Weighted second-order Poincaré inequalities: Application to RSA models

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Abstract. Consider an ergodic stationary random field A on the ambient space $\mathbb{R}^d$. In a recent work we introduced the notion of *weighted* (first-order) functional inequalities, which extend standard functional inequalities like spectral gap, covariance, and logarithmic Sobolev inequalities, while still ensuring strong concentration properties. We also developed a constructive approach to these weighted inequalities, proving their validity for prototypical examples like Gaussian fields with arbitrary covariance function, Voronoi and Delaunay tessellations of Poisson point sets, and random sequential adsorption (RSA) models, which do not satisfy *standard* functional inequalities. In the present contribution, we turn to second-order Poincaré inequalities à la Chatterjee: while first-order inequalities quantify the distance to constants for nonlinear functions $X(A)$ in terms of their local dependence on the random field A, second-order inequalities quantify their distance to normality. For the above-mentioned examples, we prove the validity of suitable *weighted* second-order Poincaré inequalities. Applied to RSA models, these functional inequalities allow us to complete and improve previous results by Schreiber, Penrose, and Yukich on the jamming limit, and to propose and fully analyze a more efficient algorithm to approximate the latter.

Contents

1. Introduction

Stein's method and its refinement by Chatterjee [3, 4] in the form of second-order Poincaré inequalities are a powerful tool to quantify the distance of a random variable to normality. In this contribution, we are motivated by two different applications: fluctuations in stochastic homogenization on the one hand, and fluctuations in stochastic geometry on the other hand. The first use of second-order Poincaré inequalities in stochastic homogenization is due to Nolen [17] (see also [1, 26] for earlier qualitative approaches to this problem), successfully followed by [8, 10, 16, 7]. Regarding stochastic geometry, we are more precisely interested in random sequential adsorption (RSA) models and in fluctuations of the jamming limit. In that context, Stein's method was first used in combination with stabilization properties by Penrose and Yukich [24], and followed by [27, 12].

Like first-order functional inequalities, second-order Poincaré inequalities are very restrictive, and essentially hold true only for product structures and for Gaussian random fields with integrable covariance function. The aim of the present contribution is to go beyond these examples, and complete our previous articles [5, 6] by proving the validity of suitable *weighted* versions of second-order Poincaré inequalities for various prototypical random fields with strong correlations.

All random fields A on $\mathbb{R}^d$ considered in this contribution can be obtained as the image $A = \Phi(A_0)$ by some "projection" Φ of some higher-dimensional random field A_0 on $\mathbb{R}^d \times \mathbb{R}^l$ that is known to satisfy a standard (not weighted) second-order Poincaré inequality. In [6] we developed an abstract yet constructive approach to weighted first-order functional inequalities under suitable assumptions on the "projection operator" Φ, and made use of this constructive approach to prove the validity of weighted functional inequalities for various examples of strongly correlated random fields considered in the literature. In the present contribution we similarly establish in Section 2 weighted second-order Poincaré inequalities for these examples.

In Section 3 we use these inequalities to study (linear) spatial averages of the random field A. Although the point of first- and second-order functional inequalities is to address concentration and approximate normality properties for general *nonlinear* functions of correlated random fields, this application to linear random variables is nontrivial, and is particularly relevant in two contexts: the analysis of the jamming limit for RSA models, and quantitative stochastic homogenization. On the one hand, in order to analyze RSA processes, Penrose and Yukich [23] introduced a crucial notion of stabilization radius having its origins in the works of Lee [14, 15] (which is also our main inspiration for the constructive approach to weighted functional inequalities that we developed in [6, Section 2.3.2]), and this paved the way to a series of strong results on the jamming limit [21, 23, 22, 24, 27, 12]. Based on weighted first- and second-order functional inequalities, we revisit and complete this series of papers. On the other hand, in the field of quantitative stochastic homogenization of random elliptic operators in divergence form (that is, operators of the form $-\nabla \cdot A\nabla$ with A a matrix-valued random coefficient field), various quantities of interest are proven to behave essentially like spatial averages of (nonlinear approximately local functions of) the random field, and applying second-order Poincaré inequalities then leads to sharp normal approximation results [17, 8, 10, 16, 7].

Notation.

- d is the dimension of the ambient space $\mathbb{R}^d$;
- C denotes various positive constants that only depend on the dimension d and possibly on other controlled quantities; we write $\lesssim$ and $\gtrsim$ for $\le$ and $\ge$ up to such multiplicative constants C; we use the notation $\simeq$ if both relations $\lesssim$ and $\gtrsim$ hold; we add a subscript in order to indicate the dependence of the multiplicative constants on other parameters;
- $Q^k := [-1/2, 1/2)^k$ denotes the unit cube centered at 0 in dimension k, and for all $x \in \mathbb{R}^k$ and $r > 0$ we set $Q^k(x) := x + Q^k$, $Q^k_r := rQ^k$ and $Q^k_r(x) := x + rQ^k$; when

$k = d$ or when there is no confusion possible on the meant dimension, we drop the superscript k;
- we use similar notation for balls, replacing Q^k by B^k (the unit ball in dimension k);
- the Euclidean distance between subsets of $\mathbb{R}^d$ is denoted by $d(\cdot,\cdot)$;
- $\mathcal{B}(\mathbb{R}^k)$ denotes the Borel σ-algebra on $\mathbb{R}^k$;
- $\mathbb{E}[\cdot]$ denotes the expectation, $\mathrm{Var}[\cdot]$ the variance, and $\mathrm{Cov}[\cdot;\cdot]$ the covariance in the underlying probability space $(\Omega, \mathcal{A}, \mathbb{P})$, and the notation $\mathbb{E}[\cdot\|\cdot]$ stands for the conditional expectation;
- $\mathcal{N}$ denotes a standard normal random variable;
- $\mathrm{d_{TV}}(\cdot,\cdot)$, $\mathrm{d_W}(\cdot,\cdot)$, and $\mathrm{d_K}(\cdot,\cdot)$ denote the total variation, the 1-Wasserstein, and the Kolmogorov distances, respectively.

2. Weighted second-order Poincaré inequalities

Chatterjee's standard second-order Poincaré inequalities are known to hold in total variation distance for Gaussian fields with integrable covariance function [4, 19], as well as in Wasserstein and Kolmogorov distance for general discrete product structures [3, 11]. Based on these results, we prove the validity of weighted second-order Poincaré inequalities for correlated random fields that display a hidden product structure (in a sense made precise below). To this aim, we first recall the constructive approach of [6] to first-order weighted functional inequalities, and then turn to the two prototypical classes of examples: deterministically localized fields (which essentially concerns Gaussian fields), and randomly localized fields (in which case localization is quantified in terms of the action radius introduced in [6]).

Before we state the main results, let us comment on the existing literature. On the one hand, for Gaussian random fields, our results can be compared with [19, Theorem 1.1] (see also [18]), which establishes a similar (infinite-dimensional) second-order Poincaré inequality in terms of Malliavin calculus in abstract Wiener space (where the covariance structure is encoded in some Hilbert norm). The interest of our formulation is the explicit structure of the right-hand side in the form of a weighted inequality, in line with the first order functional inequalities that we obtained in [5, 6].
On the other hand, for randomly localized fields, our approach to control distance to normality can be compared to [12], which develops a general strategy to prove approximate normality results for functionals of Poisson processes based on stabilization properties. In particular, this approach requires stabilization properties to be checked explicitly each time a normal approximation result is to be proved. In contrast, given a random field A which is a transformation of a Poisson process, our approach consists in exploiting stabilization properties of the transformation (in the form of a control on the action radius) to derive a "generalized" second-order functional inequality. This weighted second-order Poincaré inequality has the advantage to be intrinsic for the field A, and as such it can be subsequently applied to any random variable $X(A)$ without having to make further use of the stabilization properties of the transformation.

2.1. Weighted first-order functional inequalities.

Let $A : \mathbb{R}^d \times \Omega \to \mathbb{R}$ be a jointly measurable random field on $\mathbb{R}^d$, constructed on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$. We start with the definition of first-order weighted functional inequalities (cf. [5]), and first recall two important possible choices of (wide-sense) derivatives with respect to the (continuum) random field A, which we generically denote by $\tilde{\partial}$.

- The *oscillation* ∂^{osc} is formally defined by

$$\begin{aligned}\partial^{\mathrm{osc}}_{A,S}\, X(A) \;&:= \sup_{A,S}\operatorname{ess} X(A) - \inf_{A,S}\operatorname{ess} X(A)\\ &= \sup\operatorname{ess}\Big\{X(A') : A' \in \mathrm{Mes}(\mathbb{R}^d;\mathbb{R}),\ A'|_{\mathbb{R}^d\setminus S} = A|_{\mathbb{R}^d\setminus S}\Big\}\\ &\qquad - \inf\operatorname{ess}\Big\{X(A') : A' \in \mathrm{Mes}(\mathbb{R}^d;\mathbb{R}),\ A'|_{\mathbb{R}^d\setminus S} = A|_{\mathbb{R}^d\setminus S}\Big\}, \qquad (2.1)\end{aligned}$$

where the essential supremum and infimum are taken with respect to the measure induced by the field A on the space $\mathrm{Mes}(\mathbb{R}^d;\mathbb{R})$ (endowed with the cylindrical σ-algebra). We refer to [5, Subsection 2.1] for more careful definitions.
- The (integrated) *functional (or Malliavin) derivative* ∂^{fct}, defined as follows. Let us denote by $M \subset \mathrm{L}^\infty(\mathbb{R}^d)$ some open set such that the random field A takes its values in M. Given a $\sigma(A)$-measurable random variable $X(A)$, and given an extension $\tilde X : M \to \mathbb{R}$, its Fréchet derivative $\partial\tilde X(A)/\partial A \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d)$ is defined for all compactly supported perturbation $\delta A \in \mathrm{L}^\infty(\mathbb{R}^d)$ by

$$\lim_{t\to 0}\frac{\tilde X(A+t\delta A)-\tilde X(A)}{t} = \int_{\mathbb{R}^d}\delta A(x)\frac{\partial\tilde X(A)}{\partial A}(x)\,dx,$$

if the limit exists. Since we are interested in the local averages of this derivative, we rather define for all bounded Borel subset $S \subset \mathbb{R}^d$

$$\partial^{\mathrm{fct}}_{A,S}X(A) = \int_S\Big|\frac{\partial\tilde X(A)}{\partial A}(x)\Big|dx.$$

This derivative is additive with respect to the set S: for all disjoint Borel subsets $S_1, S_2 \subset \mathbb{R}^d$, we have $\partial^{\mathrm{fct}}_{A,S_1\cup S_2}X(A) = \partial^{\mathrm{fct}}_{A,S_1}X(A) + \partial^{\mathrm{fct}}_{A,S_2}X(A)$. The second-order functional derivative is defined similarly (and will be used for the second-order Poincaré inequality).

Definition 2.1. Given an integrable function $\pi : \mathbb{R}_+ \to \mathbb{R}_+$, we say that A satisfies the *weighted spectral gap* ($\tilde\partial$-WSG) *with weight* π if for all $\sigma(A)$-measurable random variable $X(A)$ we have

$$\mathrm{Var}\left[X(A)\right] \le \mathbb{E}\left[\int_0^\infty\int_{\mathbb{R}^d}\Big(\tilde\partial_{A,B_{\ell+1}(x)}X(A)\Big)^2 dx\,(\ell+1)^{-d}\pi(\ell)\,d\ell\right], \qquad (2.2)$$

and that it satisfies the *weighted covariance inequality* ($\tilde\partial$-WCI) *with weight* π if for all $\sigma(A)$-measurable random variables $X(A)$ and $Y(A)$ we have

$$\begin{aligned}&\mathrm{Cov}\left[X(A);Y(A)\right]\\ &\le \int_0^\infty\int_{\mathbb{R}^d}\mathbb{E}\left[\Big(\tilde\partial_{A,B_{\ell+1}(x)}X(A)\Big)^2\right]^{\frac12}\mathbb{E}\left[\Big(\tilde\partial_{A,B_{\ell+1}(x)}Y(A)\Big)^2\right]^{\frac12}dx\,(\ell+1)^{-d}\pi(\ell)\,d\ell. \qquad (2.3)\end{aligned}$$

□

In [6] we have developed a constructive approach to such inequalities. Let us be more specific: Let the random field A on $\mathbb{R}^d$ be $\sigma(\mathcal{X})$-measurable for some random field $\mathcal{X}$ defined on some measure space X and with values in some measurable space M. Assume that we have a partition $X = \biguplus_{x\in\mathbb{Z}^d,t\in\mathbb{Z}^l}X_{x,t}$ on which $\mathcal{X}$ is *completely independent*, that is, the family of restrictions $(\mathcal{X}|_{X_{x,t}})_{x\in\mathbb{Z}^d,t\in\mathbb{Z}^l}$ are all independent.

The random field $\mathcal{X}$ can be e.g. a random field on $\mathbb{R}^d \times \mathbb{R}^l$ with values in some measure space (choosing $X = \mathbb{R}^d \times \mathbb{R}^l$, $X_{x,t} = Q^d(x) \times Q^l(t)$, and M the space of values), or a random point process (or more generally a random measure) on $\mathbb{R}^d \times \mathbb{R}^l \times X'$ for some measure space X' (choosing $X = \mathbb{Z}^d \times \mathbb{Z}^l \times X'$, $X_{x,t} = \{x\} \times \{t\} \times X'$, and M the space of measures on $Q^d \times Q^l \times X'$).

Let $\mathcal{X}'$ be some given i.i.d. copy of $\mathcal{X}$. For all x, t, we define a perturbed random field $\mathcal{X}^{x,t}$ by setting $\mathcal{X}^{x,t}|_{X \setminus X_{x,t}} = \mathcal{X}|_{X \setminus X_{x,t}}$ and $\mathcal{X}^{x,t}|_{X_{x,t}} = \mathcal{X}'|_{X_{x,t}}$. By complete independence, the random fields $\mathcal{X}$ and $\mathcal{X}^{x,t}$ (resp. $A = A(\mathcal{X})$ and $A(\mathcal{X}^{x,t})$) have the same law. The following first-order functional inequalities are standard (cf. [6, Proposition 2.4]).

Proposition 2.2. *For all $\sigma(\mathcal{X})$-measurable random variables $Y(\mathcal{X})$ and $Z(\mathcal{X})$, we have*

$$\mathrm{Var}\left[Y(\mathcal{X})\right] \le \frac{1}{2} \sum_{x \in \mathbb{Z}^d} \sum_{t \in \mathbb{Z}^l} \mathbb{E}\left[\left(Y(\mathcal{X}) - Y(\mathcal{X}^{x,t})\right)^2\right],$$

$$\mathrm{Cov}\left[Y(\mathcal{X}); Z(\mathcal{X})\right] \le \frac{1}{2} \sum_{x \in \mathbb{Z}^d} \sum_{t \in \mathbb{Z}^l} \mathbb{E}\left[\left(Y(\mathcal{X}) - Y(\mathcal{X}^{x,t})\right)^2\right]^{\frac{1}{2}} \mathbb{E}\left[\left(Z(\mathcal{X}) - Z(\mathcal{X}^{x,t})\right)^2\right]^{\frac{1}{2}}. \qquad \square$$

We now describe general situations for which the above standard functional inequalities for the "hidden product structure" $\mathcal{X}$ are deformed into weighted functional inequalities of the form (2.2) and (2.3) for the random field A. As pointed out above, we distinguish two situations:

- *deterministic localization*, that is, when the random field A is a deterministic convolution of some product structure, so that the dependence pattern is prescribed deterministically a priori; it leads to weighted functional inequalities with the functional derivative ∂^{fct}, and essentially concerns Gaussian fields;
- *random localization*, that is, when the dependence pattern is encoded by the underlying product structure $\mathcal{X}$ itself (and therefore may depend on the realization, whence the terminology "random"); the localization of the dependence pattern is then measured in terms of what we call the *action radius*; it leads to weighted inequalities with the derivative ∂^{osc} (or with the slightly more precise derivative ∂^{dis} defined below).

These two situations are separately addressed in terms of weighted second-order Poincaré inequalities in Subsections 2.2 and 2.3.

2.2. **Deterministically localized fields.** In this subsection we treat the main example of deterministically localized fields, that is, correlated Gaussian random fields. The main result of this section is a continuum version with nontrivial covariance structure of the second-order Poincaré inequality for i.i.d. Gaussian random variables due to Chatterjee [4], and based on Stein's method. As already discussed, this is to be compared with [19].

Theorem 2.3. *Let G be a jointly measurable stationary Gaussian random field on $\mathbb{R}^d$, characterized by its covariance $\mathcal{C}(x) := \mathrm{Cov}\left[G(x); G(0)\right]$, and assume that $|\mathcal{C}(x)| \le c(|x|)$ for some Lipschitz non-increasing map $c : \mathbb{R}_+ \to \mathbb{R}_+$. Let $h \in C^2(\mathbb{R})$ with $h', h'' \in \mathrm{L}^\infty(\mathbb{R})$, and let A be the random field on $\mathbb{R}^d$ defined by $A(x) := h(G(x))$ for all x. Then for all*

$\sigma(A)$-measurable random variable $X(A)$ and all $R>0$ we have

$$\mathrm{d_{TV}}\left(\frac{X(A)-\mathbb{E}\left[X(A)\right]}{\sqrt{\mathrm{Var}\left[X(A)\right]}},\mathcal{N}\right)^2 \tag{2.4}$$
$$\lesssim \frac{\|h'\|^6_{\mathrm{L}^\infty}}{(\mathrm{Var}\left[X(A)\right])^2}\mathbb{E}\left[\Big(\int_0^\infty\int_{\mathbb{R}^d}\Big(\int_{B_{2(\ell+1)}(x)}\Big|\frac{\partial X(A)}{\partial A}\Big|\Big)^2dx\,(\ell+1)^{-d}(-c'(\ell))d\ell\Big)^2\right]^{\frac12}$$
$$\times\,\mathbb{E}\Bigg[\Big(\int_0^\infty\int_0^\infty\iint_{\mathbb{R}^d\times\mathbb{R}^d}\Big(\iint_{B_{2(\ell_1+1)}(x_1)\times B_{2(\ell_2+1)}(x_2)}\Big|\frac{\partial^2X(A)}{\partial A^2}\Big|\Big)^2dx_1dx_2$$
$$\times(\ell_2+1)^{-d}(-c'(\ell_2))d\ell_2\,(\ell_1+1)^{-d}(-c'(\ell_1))d\ell_1\Big)^2\Bigg]^{\frac12}$$
$$+\,\frac{\|h'\|^2_{\mathrm{L}^\infty}\|h''\|^2_{\mathrm{L}^\infty}}{(\mathrm{Var}\left[X(A)\right])^2}\int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d}c(|x_1-x_2|-R)\,c(|x_2-x_3|-R)\,c(|x_3-x_4|-R)$$
$$\times\prod_{i=1}^4\mathbb{E}\left[\Big(\fint_{B_R(x_i)}\Big|\frac{\partial X(A)}{\partial A}\Big|\Big)^4\right]^{\frac14}dx_1\ldots dx_4.$$

If the covariance is integrable in the sense of $\|\bar{\mathcal{C}}\|_{\mathrm{L}^1}:=\int(\sup_{B(x)}|\mathcal{C}|)dx<\infty$, then the above reduces to

$$\mathrm{d_{TV}}\left(\frac{X(A)-\mathbb{E}\left[X(A)\right]}{\sqrt{\mathrm{Var}\left[X(A)\right]}},\mathcal{N}\right)^2 \tag{2.5}$$
$$\lesssim \frac{\|h'\|^6_{\mathrm{L}^\infty}}{(\mathrm{Var}\left[X(A)\right])^2}\|\bar{\mathcal{C}}\|^3_{\mathrm{L}^1}\,\mathbb{E}\left[\Big(\int_{\mathbb{R}^d}\Big(\int_{B(x)}\Big|\frac{\partial X(A)}{\partial A}\Big|\Big)^2dx\Big)^2\right]^{\frac12}$$
$$\times\,\mathbb{E}\left[\Big(\iint_{\mathbb{R}^d\times\mathbb{R}^d}\Big(\iint_{B(x)\times B(y)}\Big|\frac{\partial^2X(A)}{\partial A^2}\Big|\Big)^2dxdy\Big)^2\right]^{\frac12}$$
$$+\,\frac{\|h'\|^2_{\mathrm{L}^\infty}\|h''\|^2_{\mathrm{L}^\infty}}{(\mathrm{Var}\left[X(A)\right])^2}\|\bar{\mathcal{C}}\|^3_{\mathrm{L}^1}\,\mathbb{E}\left[\int_{\mathbb{R}^d}\Big(\int_{B(x)}\Big|\frac{\partial X(A)}{\partial A}\Big|\Big)^4dx\right].\qquad\square$$

Proof. By scaling it does not restrict generality to assume $\mathbb{E}\left[X(A)\right]=0$ and $\mathrm{Var}\left[X(A)\right]=1$. We split the proof into three steps.

Step 1. Discrete Gaussian field.
In this step, we establish the discrete counterpart of the desired result, that is, a second-order Poincaré inequality à la Chatterjee for correlated Gaussian vectors. Let $V=(V_1,\ldots,V_N)$ denote a Gaussian random vector with covariance $\Sigma:=\mathrm{Var}\left[V\right]\in\mathbb{R}^{N\times N}$. Let $h\in C^2(\mathbb{R})$, and for all i let $W_i:=h(V_i)$. Given a smooth transformation $g:\mathbb{R}^N\to\mathbb{R}$, we consider the random variable $Z:=g(W)$, which can also be represented as $Z:=f(V)$ for some map $f:\mathbb{R}^N\to\mathbb{R}$. Assume that $\mathbb{E}\left[Z\right]=0$ and $\mathrm{Var}\left[Z\right]=1$. Let V' denote an i.i.d. copy of V, and for all $t\in[0,1]$ define $U_t:=\sqrt{t}V+\sqrt{1-t}V'$ and $(Y_t)_i:=h((U_t)_i)$.

In this step, we establish the following variant of [4, Theorem 2.2],

$$\begin{aligned}\frac{1}{2}\,\mathrm{d_{TV}}\left(Z,\mathcal{N}\right)^2 \le 2\|h'\|^6_{\mathrm{L}^\infty}\int_0^1\Big(1+\frac{2}{\sqrt{t}}\Big)&\sum_{i,j,k,l,m,n}|\Sigma_{ij}||\Sigma_{kl}||\Sigma_{mn}|\\ &\times\mathbb{E}\left[|\nabla_i g(Y_t)||\nabla^2_{jk}g(W)||\nabla^2_{lm}g(W)||\nabla_n g(Y_t)|\right]dt\\ +2\|h'\|^2_{\mathrm{L}^\infty}\|h''\|^2_{\mathrm{L}^\infty}\int_0^1\Big(1+\frac{2}{\sqrt{t}}\Big)&\sum_{i,j,k,l}|\Sigma_{ij}||\Sigma_{jk}||\Sigma_{kl}|\\ &\times\mathbb{E}\left[|\nabla_i g(Y_t)||\nabla_j g(W)||\nabla_k g(W)||\nabla_l g(Y_t)|\right]dt.\end{aligned}\tag{2.6}$$

For that purpose, we simply adapt the strategy of [4] to the case with a nontrivial covariance. Using the i.i.d. copy V' of V, we may decompose, for any smooth $\psi:\mathbb{R}\to\mathbb{R}$,

$$\begin{aligned}\mathbb{E}\left[Z\psi(Z)\right]&=\mathbb{E}\left[f(V)\psi(f(V))-f(V')\psi(f(V))\right]\\ &=\mathbb{E}\left[\psi(f(V))\int_0^1\frac{d}{dt}\big(f(\sqrt{t}V+\sqrt{1-t}V')\big)dt\right]\\ &=\frac{1}{2}\mathbb{E}\left[\psi(f(V))\int_0^1\Big(\frac{V}{\sqrt{t}}-\frac{V'}{\sqrt{1-t}}\Big)\cdot\nabla f(\sqrt{t}V+\sqrt{1-t}V')dt\right],\end{aligned}$$

or alternatively, in terms of $U_t:=\sqrt{t}V+\sqrt{1-t}V'$ and $V_t:=\sqrt{1-t}V-\sqrt{t}V'$,

$$\mathbb{E}\left[Z\psi(Z)\right]=\frac{1}{2}\int_0^1\frac{1}{\sqrt{t}\sqrt{1-t}}\mathbb{E}\left[\psi(f(\sqrt{t}U_t+\sqrt{1-t}V_t))V_t\cdot\nabla f(U_t)\right]dt.$$

Noting that the Gaussian vectors U_t and V_t are independent of each other and have the same law as V, and that Gaussian integration by parts takes the form

$$\mathbb{E}\left[V\zeta(V)\right]=\Sigma\,\mathbb{E}\left[\nabla\zeta(V)\right],\qquad \zeta\in C^1_b(\mathbb{R}^N),$$

we deduce from the above,

$$\mathbb{E}\left[Z\psi(Z)\right]=\int_0^1\frac{1}{2\sqrt{t}}\mathbb{E}\left[\psi'(f(\sqrt{t}U_t+\sqrt{1-t}V_t))\nabla f(\sqrt{t}U_t+\sqrt{1-t}V_t)\cdot\Sigma\nabla f(U_t)\right]dt.$$

Defining

$$T(V,V'):=\int_0^1\frac{1}{2\sqrt{t}}\nabla f(V)\cdot\Sigma\,\nabla f(U_t)dt,\tag{2.7}$$

we have thus proven the identity

$$\mathbb{E}\left[Z\psi(Z)\right]=\mathbb{E}\left[\psi'(Z)T(V,V')\right]=\mathbb{E}\left[\psi'(Z)\mathbb{E}\left[T(V,V')\,\|\,Z\right]\right].$$

In other words, we have constructed the so-called Stein factor $\mathbb{E}\left[T(V,V')\,\|\,Z\right]$ for Z. A standard use of Stein's method (see e.g. [4, Lemma 5.1]) then yields

$$\mathrm{d_{TV}}\left(Z,\mathcal{N}\right)\le 2\mathbb{E}\left[\left|\mathbb{E}\left[T(V,V')\,\|\,Z\right]-1\right|\right]\le 2\mathrm{Var}\left[\mathbb{E}\left[T(V,V')\,\|\,V\right]\right]^{\frac{1}{2}}.$$

In order to estimate this last variance, we use the Gaussian Brascamp-Lieb inequality (see e.g. [6, Proposition B.1]),

$$\begin{aligned}\frac{1}{2}\,\mathrm{d_{TV}}\left(Z,\mathcal{N}\right)^2 &\le 2\mathbb{E}\left[\nabla_V\mathbb{E}\left[T(V,V')\,\|\,V\right]\cdot\Sigma\nabla_V\mathbb{E}\left[T(V,V')\,\|\,V\right]\right]\\ &= 2\mathbb{E}\left[\left|\Sigma^{1/2}\mathbb{E}\left[\nabla_V T(V,V')\,\|\,V\right]\right|^2\right]\\ &\le 2\mathbb{E}\left[\left|\Sigma^{1/2}\nabla_V T(V,V')\right|^2\right].\end{aligned}$$

An explicit computation of the gradient $\nabla_V T(V,V')$ based on definition (2.7) yields

$$\nabla_V T(V,V') = \int_0^1 \frac{1}{2\sqrt{t}}\nabla^2 f(V)\cdot\Sigma\nabla f(U_t)dt + \frac{1}{2}\int_0^1 \nabla f(V)\cdot\Sigma\,\nabla^2 f(U_t)dt.$$

Combined with the inequality $(a+b)^2\le 2(a^2+b^2)$ for all $a,b\in\mathbb{R}$, we obtain

$$\begin{aligned}\frac{1}{2}\,\mathrm{d_{TV}}\left(Z,\mathcal{N}\right)^2 \le& \int_0^1\frac{1}{\sqrt{t}}\int_0^1\frac{1}{\sqrt{s}}\,\mathbb{E}\left[\nabla f(U_t)\cdot\Sigma\nabla^2 f(V)\Sigma\nabla^2 f(V)\Sigma\nabla f(U_s)\right]dsdt\\ &+\int_0^1\int_0^1\mathbb{E}\left[\nabla f(V)\cdot\Sigma\nabla^2 f(U_t)\Sigma\nabla^2 f(U_s)\Sigma\nabla f(V)\right]dsdt.\end{aligned}$$

Using successively the inequality $x\cdot\Sigma y\le\frac{1}{2}(x\cdot\Sigma x+y\cdot\Sigma y)$, the identity $\int_0^1 t^{-1/2}dt=2$, and noting that (V,U_t) has the same distribution as (U_t,V), we are left with

$$\frac{1}{2}\,\mathrm{d_{TV}}\left(Z,\mathcal{N}\right)^2 \le \int_0^1\Big(1+\frac{2}{\sqrt{t}}\Big)\mathbb{E}\left[\nabla f(U_t)\cdot\Sigma\nabla^2 f(V)\Sigma\nabla^2 f(V)\Sigma\nabla f(U_t)\right]dt.$$

By definition $Z=f(V)=g(W)$ with $W_i=h(V_i)$, so that $\nabla_i f(V)=h'(V_i)\nabla_i g(W)$ and $\nabla^2_{ij}f(V)=h'(V_i)h'(V_j)\nabla^2_{ij}g(W)+\delta_{ij}h''(V_i)\nabla_i g(W)$, and the result (2.6) follows.

Step 2. Continuum counterparts.
By an approximation argument, the result (2.6) of Step 1 yields for all $\sigma(A)$-measurable random variables $X(A)$ with $\mathbb{E}\left[X(A)\right]=0$ and $\mathrm{Var}\left[X(A)\right]=1$,

$$\begin{aligned}\frac{1}{2}\,\mathrm{d_{TV}}\left(X(A),\mathcal{N}\right)^2 \;\le\; & 2\|h'\|^6_{\mathrm{L}^\infty}\int_0^1\Big(1+\frac{2}{\sqrt{t}}\Big)\int\!\!\ldots\!\!\int|\mathcal{C}(x_1-x_2)||\mathcal{C}(x_3-x_4)||\mathcal{C}(x_5-x_6)|\\ &\times\mathbb{E}\left[\Big|\frac{\partial X(A_t)}{\partial A_t}(x_1)\Big|\Big|\frac{\partial^2 X(A)}{\partial A^2}(x_2,x_3)\Big|\Big|\frac{\partial^2 X(A)}{\partial A^2}(x_4,x_5)\Big|\Big|\frac{\partial X(A_t)}{\partial A_t}(x_6)\Big|\right]dx_1\ldots dx_6dt\\ &+2\|h'\|^2_{\mathrm{L}^\infty}\|h''\|^2_{\mathrm{L}^\infty}\int_0^1\Big(1+\frac{2}{\sqrt{t}}\Big)\int\!\!\ldots\!\!\int|\mathcal{C}(x_1-x_2)||\mathcal{C}(x_2-x_3)||\mathcal{C}(x_3-x_4)|\\ &\times\mathbb{E}\left[\Big|\frac{\partial X(A_t)}{\partial A_t}(x_1)\Big|\Big|\frac{\partial X(A)}{\partial A}(x_2)\Big|\Big|\frac{\partial X(A)}{\partial A}(x_3)\Big|\Big|\frac{\partial X(A_t)}{\partial A_t}(x_4)\Big|\right]dx_1\ldots dx_4dt,\end{aligned}\tag{2.8}$$

where we have set $A_t(x):=h(\sqrt{t}G(x)+\sqrt{1-t}G'(x))$ for an i.i.d. copy G' of the Gaussian random field G (in particular note that A and A_t have the same law). This result is to be compared with [19].

Step 3. Conclusion.
In this step, we argue that (2.8) yields the desired second-order weighted Poincaré inequality. For all smooth $\zeta:\mathbb{R}^d\to\mathbb{R}$ and $\xi:\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$, we claim that the following estimate

holds,

$$
\begin{aligned}
T := \int \cdots \int |\zeta(x_1)||\xi(x_2,x_3)||\xi(x_4,x_5)||\zeta(x_6)||\mathcal{C}(x_1-x_2)||\mathcal{C}(x_3-x_4)||\mathcal{C}(x_5-x_6)|dx_1\dots dx_6 \\
\le \Big(\int_0^\infty d\ell_1(\ell_1+1)^{-d}(-c'(\ell_1))\int_0^\infty d\ell_2(\ell_2+1)^{-d}(-c'(\ell_2)) \\
\times \iint dx_1dx_2\Big(\iint_{B_{2(\ell_1+1)}(x_1)\times B_{2(\ell_2+1)}(x_2)} |\xi|\Big)^2\Big) \\
\times \Big(\int_0^\infty d\ell(\ell+1)^{-d}(-c'(\ell))\int dx\Big(\int_{B_{2(\ell+1)}(x)}|\zeta|\Big)^2\Big). \qquad (2.9)
\end{aligned}
$$

We postpone the proof of this estimate to the end of this step, and first show how it implies the desired result. We denote the two RHS terms of (2.8) by S_1 and S_2, respectively, and we start with the estimation of S_1. We apply inequality (2.9) to $\zeta(x) := (\partial X(A_t)/\partial A_t)(x)$ and $\xi(x,y) := (\partial^2 X(A)/\partial A^2)(x,y)$, use Cauchy-Schwarz' inequality in probability, and note that A_t has the same law as A for all t, so that

$$
\begin{aligned}
S_1 \le 2\|h'\|_{\mathrm{L}^\infty}^6 \int_0^1 \Big(1+\frac{2}{\sqrt{t}}\Big)\mathbb{E}\Big[\Big(\int_0^\infty d\ell(\ell+1)^{-d}(-c'(\ell))\int dx\Big(\int_{B_{2(\ell+1)}(x)}\Big|\frac{\partial X(A_t)}{\partial A_t}\Big|\Big)^2\Big) \\
\times\Big(\int_0^\infty d\ell_1(\ell_1+1)^{-d}(-c'(\ell_1))\int_0^\infty d\ell_2(\ell_2+1)^{-d}(-c'(\ell_2)) \\
\times\iint dxdy\Big(\iint_{B_{2(\ell_1+1)}(x)\times B_{2(\ell_2+1)}(y)}\Big|\frac{\partial^2 X(A)}{\partial A^2}\Big|\Big)^2\Big)\Big]dt \\
\le 10\|h'\|_{\mathrm{L}^\infty}^6\,\mathbb{E}\Big[\Big(\int_0^\infty d\ell(\ell+1)^{-d}(-c'(\ell))\int dx\Big(\int_{B_{2(\ell+1)}(x)}\Big|\frac{\partial X(A)}{\partial A}\Big|\Big)^2\Big)^2\Big]^{\frac12} \\
\times\mathbb{E}\Big[\Big(\int_0^\infty d\ell_1(\ell_1+1)^{-d}(-c'(\ell_1))\int_0^\infty d\ell_2(\ell_2+1)^{-d}(-c'(\ell_2)) \\
\times\iint dxdy\Big(\iint_{B_{2(\ell_1+1)}(x)\times B_{2(\ell_2+1)}(y)}\Big|\frac{\partial^2 X(A)}{\partial A^2}\Big|\Big)^2\Big)^2\Big]^{\frac12}.
\end{aligned}
$$

We now turn to the second term S_2. Taking local spatial averages, using Hölder's inequality in probability, and recalling that A_t has the same law as A for all t, we obtain

$$
\begin{aligned}
S_2 \le 10\|h'\|_{\mathrm{L}^\infty}^2\|h''\|_{\mathrm{L}^\infty}^2\int\cdots\int \bar{c}_R(|x_1-x_2|)\bar{c}_R(|x_2-x_3|)\bar{c}_R(|x_3-x_4|) \\
\times\prod_{i=1}^4\mathbb{E}\Big[\Big(\fint_{B_{R/2}(x_i)}\Big|\frac{\partial X(A)}{\partial A}\Big|\Big)^4\Big]^{\frac14}dx_1\dots dx_4,
\end{aligned}
$$

where we have set $\bar c_R(t) := \sup_{|u|\le R} c(t+u)$. Hence, since c is non-increasing,

$$S_2 \le 2^{4d} 10 \|h'\|^2_{\mathrm{L}^\infty} \|h''\|^2_{\mathrm{L}^\infty} \int\ldots\int c(|x_1-x_2|-R)\, c(|x_2-x_3|-R)\, c(|x_3-x_4|-R) \\ \times \prod_{i=1}^4 \mathbb{E}\Big[\Big(\fint_{B_R(x_i)} \Big|\frac{\partial X(A)}{\partial A}\Big|\Big)^4\Big]^{\frac14} dx_1\ldots dx_4.$$

The result (2.4) follows by inserting the above estimates for S_1 and S_2 into (2.8).

We now prove the result (2.5) in the case when $\int \bar{\mathcal C} < \infty$, where we have set $\bar{\mathcal C}(x) := \sup_{B_2(x)} |\mathcal C|$. Using the inequality $2ab \le a^2+b^2$ for all $a,b\in\mathbb R$, we obtain

$$\begin{aligned} S_1 &\lesssim \|h'\|^6_{\mathrm{L}^\infty} \int_0^1 \Big(1+\frac{2}{\sqrt t}\Big) \int\ldots\int \mathbb{E}\Big[\Big(\int_{B(x_1)} \Big|\frac{\partial X(A_t)}{\partial A_t}\Big|\Big)^2 \Big(\iint_{B(x_4)\times B(x_5)} \Big|\frac{\partial^2 X(A)}{\partial A^2}\Big|\Big)^2\Big] \\ &\qquad\qquad \times \bar{\mathcal C}(x_1-x_2)\bar{\mathcal C}(x_3-x_4)\bar{\mathcal C}(x_5-x_6) dx_1\ldots dx_6 dt \\ &\le \|h'\|^6_{\mathrm{L}^\infty} \|\bar{\mathcal C}\|^3_{\mathrm{L}^1} \int_0^1 \Big(1+\frac{2}{\sqrt t}\Big) \\ &\qquad \times \iiint \mathbb{E}\Big[\Big(\int_{B(x_1)} \Big|\frac{\partial X(A_t)}{\partial A_t}\Big|\Big)^2 \Big(\iint_{B(x_2)\times B(x_3)} \Big|\frac{\partial^2 X(A)}{\partial A^2}\Big|\Big)^2\Big] dx_1 dx_2 dx_3 dt \\ &\lesssim \|h'\|^6_{\mathrm{L}^\infty} \|\bar{\mathcal C}\|^3_{\mathrm{L}^1}\, \mathbb{E}\Big[\Big(\int\Big(\int_{B(x)} \Big|\frac{\partial X(A)}{\partial A}\Big|\Big)^2 dx\Big)^2\Big]^{\frac12} \\ &\qquad\qquad \times \mathbb{E}\Big[\Big(\iint\Big(\iint_{B(x)\times B(y)} \Big|\frac{\partial^2 X(A)}{\partial A^2}\Big|\Big)^2 dxdy\Big)^2\Big]^{\frac12}. \end{aligned}$$

Likewise, using the inequality $a_1a_2a_3a_4 \le \frac14 \sum_{i=1}^4 a_i^4$, we obtain

$$S_2 \lesssim \|h'\|^2_{\mathrm{L}^\infty} \|h''\|^2_{\mathrm{L}^\infty} \|\bar{\mathcal C}\|^3_{\mathrm{L}^1} \mathbb{E}\Big[\int\Big(\int_{B(x)} \Big|\frac{\partial X(A)}{\partial A}\Big|\Big)^4 dx\Big].$$

Combined with (2.8), these estimates yield the desired result (2.5).

It remains to prove the general estimate (2.9). Using radial coordinates, the LHS T takes the form

$$T \le \iiint dx_1dx_2dx_3 \int_0^\infty d\ell_1 c(\ell_1) \int_{\partial B_{\ell_1}} d\sigma(u_1)\ldots \int_0^\infty d\ell_3 c(\ell_3) \int_{\partial B_{\ell_3}} d\sigma(u_3) \\ \times |\zeta(x_1)||\xi(x_1+u_1,x_2)||\xi(x_2+u_2,x_3+u_3)||\zeta(x_3)|,$$

which, by integration by parts, turns into

$$T \le \iiint dx_1dx_2dx_3 \int_0^\infty d\ell_1(-c'(\ell_1)) \int_0^\infty d\ell_2(-c'(\ell_2)) \int_0^\infty d\ell_3(-c'(\ell_3)) \\ \times |\zeta(x_1)||\zeta(x_3)|\Big(\int_{B_{\ell_1}(x_1)} |\xi(\cdot,x_2)|\Big)\Big(\int_{B_{\ell_2}(x_2)\times B_{\ell_3}(x_3)} |\xi|\Big).$$

Taking local averages, and bounding $\int_{B_{\ell_1}(y_1)}$ by $\int_{B_{2(\ell_1+1)}(x_1)}$ for all $y_1 \in B_{\ell_1+1}(x_1)$, we directly deduce

$$T \le \iiint dx_1dx_2dx_3 \int_0^\infty d\ell_1(\ell_1+1)^{-d}(-c'(\ell_1))\dots\int_0^\infty d\ell_3(\ell_3+1)^{-d}(-c'(\ell_3)) \times\Big(\int_{B_{2(\ell_1+1)}(x_1)\times B_{2(\ell_2+1)}(x_2)}|\xi|\Big)\Big(\int_{B_{2(\ell_2+1)}(x_2)\times B_{2(\ell_3+1)}(x_3)}|\xi|\Big) \times\Big(\int_{B_{2(\ell_1+1)}(x_1)}|\zeta|\Big)\Big(\int_{B_{2(\ell_3+1)}(x_3)}|\zeta|\Big),$$

which, by the inequality $ab\le\frac12(a^2+b^2)$ for all $a,b\in\mathbb{R}$, yields

$$T \le \iiint dx_1dx_2dx_3 \int_0^\infty d\ell_1(\ell_1+1)^{-d}(-c'(\ell_1))\dots\int_0^\infty d\ell_3(\ell_3+1)^{-d}(-c'(\ell_3)) \times\Big(\int_{B_{2(\ell_1+1)}(x_1)}|\zeta|\Big)^2\Big(\int_{B_{2(\ell_2+1)}(x_2)\times B_{2(\ell_3+1)}(x_3)}|\xi|\Big)^2,$$

that is, (2.9). ☐

2.3. **Randomly localized fields.** Let A be a $\sigma(\mathcal{X})$-measurable random field on $\mathbb{R}^d$, where $\mathcal{X}$ is a completely independent random field on some measure space $X=\biguplus_{x\in\mathbb{Z}^d,t\in\mathbb{Z}^l}X_{x,t}$ with values in some measurable space M. In this subsection, we address the situation when the dependence pattern of A with respect to $\mathcal{X}$ is random in the sense that it is determined by the underlying product structure $\mathcal{X}$ itself. In this context, we first recall the crucial notion of *action radius* (cf. [6]), which is a probabilistic measure of the localization of this dependence pattern (that is inspired by the stabilization radius first introduced by Lee [14, 15] and crucially used in the works by Penrose, Schreiber, and Yukich on RSA processes [23, 22, 24, 27]).

Definition 2.4. Given an i.i.d. copy $\mathcal{X}'$ of the field $\mathcal{X}$, an *action radius for A with respect to $\mathcal{X}$ on $X_{x,t}$* (with reference perturbation $\mathcal{X}'$), if it exists, is defined as a nonnegative $\sigma(\mathcal{X},\mathcal{X}')$-measurable random variable ρ such that we have a.s.,

$$A(\mathcal{X}^{x,t})|_{\mathbb{R}^d\setminus(Q(x)+B_\rho)} = A(\mathcal{X})|_{\mathbb{R}^d\setminus(Q(x)+B_\rho)}\,,$$

where we recall that the perturbed random field $\mathcal{X}^{x,t}$ is defined by $\mathcal{X}^{x,t}|_{X\setminus X_{x,t}}:=\mathcal{X}|_{X\setminus X_{x,t}}$ and $\mathcal{X}^{x,t}|_{X_{x,t}}:=\mathcal{X}'|_{X_{x,t}}$. ☐

The following theorem establishes weighted second-order Poincaré inequalities for A, based on assumptions on a slightly stronger notion of action radius. The strategy consists in applying the standard second-order Poincaré inequality for $\mathcal{X}$ due to Chatterjee [3], and then exploiting the localization properties of the action radius to devise an approximate chain rule and deduce a functional inequality for $A=A(\mathcal{X})$ itself. As already discussed, this is to be compared with [12].

Theorem 2.5. *Let A be a $\sigma(\mathcal{X})$-measurable random field on $\mathbb{R}^d$, where $\mathcal{X}$ is a completely independent random field on some measure space $X=\biguplus_{x\in\mathbb{Z}^d,t\in\mathbb{Z}^l}X_{x,t}$ with values in some measurable space M. Let $\mathcal{X}'$ be an i.i.d. copy of $\mathcal{X}$. For all $B\subset\mathbb{Z}^d\times\mathbb{Z}^l$, let the perturbed random field $\mathcal{X}^B$ be defined by*

$$\mathcal{X}^B|_{\cup_{(x,t)\in B}X_{x,t}}=\mathcal{X}'|_{\cup_{(x,t)\in B}X_{x,t}},\qquad \mathcal{X}^B|_{\cup_{(x,t)\notin B}X_{x,t}}=\mathcal{X}|_{\cup_{(x,t)\notin B}X_{x,t}},$$

and for all $x,x'\in\mathbb{Z}^d$ and $t,t'\in\mathbb{Z}^l$ we set for simplicity $\mathcal{X}^{x,t}:=\mathcal{X}^{\{(x,t)\}}$ and $\mathcal{X}^{x,t;x',t'}:=\mathcal{X}^{\{(x,t),(x',t')\}}$. Assume that

(a) For all x,t and all $B\subset\mathbb{Z}^d\times\mathbb{Z}^l$, there exists an action radius $\rho_{x,t}(\mathcal{X}^B)$ for $A(\mathcal{X}^B)$ with respect to $\mathcal{X}^B$ in $X_{x,t}$ with reference perturbation $\mathcal{X}'$ (in the sense of Definition 2.4), and set

$$\tilde\rho_{x,t}:=\sup\big\{\rho_{x,t}(\mathcal{X}^B)\,:\,B\subset\mathbb{Z}^d\times\mathbb{Z}^l\big\}.$$

(b) The transformation A of $\mathcal{X}$ is stationary, that is, the random fields $A(\mathcal{X}(\cdot+z,\cdot))$ and $A(\mathcal{X})(\cdot+z)$ have the same law for all $z\in\mathbb{Z}^d$. Moreover, for all t,B, the law of the action radius $\rho_{x,t}(\mathcal{X}^B)$ is independent of x. In particular, for all t, the law of $\tilde\rho_{x,t}$ is independent of x.

For all $t\in\mathbb{Z}^l$ and $\ell\geq 1$, define the weight

$$\pi(t,\ell):=\mathbb{P}\left[\ell-1\leq\tilde\rho_{0,t}<\ell\,,\;\mathcal{X}\neq\mathcal{X}^{0,t}\right].$$

Then the following results hold.

(i) For all $\sigma(A)$-measurable random variables $X=X(A)$, we have

$$\begin{aligned}
&\mathrm{d_W}\left(\frac{X-\mathbb{E}[X]}{\sqrt{\mathrm{Var}[X]}},\mathcal{N}\right)\\
&\lesssim\frac{1}{\mathrm{Var}[X]}\inf_{0<\lambda<1}\Bigg(\sum_{x,x',x''}\sum_{t,t',t''}\sum_{\ell,\ell',\ell''=1}^{\infty}\big(\pi(t,\ell)^{\frac13}\pi(t',\ell')^{\frac13}\pi(t'',\ell'')^{\frac13}\big)^{\lambda}\\
&\qquad\times\mathbb{E}\left[\Big(\partial^{\mathrm{dis}}_{\ell,x,t}\partial^{\mathrm{dis}}_{\ell',x',t'}X\Big)^{\frac{4}{1-\lambda}}\right]^{\frac{1-\lambda}{4}}\mathbb{E}\left[\Big(\partial^{\mathrm{dis}}_{\ell,x,t}\partial^{\mathrm{dis}}_{\ell'',x'',t''}X\Big)^{\frac{4}{1-\lambda}}\right]^{\frac{1-\lambda}{4}}\\
&\qquad\times\mathbb{E}\left[\Big(\partial^{\mathrm{dis}}_{\ell',x',t'}X\Big)^{\frac{4}{1-\lambda}}\right]^{\frac{1-\lambda}{4}}\mathbb{E}\left[\Big(\partial^{\mathrm{dis}}_{\ell'',x'',t''}X\Big)^{\frac{4}{1-\lambda}}\right]^{\frac{1-\lambda}{4}}\Bigg)^{\frac12}\\
&\quad+\frac{1}{\mathrm{Var}[X]}\inf_{0<\lambda<1}\Bigg(\sum_{x,x'}\sum_{t,t'}\sum_{\ell,\ell'=1}^{\infty}\big(\pi(t,\ell)^{\frac12}\pi(t',\ell')^{\frac12}\big)^{\lambda}\\
&\qquad\times\mathbb{E}\left[\Big(\partial^{\mathrm{dis}}_{\ell,x,t}\partial^{\mathrm{dis}}_{\ell',x',t'}X\Big)^{\frac{4}{1-\lambda}}\right]^{\frac{1-\lambda}{2}}\mathbb{E}\left[\Big(\partial^{\mathrm{dis}}_{\ell',x',t'}X\Big)^{\frac{4}{1-\lambda}}\right]^{\frac{1-\lambda}{2}}\Bigg)^{\frac12}\\
&\quad+\frac{1}{\mathrm{Var}[X]}\inf_{0<\lambda<1}\Bigg(\sum_{x}\sum_{t}\sum_{\ell=1}^{\infty}\pi(t,\ell)^{\lambda}\mathbb{E}\left[(\partial^{\mathrm{dis}}_{\ell,x,t}X)^{\frac{4}{1-\lambda}}\right]^{1-\lambda}\Bigg)^{\frac12}\\
&\quad+\frac{1}{\mathrm{Var}[X]^{3/2}}\inf_{0<\lambda<1}\sum_{x}\sum_{t}\sum_{\ell=1}^{\infty}\pi(t,\ell)^{\lambda}\mathbb{E}\left[\Big(\partial^{\mathrm{dis}}_{\ell,x,t}X\Big)^{\frac{3}{1-\lambda}}\right]^{1-\lambda},
\end{aligned}\tag{2.10}$$

where the sums in x,x',x'' (resp. in t,t',t'') implicitly run over $\mathbb{Z}^d$ (resp. over $\mathbb{Z}^l$), and where for all $x\in\mathbb{Z}^d$ and $t\in\mathbb{Z}^l$ we have defined the discrete derivative

$$\partial^{\mathrm{dis}}_{\ell,x,t}X:=\big(X(A)-X(A(\mathcal{X}^{x,t}))\big)\mathbb{1}_{A(\mathcal{X}^{x,t})|_{\mathbb{R}^d\setminus Q_{2\ell+1}(x)}=A|_{\mathbb{R}^d\setminus Q_{2\ell+1}(x)}}$$

and the discrete second derivative

$$\partial^{\mathrm{dis}}_{\ell,x,t}\partial^{\mathrm{dis}}_{\ell',x',t'}X := \big(X(A) - X(A(\mathcal{X}^{x,t})) - X(A(\mathcal{X}^{x',t'})) + X(A(\mathcal{X}^{x,t;x',t'}))\big) \times \mathbb{1}_{A(\mathcal{X}^{x,t})|_{\mathbb{R}^d\setminus Q_{2\ell+1}(x)}=A|_{\mathbb{R}^d\setminus Q_{2\ell+1}(x)}}\mathbb{1}_{A(\mathcal{X}^{x,t;x',t'})|_{\mathbb{R}^d\setminus Q_{2\ell+1}(x)}=A(\mathcal{X}^{x',t'})|_{\mathbb{R}^d\setminus Q_{2\ell+1}(x)}} \times \mathbb{1}_{A(\mathcal{X}^{x',t'})|_{\mathbb{R}^d\setminus Q_{2\ell'+1}(x')}=A|_{\mathbb{R}^d\setminus Q_{2\ell'+1}(x')}}\mathbb{1}_{A(\mathcal{X}^{x,t;x',t'})|_{\mathbb{R}^d\setminus Q_{2\ell'+1}(x')}=A(\mathcal{X}^{x,t})|_{\mathbb{R}^d\setminus Q_{2\ell'+1}(x')}}.$$

(ii) *For all $\sigma(A)$-measurable random variables $X = X(A)$, we have*

$$\mathrm{d}_{\mathrm{K}}\left(\frac{X-\mathbb{E}[X]}{\sqrt{\mathrm{Var}[X]}},\mathcal{N}\right) \lesssim \mathrm{RHS}_{(2.10)}(X) + G_1(X), \tag{2.11}$$

where $\mathrm{RHS}_{(2.10)}(X)$ *denote the RHS of* (2.10)*, and where we have set*

$$G_1(X) := \frac{1}{\mathrm{Var}[X]^{3/2}}\inf_{0<\lambda<1}\sum_x\sum_t\left(\sum_{\ell=1}^{\infty}\pi(t,\ell)^{\lambda}\,\mathbb{E}\left[\left(\partial^{\mathrm{dis}}_{\ell,x,t}X\right)^{\frac{6}{1-\lambda}}\right]^{1-\lambda}\right)^{\frac12}.$$

If in addition for all x,t there exists a $\sigma(\mathcal{X}|_{X_{x,t}},\mathcal{X}'|_{X_{x,t}})$-measurable action radius $\rho_{x,t}$ for $A(\mathcal{X})$ with respect to $\mathcal{X}$ on $X_{x,t}$, then we simply have $\tilde\rho_{x,t} = \rho_{x,t}$ for all x,t, the weights $\pi^{\frac13}$ and $\pi^{\frac12}$ can both be replaced by π in the first two RHS terms of (2.10) *and in the corresponding terms in* $\mathrm{RHS}_{(2.10)}(X)$ *in* (2.11)*, and the term $G_1(X)$ in* (2.11) *can be replaced by*

$$G_2(X) := \frac{1}{\mathrm{Var}[X]^{3/2}}\inf_{0<\lambda<1}\sum_x\sum_t\sum_{\ell=1}^{\infty}\pi(t,\ell)^{\lambda}\,\mathbb{E}\left[\left(\partial^{\mathrm{dis}}_{\ell,x,t}X\right)^{\frac{6}{1-\lambda}}\right]^{\frac{1-\lambda}{2}}. \qquad \square$$

Remark 2.6. The additional term $G_1(X)$ in (2.11) typically dominates the RHS terms of (2.10). However they become of the same order if the weight π is super-algebraically decaying, or if the improved form of the above result holds (that is, with $G_1(x)$ replaced by $G_2(X)$). In each of the examples below, we are in one of these two situations, hence the above bounds on the Kolmogorov and on the Wasserstein distances essentially coincide. Otherwise, it might be advantageous to rather bound the Kolmogorov distance by the square-root of the Wasserstein distance and then use the above estimate for the latter. $\square$

Before we turn to the proof of Theorem 2.5, we recall representative examples analyzed in [6, Section 3], and to which it applies. In each case, we quickly discuss the existence and properties of the action radius $\tilde\rho$ (which is a slightly stronger notion of action radius than the one ρ given in Definition 2.4 and needed for first-order weighted functional inequalities). For technical details we refer the reader to [6, Section 3], where the action radii ρ are constructed.

(A) *Poisson unbounded spherical inclusion model.* Consider a Poisson point process $\mathcal{P}$ of unit intensity on $\mathbb{R}^d$. For each Poisson point $x \in \mathcal{P}$ consider a random radius $r(x)$ (independent of the radii of other points and identically distributed according to some given law ν on $\mathbb{R}^+$), and define the inclusion $C_x := B_{r(x)}(x)$. Consider the inclusion set $\mathcal{I} := \cup_{x\in\mathcal{P}}C_x$, let $A_0, A_1 \in \mathbb{R}$ be given values, and define a random field A on $\mathbb{R}^d$ by

$$A(x) := A_0\mathbb{1}_{x\notin\mathcal{I}} + A_1\mathbb{1}_{x\in\mathcal{I}},$$

that is, A takes value A_1 in the inclusions and A_0 outside. As argued in [6, Subsection 3.4], A can be reformulated in the form addressed in Theorem 2.5 above with $l=1$, and for all x,t there exists a $\sigma(\mathcal{X}|_{X_{x,t}},\mathcal{X}'|_{X_{x,t}})$-measurable action radius $\rho_{x,t}:=t\,\mathbb{1}_{\mathcal{X}\neq\mathcal{X}^{x,t}}$ (cf. [6, proof of Proposition 3.4(i)]). The improved form of the above result therefore holds with

$$\pi(t,\ell):=\mathbb{1}_{\ell-1\le t<\ell}\,\mathbb{P}\left[\mathcal{X}\neq\mathcal{X}^{0,t}\right]\le 2\,\nu([t-\tfrac12,t+\tfrac12))\,\mathbb{1}_{\ell-1\le t<\ell}.$$

(B) *Random parking process.* Consider the random parking point process $\mathcal{R}$ with unit radius on $\mathbb{R}^d$ (see Subsection 3.2 below for a precise construction based on an underlying Poisson point process $\mathcal{P}_0$ of unit intensity on $\mathbb{R}^d\times\mathbb{R}_+$). As above, for all $x\in\mathcal{R}$ we denote by $C_x:=B(x)$ the unit spherical inclusion centered at x (so that by definition of $\mathcal{R}$ all the inclusions are disjoint), we consider the inclusion set $\mathcal{I}:=\cup_{x\in\mathcal{R}}C_x$, and we define a random field A on $\mathbb{R}^d$ by

$$A(x)\;:=\;A_0\mathbb{1}_{x\notin\mathcal{I}}+A_1\mathbb{1}_{x\in\mathcal{I}}.$$

In [6, proof of Proposition 3.3], for all x we have constructed an action radius ρ_x with respect to the underlying Poisson point process $\mathcal{P}_0$ on $Q(x)\times\mathbb{R}_+$. By definition, this action radius satisfies $\rho_x(\mathcal{P}_0^B)\le\rho_x(\mathcal{P}_0\cup\mathcal{P}_0')$ for all $B\subset\mathbb{Z}^d$: indeed, adding points in the Poisson point process $\mathcal{P}_0$ adds possible causal chains, hence increases the defined action radius. Therefore, we deduce $\tilde\rho_x\le\rho_x(\mathcal{P}_0\cup\mathcal{P}_0')$. As $\mathcal{P}_0\cup\mathcal{P}_0'$ is itself a Poisson point process on $\mathbb{R}^d\times\mathbb{R}_+$ with doubled intensity, we conclude $\mathbb{P}\left[\tilde\rho_x\ge\ell\right]\le C\exp(-\frac1C\ell)$ as in [6, Proposition 3.3], and we may apply Theorem 2.5 with $l=0$ and exponential weight $\pi(\ell)\le C\exp(-\frac1C\ell)$.

(C) *Poisson random tessellations.* Consider a Poisson point process $\mathcal{P}$ on $\mathbb{R}^d$, and let $\mathcal{V}$ denote the associated Voronoi tessellation of $\mathbb{R}^d$, that is, a partition of $\mathbb{R}^d$ into convex polyhedra $V_x\in\mathcal{V}$ centered at the Poisson points $x\in\mathcal{P}$. For each point $x\in\mathcal{P}$ consider a random value $\alpha(x)$ (independent of the values at other points and identically distributed), and we define a random field A on $\mathbb{R}^d$ by

$$A(x)\;:=\;\sum_{y\in\mathcal{P}}\alpha(y)\mathbb{1}_{x\in V_y}.$$

As argued in [6, proof of Proposition 3.3], A can be reformulated in the form addressed in Theorem 2.5 above with $l=0$ and with weight

$$\pi(\ell)\le\mathbb{P}\left[\tilde\rho_x\ge\ell-1\right]\le C\exp\Big(-\frac1C\ell^d\Big).\tag{2.12}$$

(More precisely, we argue as follows: Denote by $C_i:=\{x\in\mathbb{R}^d: x_i\ge\frac56|x|\}$, $1\le i\le d$, the d cones in the canonical directions e_i of $\mathbb{R}^d$, and consider the $2d$ cones $C_i^\pm:=\pm(2e_i+C_i)$. For all x, let $\rho_x:=\rho_x^0$ denote the action radius for A defined in [6, proof of Proposition 3.2], and let $\tilde\rho_x$ be defined as in the statement of Theorem 2.5 above. By construction, the inequality $\tilde\rho_x\le CL$ holds if for each cone $C_i^\pm$ there exists a cube $Q\subset C_i^\pm\cap\{x:|x_i|\le L\}$ such that $\mathcal{P}_0\cap Q\neq\varnothing\neq\mathcal{P}_0'\cap Q$. By independence of $\mathcal{P}_0$ and $\mathcal{P}_0'$, and by a union bound, the claim (2.12) follows.)

Proof of Theorem 2.5. We split the proof into two steps. First note that by approximation it is enough to prove the result for $\sigma(\mathcal{X}|_{\cup_{(x,t)\in E}(Q(x)\times Q(t))})$-measurable random variables $X=X(\mathcal{X})$ for a finite set $E\subset\mathbb{Z}^d\times\mathbb{Z}^l$. Let such a finite set E and such a random variable X be fixed.

Step 1. Application of a result by Chatterjee.
By [3, Theorem 2.2] (together with the standard spectral gap (2.2)), we have

$$\mathrm{d_W}\left(\frac{X-\mathbb{E}[X]}{\sqrt{\mathrm{Var}[X]}},\mathcal{N}\right)\lesssim\frac{1}{\mathrm{Var}[X]^{3/2}}\sum_{x,t}\mathbb{E}\left[|\Delta_{x,t}X|^3\right]$$
$$+\frac{1}{\mathrm{Var}[X]}\left(\sum_{x,t}\mathbb{E}\Big[\Big|\sum_{x',t'}(\Delta_{x,t}\Delta_{x',t'}X)\overline{\Delta_{x',t'}X}\Big|^2\Big]\right)^{\frac12}$$
$$+\frac{1}{\mathrm{Var}[X]}\left(\sum_{x,t}\mathbb{E}\Big[\Big|\sum_{x',t'}(\Delta_{x,t}\overline{\Delta_{x',t'}X})\Delta_{x',t'}X\Big|^2\Big]\right)^{\frac12},\quad (2.13)$$

where the sums in (x,t) and (x',t') implicitly run over E, and where we have set

$$\Delta_{x,t}X(\mathcal{X}^B):=X(\mathcal{X}^B)-X(\mathcal{X}^{B\cup\{(x,t)\}}),$$
$$\overline{\Delta_{x,t}X}:=\sum_{\substack{B\subset E\\(x,t)\notin B}}K_B\Delta_{x,t}X(\mathcal{X}^B),\qquad K_B:=\frac{|B|!(|E|-|B|-1)!}{|E|!}.$$

Note that by definition $\sum_{B\subset E:(x,t)\notin B}K_B=1$. By [11, Theorem 4.2] (together with the standard spectral gap (2.2)), the following estimate on the Kolmogorov distance also holds

$$\mathrm{d_K}\left(\frac{X-\mathbb{E}[X]}{\sqrt{\mathrm{Var}[X]}},\mathcal{N}\right)\lesssim\mathrm{RHS}(X)+\frac{1}{\mathrm{Var}[X]^{3/2}}\mathbb{E}\Big[\Big(\sum_{x,t}|\Delta_{x,t}X|^2\,\overline{\overline{\Delta_{x,t}X}}\Big)^2\Big]^{\frac12}$$
$$+\frac{1}{\mathrm{Var}[X]}\left(\sum_{x,t}\mathbb{E}\Big[\Big|\sum_{x',t'}(\Delta_{x,t}\Delta_{x',t'}X)\overline{\overline{\Delta_{x',t'}X}}\Big|^2\Big]\right)^{\frac12}$$
$$+\frac{1}{\mathrm{Var}[X]}\left(\sum_{x,t}\mathbb{E}\Big[\Big|\sum_{x',t'}(\Delta_{x,t}\overline{\overline{\Delta_{x',t'}X}})\Delta_{x',t'}X\Big|^2\Big]\right)^{\frac12},\quad (2.14)$$

where $\mathrm{RHS}(X)$ stands for the RHS of (2.13) above, and

$$\overline{\overline{\Delta_{x,t}X}}:=\sum_{\substack{B\subset E\\(x,t)\notin B}}K_B|\Delta_{x,t}X(\mathcal{X}^B)|.$$

Only the first RHS term of (2.14) (after $\mathrm{RHS}(X)$) will lead the correction $G_1(X)$ in (2.11) with respect to (2.10).

Step 2. Conditioning with respect to the action radius.
In this step we reformulate the RHSs of (2.13) and (2.14) by introducing the action radius $\rho_{x,t}$ for A with respect to $\mathcal{X}$. We only address the second RHS term in (2.13) since all the other terms can be treated similarly. To simplify notation, we write $z:=(x,t)$ and $Q(z):=Q(x)\times Q(t)$. We start by expanding the square and by distinguishing cases when

the differences Δ_z are taken at the same points,

$$\begin{aligned}
&\sum_z \mathbb{E}\left[\Big|\sum_{z'}(\Delta_z\Delta_{z'}X)\overline{\Delta_{z'}X}\Big|^2\right] \\
&\quad\le \sum_{z,z',z''} \mathbb{E}\left[|\Delta_z\Delta_{z'}X||\Delta_z\Delta_{z''}X||\overline{\Delta_{z'}X}||\overline{\Delta_{z''}X}|\right] \\
&\quad= \sum_z \mathbb{E}\left[|\Delta_z X|^2|\overline{\Delta_z X}|^2\right] + 2\sum_{z\ne z'}\mathbb{E}\left[|\Delta_z\Delta_{z'}X||\Delta_z X||\overline{\Delta_z X}||\overline{\Delta_{z'}X}|\right] \qquad (2.15)\\
&\qquad + \sum_{z\ne z'}\mathbb{E}\left[|\Delta_z\Delta_{z'}X|^2|\overline{\Delta_{z'}X}|^2\right] + \sum_{\substack{z,z',z''\\ \text{distinct}}}\mathbb{E}\left[|\Delta_z\Delta_{z'}X||\Delta_z\Delta_{z''}X||\overline{\Delta_{z'}X}||\overline{\Delta_{z''}X}|\right],
\end{aligned}$$

where we used the fact that $\Delta_z\Delta_z X = \Delta_z X$. We then reformulate the four RHS terms by introducing the action radius. We only treat the last term in detail (the other terms are similar). Since the product $|\Delta_z\Delta_{z'}X||\Delta_z\Delta_{z''}X||\overline{\Delta_{z'}X}||\overline{\Delta_{z''}X}|$ vanishes whenever $\mathcal{X}|_{Q(z)} = \mathcal{X}'|_{Q(z)}$ or $\mathcal{X}|_{Q(z')} = \mathcal{X}'|_{Q(z')}$ or $\mathcal{X}|_{Q(z'')} = \mathcal{X}'|_{Q(z'')}$, we obtain after conditioning with respect to the values of $\tilde\rho_z$, $\tilde\rho_{z'}$ and $\tilde\rho_{z''}$ (that is, the stronger notion of action radii defined in the statement),

$$\begin{aligned}
&\sum_{\substack{z,z',z''\\ \text{distinct}}}\mathbb{E}\left[|\Delta_z\Delta_{z'}X||\Delta_z\Delta_{z''}X||\overline{\Delta_{z'}X}||\overline{\Delta_{z''}X}|\right] \\
&\qquad\qquad\le \sum_{\ell,\ell',\ell''=1}^{\infty}\sum_{\substack{z,z',z''\\ \text{distinct}}}\mathbb{E}\Big[|\Delta_z\Delta_{z'}X||\Delta_z\Delta_{z''}X||\overline{\Delta_{z'}X}||\overline{\Delta_{z''}X}| \\
&\times \mathbb{1}_{\ell-1\le\tilde\rho_z<\ell}\,\mathbb{1}_{\mathcal{X}|_{Q(z)}\ne\mathcal{X}'|_{Q(z)}}\mathbb{1}_{\ell'-1\le\tilde\rho_{z'}<\ell'}\,\mathbb{1}_{\mathcal{X}|_{Q(z')}\ne\mathcal{X}'|_{Q(z')}}\mathbb{1}_{\ell''-1\le\tilde\rho_{z''}<\ell''}\,\mathbb{1}_{\mathcal{X}|_{Q(z'')}\ne\mathcal{X}'|_{Q(z'')}}\Big].
\end{aligned}$$

Note that the event $\tilde\rho_z < \ell$ entails by definition $A(\mathcal{X}^B)|_{\mathbb{R}^d\setminus Q_{2\ell+1}(x)} = A(\mathcal{X}^{B\cup\{z\}})|_{\mathbb{R}^d\setminus Q_{2\ell+1}(x)}$ for all $B\subset E$. By Hölder's inequality and by definition of ∂^{dis} and $\partial^{\text{dis}}\partial^{\text{dis}}$, we then obtain for all $0<\lambda<1$,

$$\begin{aligned}
&\sum_{\substack{z,z',z''\\ \text{distinct}}}\mathbb{E}\left[|\Delta_z\Delta_{z'}X||\Delta_z\Delta_{z''}X||\overline{\Delta_{z'}X}||\overline{\Delta_{z''}X}|\right] \le \sum_{\ell,\ell',\ell''=1}^{\infty}\sum_{\substack{z,z',z''\\ \text{distinct}}}\sum_{\substack{B'\subset E\\ z'\notin B'}}K_{B'}\sum_{\substack{B''\subset E\\ z''\notin B''}}K_{B''} \\
&\times \mathbb{E}\left[\mathbb{1}_{\ell-1\le\tilde\rho_z<\ell}\,\mathbb{1}_{\mathcal{X}|_{Q(z)}\ne\mathcal{X}'|_{Q(z)}}\mathbb{1}_{\ell'-1\le\tilde\rho_{z'}<\ell'}\,\mathbb{1}_{\mathcal{X}|_{Q(z')}\ne\mathcal{X}'|_{Q(z')}}\mathbb{1}_{\ell''-1\le\tilde\rho_{z''}<\ell''}\,\mathbb{1}_{\mathcal{X}|_{Q(z'')}\ne\mathcal{X}'|_{Q(z'')}}\right]^{\lambda} \\
&\qquad\times \mathbb{E}\left[\Big(\big|\partial^{\text{dis}}_{\ell,z}\partial^{\text{dis}}_{\ell',z'}X(\mathcal{X})\big|\big|\partial^{\text{dis}}_{\ell,z}\partial^{\text{dis}}_{\ell'',z''}X(\mathcal{X})\big|\big|\partial^{\text{dis}}_{\ell',z'}X(\mathcal{X}^{B'})\big|\big|\partial^{\text{dis}}_{\ell'',z''}X(\mathcal{X}^{B''})\big|\Big)^{\frac{1}{1-\lambda}}\right]^{1-\lambda}.
\end{aligned}$$

Again applying Hölder's inequality, noting that $\sum_{B\subset E:z\notin B} K_B = 1$, and recalling that $\mathcal{X}$ and $\mathcal{X}^B$ have the same law for all $B \subset E$, we conclude

$$\sum_{\substack{z,z',z''\\ \text{distinct}}} \mathbb{E}\left[|\Delta_z\Delta_{z'}X||\Delta_z\Delta_{z''}X||\overline{\Delta_{z'}X}||\overline{\Delta_{z''}X}|\right]$$

$$\le \sum_{\ell,\ell',\ell''=1}^{\infty}\sum_{z,z',z''}\big(\pi(t,\ell)\pi(t',\ell')\pi(t'',\ell'')\big)^{\frac{\lambda}{3}}\,\mathbb{E}\left[\big|\partial^{\text{dis}}_{\ell,z}\partial^{\text{dis}}_{\ell',z'}X\big|^{\frac{4}{1-\lambda}}\right]^{\frac{1-\lambda}{4}}\mathbb{E}\left[\big|\partial^{\text{dis}}_{\ell,z}\partial^{\text{dis}}_{\ell'',z''}X\big|^{\frac{4}{1-\lambda}}\right]^{\frac{1-\lambda}{4}}$$

$$\times\,\mathbb{E}\left[\big|\partial^{\text{dis}}_{\ell',z'}X\big|^{\frac{4}{1-\lambda}}\right]^{\frac{1-\lambda}{4}}\mathbb{E}\left[\big|\partial^{\text{dis}}_{\ell'',z''}X\big|^{\frac{4}{1-\lambda}}\right]^{\frac{1-\lambda}{4}}. \quad (2.16)$$

The other terms in (2.15) can be treated similarly, and the results (i)–(ii) follow. Finally note that if for all z there is an action radius ρ_z for A with respect to $\mathcal{X}$ on $Q(z)$ which is $\sigma(\mathcal{X}|_{Q(z)},\mathcal{X}'|_{Q(z)})$-measurable, then the complete independence of $\mathcal{X}$ ensures that $\tilde\rho_z$, $\tilde\rho_{z'}$ and $\tilde\rho_{z''}$ are independent for z,z',z'' distinct, so that we simply obtain

$$\mathbb{E}\Big[\mathbb{1}_{\ell-1\le\tilde\rho_z<\ell}\,\mathbb{1}_{\mathcal{X}|_{Q(z)}\neq\mathcal{X}'|_{Q(z)}}\mathbb{1}_{\ell'-1\le\tilde\rho_{z'}<\ell'}\,\mathbb{1}_{\mathcal{X}|_{Q(z')}\neq\mathcal{X}'|_{Q(z')}}\mathbb{1}_{\ell''-1\le\tilde\rho_{z''}<\ell''}\,\mathbb{1}_{\mathcal{X}|_{Q(z'')}\neq\mathcal{X}'|_{Q(z'')}}\Big] = \pi(t,\ell)\pi(t',\ell')\pi(t'',\ell'').$$

The exponent $\frac{1}{3}$ can then be removed from the weights in (2.16), and the corresponding improved result follows. ☐

3. Application to spatial averages and to RSA models

3.1. **Spatial averages of the random field.** In this subsection, we investigate the approximate normality of the spatial averages $X_L := X_L(A) := \fint_{Q_L}(A-\mathbb{E}[A])$ of the random field (more general X_L can be considered as well, replacing the field A by an approximately local function thereof, at the price of further assumptions on second derivatives). We focus on two prototypical examples: Gaussian random fields, and Poisson random inclusions with (unbounded) random radii.

Proposition 3.1. *We consider the two examples separately.*

(i) Let G be a jointly measurable stationary Gaussian random field on $\mathbb{R}^d$, characterized by its covariance $\mathcal{C}(x) := \mathrm{Cov}[G(x);G(0)]$, and assume that $\sup_{B(x)}|\mathcal{C}| \le c(|x|)$ for some Lipschitz non-increasing map $c:\mathbb{R}_+\to\mathbb{R}_+$. Let $h\in C^2(\mathbb{R})$ with $h',h''\in \mathrm{L}^\infty(\mathbb{R})$, and let A be the random field on $\mathbb{R}^d$ defined by $A(x) := h(G(x))$ for all x. Set $\pi(\ell) = -c'(\ell)$, and define

$$\pi_*(\ell) := \Big(\fint_{B_\ell}\int_{|x|}^{\infty}\pi(s)dsdx\Big)^{-1}.$$

Then the results in [5, Proposition 4.1] *ensure that the rescaled random variable $Z_L := \pi_*(L)^{1/2}X_L$ satisfies $\sigma_L^2 := \mathrm{Var}[Z_L] \lesssim 1$. Moreover we have for all $L\ge 1$,*

$$\mathrm{d_{TV}}\left(\frac{Z_L}{\sigma_L},\mathcal{N}\right) \lesssim \sigma_L^{-2}\pi_*(L)^{-\frac{1}{2}}. \quad (3.1)$$

(ii) Let the random field A be given by Poisson unbounded spherical inclusion model with radius law ν (cf. example (A) in Subsection 2.3), and assume that the law ν satisfies for some $\beta>0$,

$$\gamma(\ell) := \nu([\ell,\ell+1)) \lesssim \ell^{-3d-\beta-1}.$$

Then the results in [5, Proposition 4.1] *hold with weight* $\pi(\ell) = (\ell+1)^{-2d-\beta-1}$ *and* $\pi_*(L) = L^d$*, and the rescaled random variable* $Z_L := L^{d/2}X_L$ *satisfies* $\sigma_L^2 := \operatorname{Var}[Z_L] \lesssim 1$*. Moreover we have for all* $L \ge 1$*,*

$$\mathrm{d_W}\left(\frac{Z_L}{\sigma_L},\mathcal{N}\right) + \mathrm{d_K}\left(\frac{Z_L}{\sigma_L},\mathcal{N}\right) \lesssim L^{-\frac{d}{2}}\big(1+L^{d-\beta}\big)^{\frac12}\big(1+\sigma_L^{-3}\big). \qquad \square$$

A similar result as above holds in stochastic homogenization, where Z_L is replaced by the spatial average of the homogenization commutator [7]. As e.g. in [3], we consider that estimating $\sigma_L \lesssim 1$ from below is a separate issue. In the Gaussian case with integrable covariance function, we do not believe this is essential. In that case, if h is for instance an increasing function, then one can indeed prove that $\sigma_L \gtrsim 1$ (see for instance [8, Proposition 2.1] for a similar argument in stochastic homogenization, starting from a lower bound for variances proved in [31]). In the Gaussian case with non-integrable covariance, the question of bounding σ_L from below is more subtle. It is typically related to the Hermite rank of the function h and may lead to different scalings than π_*, in which case approximate normality may fail. We refer the reader to the recent works [9, 13] in the context of one-dimensional stochastic homogenization, and more generally to [28].

Before we turn to the proof of this result, let us discuss its optimality. We believe that for Gaussian random fields Proposition 3.1(i) is generically optimal. Optimality is less clear for Proposition 3.1(ii), as the comparison to results based on α-mixing suggests. Let us briefly recall the definition of α-mixing first introduced by Rosenblatt [25]. For all sub-σ-algebras $\mathcal{G}_1,\mathcal{G}_2 \subset \mathcal{A}$, their α-mixing coefficient $\alpha(\mathcal{G}_1,\mathcal{G}_2)$ is given by

$$\alpha(\mathcal{G}_1,\mathcal{G}_2) := \sup\big\{|\mathbb{P}[G_1\cap G_2]-\mathbb{P}[G_1]\mathbb{P}[G_2]| \,:\, G_1\in\mathcal{G}_1, G_2\in\mathcal{G}_2\big\},$$

and the α-mixing coefficient for the random field A is then defined as follows, for all diameters $D\in(0,\infty]$ and distances $R>0$,

$$\tilde\alpha(R,D;A) := \sup\big\{\alpha(\sigma(A|_{S_1}),\sigma(A|_{S_2})) \,:\, S_1,S_2\in\mathcal{B}(\mathbb{R}^d),\, d(S_1,S_2)\ge R,\\ \operatorname{diam}(S_1),\operatorname{diam}(S_2)\le D\big\}.$$

For this discussion, we restrict to the more documented case of dimension $d=1$. Two results are available on approximate normality for spatial averages of α-mixing random fields. The first result is classical and due to Ibragimov (see e.g. [2]): it ensures that a *qualitative* central limit theorem (CLT) holds for $Z_L := L^{1/2}X_L$ whenever for some $\kappa>1$ the field A satisfies $\tilde\alpha(R,\infty;A)\lesssim R^{-\kappa}$ for all $R\ge1$. The second result is due to Pène [20, Theorem 1.1] and essentially shows that Z_L satisfies a *quantitative* CLT in 1-Wasserstein distance with optimal rate $L^{-1/2}$ whenever for some $\kappa>2$ there holds $\tilde\alpha(R,\infty;A)\lesssim R^{-\kappa}$ for all $R\ge1$. Let us compare these results with the statement of Proposition 3.1(ii) above. For the Poisson unbounded spherical inclusion model with radius law ν (cf. example (A) in Subsection 2.3), assuming that $\gamma(\ell) := \nu([\ell,\ell+1)) \simeq (\ell+1)^{-\kappa-d-1}$ with $\kappa>0$, we proved in [5, Proposition 2.5(iii)] and [6, Proposition 3.4(i)] that for any fixed diameter $D>0$ the α-mixing coefficient satisfies $\tilde\alpha(R,D;A)\lesssim_D R^{-\kappa}$ for all $R\ge1$, while Proposition 3.1(ii) above for $d=1$ yields a qualitative CLT whenever $\kappa>2$, and a CLT in 1-Wasserstein distance with optimal rate $L^{-1/2}$ whenever $\kappa>3$. Comparing this with the results by Ibragimov and by Pène, there is thus a discrepancy in the critical values of κ, which suggests that Proposition 3.1(ii) might not be optimal. Nevertheless, in the Poisson unbounded spherical model under consideration one can prove that $\inf_{R\ge1}\tilde\alpha(R,\infty;A)>0$, so that strictly speaking the results by Ibragimov and Pène do not apply — no general CLT result seems to be known based on the decay of α-mixing coefficients on bounded sets only.

Proof of Proposition 3.1. We split the proof into two steps.

Step 1. Proof of item (i).
By [6, Corollary 3.1], we may apply [5, Proposition 4.1] with the weight $\pi(\ell) = -c'(\ell)$, which then yields $\sigma_L \lesssim 1$. We now apply Theorem 2.3 to Z_L, which greatly simplifies in this precise linear situation since second derivatives of Z_L with respect to A vanish identically. More precisely, for all $L \geq 1$ with the choice $R := 1$, it leads to

$$\mathrm{d_{TV}}\left(\frac{Z_L}{\sigma_L}, \mathcal{N}\right)^2 \lesssim \frac{1}{\sigma_L^4} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} c(|x_1 - x_2| - 1)\, c(|x_2 - x_3| - 1)\, c(|x_3 - x_4| - 1) \\ \times \prod_{i=1}^4 \mathbb{E}\left[\Big(\int_{B(x_i)} \Big|\frac{\partial Z_L}{\partial A}\Big|\Big)^4\right]^{\frac14} dx_1 \ldots dx_4.$$

For all $x \in \mathbb{R}^d$ a direct calculation yields

$$\int_{B(x)} \Big|\frac{\partial Z_L}{\partial A}\Big| \lesssim \pi_*(L)^{\frac12} L^{-d} \mathbb{1}_{|x| \lesssim L},$$

so that the above turns into

$$\begin{aligned} &\mathrm{d_{TV}}\left(\frac{Z_L}{\sigma_L}, \mathcal{N}\right)^2 \\ &\lesssim \frac{1}{\sigma_L^4} L^{-4d} \pi_*(L)^2 \int_{B_{CL}} \cdots \int_{B_{CL}} c(|x_1 - x_2|) c(|x_2 - x_3|) c(|x_3 - x_4|) dx_1 \ldots dx_4 \\ &\lesssim \frac{1}{\sigma_L^4} \pi_*(L)^2 \Big(L^{-d} \int_{B_{CL}} c(|x|) dx\Big)^3. \end{aligned}$$

Recalling that c is non-increasing and that $\pi(\ell) = -c'(\ell)$, we compute

$$L^{-d} \int_{B_{CL}} c(|x|) dx \simeq L^{-d} \int_{B_L} c(|x|) dx \simeq \fint_{B_L} \int_{|x|}^\infty \pi(s) ds dx = \pi_*(L)^{-1}.$$

The claim (3.1) then follows from the combination of these last two estimates.

Step 2. Proof of item (ii).
By [6, Proposition 3.4], we may apply [5, Proposition 4.1] with the weight

$$\pi(\ell) \simeq (\ell + 1)^d \sup_{|u| \leq 2} \gamma(\ell + u - 1) \lesssim \ell^{-2d-\beta-1},$$

which implies $\pi_*(L) \simeq L^d$ and hence $\sigma_L \lesssim 1$. We then apply Theorem 2.5 to Z_L. For all $x, x' \in \mathbb{Z}^d$ and $\ell, \ell' \in \mathbb{N}$, we have

$$|\partial^{\mathrm{dis}}_{\ell,x} Z_L| \lesssim L^{-\frac{d}{2}} |B_{\ell+1}(x) \cap Q_L| \lesssim L^{-\frac{d}{2}} \big(L \wedge (\ell+1)\big)^d \mathbb{1}_{|x| \lesssim L + \ell},$$

and also

$$\begin{aligned} |\partial^{\mathrm{dis}}_{\ell,x} \partial^{\mathrm{dis}}_{\ell',x'} Z_L| &\lesssim L^{-\frac{d}{2}} |B_{\ell'+1}(x') \cap B_{\ell+1}(x) \cap Q_L| \\ &\lesssim L^{-\frac{d}{2}} \big(L \wedge (\ell+1) \wedge (\ell'+1)\big)^d \mathbb{1}_{|x'| \lesssim L+\ell'} \mathbb{1}_{|x| \lesssim L+\ell} \mathbb{1}_{|x-x'| \lesssim \ell+\ell'}. \end{aligned}$$

As these RHS are deterministic, we may actually apply Theorem 2.5 with the borderline exponent $\lambda = 1$, which yields

$$\begin{aligned} &\mathrm{d_W}\Big(\frac{Z_L}{\sigma_L},\mathcal{N}\Big)+\mathrm{d_K}\Big(\frac{Z_L}{\sigma_L},\mathcal{N}\Big) \\ &\lesssim_\mu \quad \frac{1}{\sigma_L^2}\bigg(\sum_{x,x',x''}\sum_{\ell,\ell',\ell''=0}^\infty \gamma(\ell)\gamma(\ell')\gamma(\ell'')\operatorname*{sup\,ess}_A|\partial^{\mathrm{dis}}_{\ell,x}\partial^{\mathrm{dis}}_{\ell',x'}Z_L| \\ &\qquad\qquad \times\operatorname*{sup\,ess}_A|\partial^{\mathrm{dis}}_{\ell,x}\partial^{\mathrm{dis}}_{\ell'',x''}Z_L|\operatorname*{sup\,ess}_A|\partial^{\mathrm{dis}}_{\ell',x'}Z_L|\operatorname*{sup\,ess}_A|\partial^{\mathrm{dis}}_{\ell'',x''}Z_L|\bigg)^{\frac12} \\ &\qquad +\frac{1}{\sigma_L^2}\bigg(\sum_{x,x'}\sum_{\ell,\ell'=0}^\infty\gamma(\ell)\gamma(\ell')\operatorname*{sup\,ess}_A|\partial^{\mathrm{dis}}_{\ell,x}\partial^{\mathrm{dis}}_{\ell',x'}Z_L|^2\operatorname*{sup\,ess}_A|\partial^{\mathrm{dis}}_{\ell',x'}Z_L|^2\bigg)^{\frac12} \\ &\qquad +\frac{1}{\sigma_L^2}\bigg(\sum_x\sum_{\ell=0}^\infty\gamma(\ell)\operatorname*{sup\,ess}_A|\partial^{\mathrm{dis}}_{\ell,x}Z_L|^4\bigg)^{\frac12} \\ &\qquad +\frac{1}{\sigma_L^3}\sum_x\sum_{\ell=0}^\infty\gamma(\ell)\operatorname*{sup\,ess}_A|\partial^{\mathrm{dis}}_{\ell,x}Z_L|^3. \end{aligned}$$

We denote by $I_1,\ldots,I_4$ the four RHS terms. Given the bound $\gamma(\ell)\lesssim \ell^{-\beta'-1}$ for some $\beta'>0$, straightforward calculations left to the reader yield for all $L\ge 1$,

$$\begin{aligned} I_1 &\lesssim \frac{1}{\sigma_L^2}L^{-\frac d2}(1\vee L^{2d-\beta'})^{\frac32}, \\ I_2 &\lesssim \frac{1}{\sigma_L^2}L^{-\frac d2}(1\vee L^{3d-\beta'})^{\frac12}(1\vee L^{2d-\beta'})^{\frac12}, \\ I_3 &\lesssim \frac{1}{\sigma_L^2}L^{-\frac d2}(1\vee L^{4d-\beta'})^{\frac12}, \\ I_4 &\lesssim \frac{1}{\sigma_L^3}L^{-\frac d2}(1\vee L^{3d-\beta'}). \end{aligned}$$

The dominating term with respect to scaling in L is the third one I_3, and the claim then follows by taking $\beta' := 3d+\beta$ for $\beta>0$. □

3.2. **Random sequential adsorption and the jamming limit.** We consider the problem of sequential packing at saturation, following the presentation in [27]. Let $R>0$, and let $(U_{i,R})_{i\ge1}$ be a sequence of i.i.d. random points uniformly distributed on the cube Q_R. Let $\mathcal{S}$ be a fixed bounded closed convex set in $\mathbb{R}^d$ with non-empty interior and centered at the origin 0 of $\mathbb{R}^d$ (that is, a reference "solid"), and for $i\ge1$ let $\mathcal{S}_{i,R}$ be the translate of $\mathcal{S}$ with center at $U_{i,R}$. Then $\mathcal{S}_R := (\mathcal{S}_{i,R})_{i\ge1}$ is an infinite sequence of solids centered at uniform random positions in Q_R (the centers lie in Q_R but the solids themselves need not lie wholly inside Q_R). Let the first solid $S_{1,R}$ be packed, and recursively for $i\ge2$ let the i-th solid $\mathcal{S}_{i,R}$ be packed if it does not overlap any solid in $\{\mathcal{S}_{1,R},\ldots,\mathcal{S}_{i-1,R}\}$ which has already been packed. If not packed, the i-th solid is discarded. This process, known as random sequential adsorption (RSA) with infinite input on the domain Q_R, is irreversible and terminates when it is not possible to accept additional solids. The jamming number

$\mathcal{N}_R := \mathcal{N}_R(\mathcal{S}_R)$ denotes the number of solids packed in Q_R at termination. We are then interested in the asymptotic behavior of $R^{-d}\mathcal{N}_R$ in the infinite volume regime $R \uparrow \infty$, the limit of which (if it exists) is called the *jamming limit*.

In any dimension $d \geq 1$ and for any choice of the reference solid $\mathcal{S}$, Penrose [21] established the existence of the jamming limit, as well as the existence of the infinite volume limit for the distribution of the centers of packed solids, which defines a point process ξ on the whole of $\mathbb{R}^d$. (In the model case $\mathcal{S} := B_1$, this locally finite random measure ξ is referred to as the random parking point process with unit radius.) As we now quickly recall, the key argument in [21] relies on a graphical construction for ξ as a transformation $\xi = \Phi(\mathcal{P}_0)$ of a unit intensity Poisson point process $\mathcal{P}_0$ on the extended space $\mathbb{R}^d \times \mathbb{R}_+$. We first construct an oriented graph on the points of $\mathcal{P}_0$ in $\mathbb{R}^d \times \mathbb{R}_+$, by putting an oriented edge from (x,t) to (x',t') whenever $(x+\mathcal{S}) \cap (x'+\mathcal{S}) \neq \varnothing$ and $t < t'$ (or $t = t'$ and x precedes x' in the lexicographic order, say). We say that (x',t') is an offspring (resp. a descendant) of (x,t), if (x,t) is a direct ancestor (resp. an ancestor) of (x',t'), that is, if there is an edge (resp. a directed path) from (x,t) to (x',t'). The set $\xi := \Phi(\mathcal{P}_0)$ is then constructed as follows. Let F_1 be the set of all roots in the oriented graph (that is, the points of $\mathcal{P}_0$ without ancestor), let G_1 be the set of points of $\mathcal{P}_0$ that are offsprings of points of F_1, and let $H_1 := F_1 \cup G_1$. Now consider the oriented graph induced on $\mathcal{P}_0 \setminus H_1$, and define F_2, G_2, H_2 in the same way, and so on. By construction, the sets $(F_j)_j$ and $(G_j)_j$ are all disjoint and constitute a partition of $\mathcal{P}_0$. We finally define $\xi := \Phi(\mathcal{P}_0) := \bigcup_{j=1}^\infty F_j$.

In [27], Schreiber, Penrose, and Yukich further showed in any dimension $d \geq 1$ that the rescaled variance $R^{-d}\mathrm{Var}\,[\mathcal{N}_R]$ converges to a positive limit (without rate) and that $\mathcal{N}_R$ satisfies a CLT, that is, the fluctuations of the random variable $\mathcal{N}_R$ are asymptotically normal. They also quantified the rate of convergence to the normal, as well as the rate of convergence of $R^{-d}\mathbb{E}[\mathcal{N}_R]$ to the jamming limit. The numerical approximation of the value of the jamming limit has been the object of several works, including [29, Chapter 11.4] and [30]. As is clear from the analysis, the speed of convergence of $R^{-d}\mathbb{E}[\mathcal{N}_R]$ towards its limit is dominated by a boundary effect (the error scales like R^{-1}).

In order to avoid this boundary effect and to obtain better rates of convergence, we may replace $\mathcal{N}_R$ by the number $\tilde{\mathcal{N}}_R$ of packed solids with periodic boundary conditions on Q_R: we say that the i-th solid $\mathcal{S}_{i,R}$ is packed with periodic boundary conditions if its *periodic extension* $\mathcal{S}_{i,R}+R\mathbb{Z}^d$ does not overlap with any solid in $\{\mathcal{S}_{1,R},\ldots,\mathcal{S}_{i-1,R}\}$ which has already been packed. The following shows that this allows one to get rid of the boundary effect, yields optimal estimates, and therefore suggests a more efficient way to approximate the jamming limit numerically.

Theorem 3.2. *For all $R \geq 0$, let $\tilde{\mathcal{N}}_R := \tilde{\mathcal{N}}_R(\mathcal{S}_R)$ be the number of packed solids of $\mathcal{S}_R$ with periodic boundary conditions as defined above. There are constants $\mu := \mu(\mathcal{S},d) \in (0,\infty)$ (the jamming limit) and $\sigma^2 := \sigma^2(\mathcal{S},d) \in (0,\infty)$ such that as $R \uparrow \infty$ we have*

$$|R^{-d}\mathbb{E}[\tilde{\mathcal{N}}_R] - \mu| \;\lesssim\; e^{-\frac{1}{C}R}, \tag{3.2}$$

$$|R^{-d}\mathrm{Var}[\tilde{\mathcal{N}}_R] - \sigma^2| \;\lesssim\; e^{-\frac{1}{C}R}, \tag{3.3}$$

and

$$\mathrm{d_W}\left(R^{\frac{d}{2}}(R^{-d}\tilde{\mathcal{N}}_R - \mu)\,,\,\mathcal{N}(\sigma^2)\right) + \mathrm{d_K}\left(R^{\frac{d}{2}}(R^{-d}\tilde{\mathcal{N}}_R - \mu)\,,\,\mathcal{N}(\sigma^2)\right) \;\lesssim\; R^{-\frac{d}{2}}, \tag{3.4}$$

where $\mathcal{N}(\sigma^2)$ denotes a centered normal random variable with variance σ^2. □

Estimates (3.2) and (3.3) are a consequence of the stabilization properties established in [27]. Note that (3.4) is the best one can hope for: If we considered a Poisson point process instead of the random parking process, then $\tilde{\mathcal{N}}_R$ would be the number of Poisson points in Q_R, μ would be the intensity of the process, we would have $\sigma^2 = \mu$, and (3.4) would be sharp. The proof of (3.4) combines (3.2) and (3.3) to a normal approximation result, which is itself a slight improvement of [27, Theorem 1.1] in the sense that it avoids the spurious logarithmic correction $\log^{3d}(R)$. This improvement is a direct consequence of Theorem 2.5 (it also follows from [12, Theorem 6.1], but the proof we display here is more direct).

Proof of Theorem 3.2. Denote by ξ_R the (R-periodic extension of the) random parking measure on Q_R with periodic boundary conditions (that is, the measure obtained as the sum of Dirac masses at the centers of the periodically packed solids in Q_R). Also denote by $\xi = \xi_\infty$ the corresponding random parking measure on the whole space $\mathbb{R}^d$. Note that by definition both measures ξ_R and ξ are stationary, and we have $\xi_R(Q_R) = \tilde{\mathcal{N}}_R$.

Let us first introduce a natural pairing between ξ_R and ξ based on the graphical construction recalled above. Replacing the original Poisson point process $\mathcal{P}_0$ by $\mathcal{P}_0 \cap (Q_R \times \mathbb{R}_+) + R\mathbb{Z}^d$ (that is, the R-periodization of the restriction of $\mathcal{P}_0$ to $Q_R \times \mathbb{R}^+$), and then running the same graphical construction as above, we obtain a version of the R-periodic random parking measure ξ_R. Using this version, we view both ξ_R and ξ as $\sigma(\mathcal{P}_0)$-measurable random measures for the same underlying Poisson point process $\mathcal{P}_0$. Note however that with this coupling the pair (ξ_R, ξ) is no longer stationary.

We split the proof into three steps. In the first step we recall the construction of action radii for ξ_R and ξ. We then prove (3.2) and (3.3) using the exponentially decaying tail of the constructed action radii (or alternatively, the weighted covariance inequality of [6, Proposition 3.3]), and finally we prove (3.4) by appealing to Theorem 2.5.

Step 1. Construction and properties of action radii.
In this step we claim for all y that ξ admits an action radius ρ_y with respect to $\mathcal{P}_0$ on $Q(y) \times \mathbb{R}^+$, that the restriction $\xi_R|_{Q_R}$ admits an action radius $\rho_{R,y}$ with respect to $\mathcal{P}_0$ on $Q(y) \times \mathbb{R}^+$, and that we have

$$\mathbb{P}[\rho_y > \ell] + \mathbb{P}[\rho_{R,y} > \ell] \lesssim e^{-\frac{1}{C}\ell}.$$

In particular, we show that this implies

$$\sup_{y \in Q_{R/2}} \mathbb{P}[\xi(Q(y)) \neq \xi_R(Q(y))] \lesssim e^{-\frac{1}{C}R}. \tag{3.5}$$

The construction and tail behavior of the action radius ρ_y follows from [6, Proposition 3.3] (with $\ell = 0$). Let the action radius $\rho_{R,y}$ be constructed similarly (simply replacing $\mathcal{P}_0$ by the point set $\mathcal{P}_0 \cap (Q_R \times \mathbb{R}_+) + R\mathbb{Z}^d$). A careful inspection of the proof of [27, Lemma 3.5] reveals that the same exponential tail behavior holds for $\rho_{R,y}$ uniformly in $R > 0$. It remains to argue in favor of (3.5), which simply follows from the exponential tail behavior of the action radii in the form

$$\sup_{y \in Q_{R/2}} \mathbb{P}[\xi(Q(y)) \neq \xi_R(Q(y))] \leq \sup_{y \in Q_{R/2}} \mathbb{P}\left[Q(y) + B_{\rho_y} \not\subset Q_R\right] \lesssim e^{-\frac{1}{C}R}.$$

Step 2. Proof of (3.2) and (3.3).
By stationarity of ξ_R and ξ we find $\mathbb{E}\left[\xi_R(Q_R)\right] = R^d\mathbb{E}\left[\xi_R(Q)\right]$ and $\mathbb{E}\left[\xi(Q_R)\right] = \mu R^d$ with $\mu := \mathbb{E}\left[\xi(Q)\right]$. We define

$$\sigma^2 := \int_{\mathbb{R}^d} \mathrm{Cov}\left[\xi(Q(x)); \xi(Q)\right] dx \tag{3.6}$$

and shall prove (3.2) and (3.3) in the form

$$|R^{-d}\mathbb{E}[\tilde{\mathcal{N}}_R] - \mu| \lesssim e^{-\frac{1}{C}R} \qquad \text{and} \qquad |R^{-d}\mathrm{Var}[\tilde{\mathcal{N}}_R] - \sigma^2| \lesssim e^{-\frac{1}{C}R}. \tag{3.7}$$

The estimate for the convergence of the mean follows from (3.5) in the form

$$\begin{aligned} |R^{-d}\mathbb{E}[\tilde{\mathcal{N}}_R] - \mu| &= |\mathbb{E}[\xi_R(Q) - \xi(Q)]| \\ &\le \sup\mathrm{ess}\,\big(\xi_R(Q) + \xi(Q)\big)\,\mathbb{P}\left[\xi_R(Q) \neq \xi(Q)\right] \lesssim e^{-\frac{1}{C}R}. \end{aligned}$$

We now appeal to the covariance inequality of [6, Proposition 3.3] to prove both the existence of σ^2 (by showing that the integral (3.6) is absolutely convergent) and the estimate for the convergence of the variance in (3.7). Rather than using the complete covariance inequality, it is actually sufficient here to make direct use of the constructed action radii ρ_0 and $\rho_{R,0}$ of Step 1. For $|y| \ge \sqrt{d}+1$, noting that given $\rho_0 \vee \rho_y \le \frac{1}{2}(|y| - \sqrt{d})$ the random variables $\xi(Q(y))$ and $\xi(Q)$ are by definition independent, we obtain

$$\begin{aligned} &\mathrm{Cov}\left[\xi(Q(y)); \xi(Q)\right] \\ &= \mathbb{E}[(\xi(Q(y)) - \mu)(\xi(Q) - \mu)\mathbb{1}_{\rho_0\vee\rho_y > \frac{1}{2}(|y|-\sqrt{d})}] \\ &\quad + \mathbb{E}\big[(\xi(Q(y)) - \mu)(\xi(Q) - \mu)\,\big\|\,\rho_0 \vee \rho_y \le \tfrac{1}{2}(|y| - \sqrt{d})\big]\,\mathbb{P}\big[\rho_0 \vee \rho_y \le \tfrac{1}{2}(|y| - \sqrt{d})\big] \\ &= \mathbb{E}[(\xi(Q(y)) - \mu)(\xi(Q) - \mu)\mathbb{1}_{\rho_0\vee\rho_y > \frac{1}{2}(|y|-\sqrt{d})}] \\ &\quad + \mathbb{P}\big[\rho_0 \vee \rho_y \le \tfrac{1}{2}(|y| - \sqrt{d})\big]^{-1} \\ &\qquad \times \mathbb{E}[(\xi(Q(y)) - \mu)\mathbb{1}_{\rho_0\vee\rho_y \le \frac{1}{2}(|y|-\sqrt{d})}]\,\mathbb{E}[(\xi(Q) - \mu)\mathbb{1}_{\rho_0\vee\rho_y \le \frac{1}{2}(|y|-\sqrt{d})}] \\ &= \mathbb{E}[(\xi(Q(y)) - \mu)(\xi(Q) - \mu)\mathbb{1}_{\rho_0\vee\rho_y > \frac{1}{2}(|y|-\sqrt{d})}] \\ &\quad + \big(1 - \mathbb{P}\big[\rho_0 \vee \rho_y > \tfrac{1}{2}(|y| - \sqrt{d})\big]\big)^{-1} \\ &\qquad \times \mathbb{E}[(\xi(Q(y)) - \mu)\mathbb{1}_{\rho_0\vee\rho_y > \frac{1}{2}(|y|-\sqrt{d})}]\,\mathbb{E}[(\xi(Q) - \mu)\mathbb{1}_{\rho_0\vee\rho_y > \frac{1}{2}(|y|-\sqrt{d})}], \end{aligned}$$

and hence, for all $|y| \ge C$ with $C \simeq 1$ large enough such that

$$\mathbb{P}\big[\rho_0 \vee \rho_y > \tfrac{1}{2}(|y| - \sqrt{d})\big] \le 2\,\mathbb{P}\big[\rho_0 > \tfrac{1}{2}(|y| - \sqrt{d})\big] \le \frac{1}{2},$$

we conclude

$$|\mathrm{Cov}\left[\xi(Q(y)); \xi(Q)\right]| \lesssim \sup\mathrm{ess}\,\big(\xi(Q)^2\big)\,\mathbb{P}\big[\rho_0 > \tfrac{1}{2}(|y| - \sqrt{d})\big] \lesssim e^{-\frac{1}{C}|y|}. \tag{3.8}$$

Arguing similarly for ξ_R with ρ_0 replaced by $\rho_{R,0}$, we deduce for all $y \in Q_R$,

$$|\mathrm{Cov}\left[\xi_R(Q(y)); \xi_R(Q)\right]| \lesssim e^{-\frac{1}{C}|y|}. \tag{3.9}$$

The estimate (3.8) implies in particular that the integral for σ^2 in (3.6) is well-defined. It remains to prove the estimate for the convergence of the variance in (3.7). By R-periodicity and stationarity of ξ_R, we find

$$\begin{aligned} R^{-d}\mathrm{Var}[\tilde{\mathcal{N}}_R] &= R^{-d}\mathrm{Var}\left[\int_{Q_R}\xi_R(Q(y))dy\right] \\ &= \fint_{Q_R}\int_{Q_R}\mathrm{Cov}\left[\xi_R(Q(x-y));\xi_R(Q)\right]dxdy \\ &= \int_{Q_R}\mathrm{Cov}\left[\xi_R(Q(y));\xi_R(Q)\right]dy, \end{aligned}$$

so that we may decompose

$$\begin{aligned} \sigma^2-R^{-d}\mathrm{Var}[\tilde{\mathcal{N}}_R] \;=\; &\int_{\mathbb{R}^d\setminus Q_{R/2}}\mathrm{Cov}\left[\xi(Q(y));\xi(Q)\right]dy-\int_{Q_R\setminus Q_{R/2}}\mathrm{Cov}\left[\xi_R(Q(y));\xi_R(Q)\right]dy \\ &+\int_{Q_{R/2}}\big(\mathrm{Cov}\left[\xi(Q(y));\xi(Q)\right]-\mathrm{Cov}\left[\xi_R(Q(y));\xi_R(Q)\right]\big)dy. \quad (3.10) \end{aligned}$$

We estimate each of the three RHS terms separately. On the one hand, the estimates (3.8) and (3.9) yield

$$\Big|\int_{\mathbb{R}^d\setminus Q_{R/2}}\mathrm{Cov}\left[\xi(Q(y));\xi(Q)\right]dy\Big|\lesssim\int_{\mathbb{R}^d\setminus Q_{R/2}}e^{-\frac{1}{C}|y|}dy\lesssim e^{-\frac{1}{C}R}.$$

and

$$\Big|\int_{Q_R\setminus Q_{R/2}}\mathrm{Cov}\left[\xi_R(Q(y));\xi_R(Q)\right]dy\Big|\lesssim\int_{Q_R\setminus Q_{R/2}}e^{-\frac{1}{C}|y|}dy\lesssim e^{-\frac{1}{C}R}.$$

On the other hand, using (3.5), we obtain

$$\begin{aligned} &\Big|\int_{Q_{R/2}}\big(\mathrm{Cov}\left[\xi(Q(y));\xi(Q)\right]-\mathrm{Cov}\left[\xi_R(Q(y));\xi_R(Q)\right]\big)dy\Big| \\ &\le\;\int_{Q_{R/2}}\mathbb{E}\left[\big|\xi(Q)-\mathbb{E}\left[\xi(Q)\right]\big|\,\big|\xi(Q(y))-\xi_R(Q(y))\big|\right] \\ &\qquad\qquad+\int_{Q_{R/2}}\mathbb{E}\left[\big|\xi_R(Q(y))-\mathbb{E}\left[\xi_R(Q(y))\right]\big|\,\big|\xi(Q)-\xi_R(Q)\big|\right]dy \\ &\lesssim\; R^d\operatorname{sup\,ess}\big(\xi(Q)+\xi_R(Q)\big)\sup_{y\in Q_{R/2}}\mathbb{P}\left[\xi(Q(y))\ne\xi_R(Q(y))\right]\;\lesssim\;e^{-\frac{1}{C}R}. \end{aligned}$$

Injecting these estimates into (3.10), the conclusion (3.7) for the convergence of the variance follows.

Step 3. Proof of (3.4).
We claim that it is enough to prove the normal approximation estimate

$$\mathrm{d_W}\left(\frac{\mathcal{N}_R-\mathbb{E}\left[\mathcal{N}_R\right]}{\sqrt{\mathrm{Var}\left[\mathcal{N}_R\right]}},\mathcal{N}\right)+\mathrm{d_K}\left(\frac{\mathcal{N}_R-\mathbb{E}\left[\mathcal{N}_R\right]}{\sqrt{\mathrm{Var}\left[\mathcal{N}_R\right]}},\mathcal{N}\right)\;\lesssim\;R^{-\frac{d}{2}}. \quad (3.11)$$

Indeed, the result (3.4) then follows from (3.11), (3.2), and (3.3) by the triangle inequality. We omit the proof of (3.11), which is identical to the proof of Proposition 3.1(ii) (the correction $L^{d-\beta}$ disappears here since the weight is exponential). □

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