,9000	-199,2405
-341,4100	-324,0800	-326,9483
-341,4100	-361,2600	-357,5681

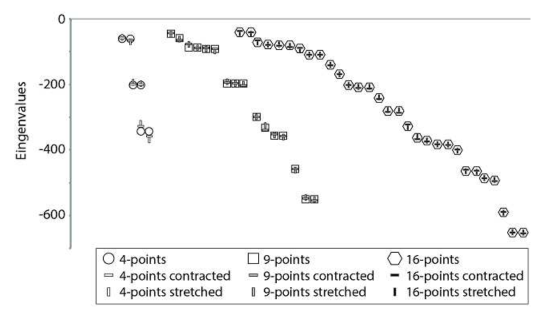

Fig. 4: Eigenvalue distribution of the QEP for overdamped mass-spring systems regular, contracted and stretched. For sake of clarity, all eigenvalues greater than -1 are not shown.

In a previous work [4], we have developed an algorithm, to a domain of material points, that establishes the set of physical bonds between any two neighbours and their geometrical relations (angles α, β, γ, θ, φ and δ), so to define an *adjacency matrix*. This is the scaffold to build up matrices K and B, used in the herein QEP example.

As an undamped dynamic system has already been addressed as an Eigenvalue Complementarity Problem (EiCP) [5], we intend to treat a broader case by formulating a Quadratic Eigenvalues Complementarity Problem (QEiCP) [6] to a visco-elastic system.

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