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WEIGHTED INEQUALITIES FOR COMMUTATORS OF FRACTIONAL AND SINGULAR INTEGRALS

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Introduction

We dedicate this paper to the memory of José Luis Rubio de Francia, who developed the theory of extrapolation and gave beautiful applications of vectorial methods in harmonic analysis.

Through this paper we shall work on $\mathbf{R}^n$, endowed with the Lebesgue measure. Given a Banach space E we shall denote by $L^p_E(\mathbf{R}^n)$ or L^p_E the Bochner-Lebesgue space of E-valued strongly measurable functions such that

$$\int \|f(x)\|_E^p dx < +\infty.$$

Given a positive measurable function $\alpha(x)$ we shall denote by $L^p_E(\alpha)$ the space of E-valued strongly measurable functions such that $\int \|f(x)\|_E^p \alpha(x) dx < \infty$ and we shall denote by $BMO_E(\alpha)$ the space of strongly measurable functions b such that

$$\int_Q \|b(x) - b_Q\|_E dx \leq C \int_Q \alpha(x) dx,$$

where

$$b_Q = |Q|^{-1} \int_Q b(x) dx.$$

Given two Banach spaces E and F, we shall denote by $\mathcal{L}(E, F)$ the Banach space of all continuous linear operators from E into F.

By a Banach lattice we mean a partially ordered Banach space F over the reals such that

(i) $x \leq y$ implies $x + z \leq y + z$ for every $x, y, z \in F$,
(ii) $ax \geq 0$ for every $x \geq 0$ in F and $a \geq 0$ in $\mathbf{R}$.
(iii) for every $x, y \in F$ there exists a least upper bound (l.u.b.) and a greatest lower bound (g.l.b.), and
(iv) if $|x|$ is defined as $|x| = \text{l.u.b. } (x, -x)$ then $\|x\| \leq \|y\|$ whenever $|x| \leq |y|$.

We shall say that a positive function α belongs to $A(p,q)$ if

$$(\frac{1}{|Q|}\int_Q \alpha^{-p'}(x)dx)^{1/p'}(\frac{1}{|Q|}\int_Q \alpha^{q}(x)dx)^{1/q} \leq C,$$

holds for any cube $Q \subset \mathbf{R}^n$ and $p' + p = p'p$, the constant C not depending on Q.

Observe that if we denote by A_p the Muckenhoupt's class, then, for $p > 1$, $\omega \in A(p,p)$ if and only if $\omega^p \in A_p$.

Finally we shall say that a Banach space E is U.M.D. if the Hilbert transform is bounded from L^2_E into L^2_E, see [**2**].

The paper is organized as follows: in section 1 we state and prove the extrapolation results, in section 2 we state the commutator theorems, these theorems are proved in section 4, we give several applications in section 3.

1. Two extrapolation results

Let $\nu \geq 0$ be a measurable function , $1 < p \leq q < \infty$, $1 < \lambda \leq \infty$ and $\frac{1}{p} - \frac{1}{q} = \frac{1}{\lambda}$. We shall say that a weight ω belongs to the class $A^{(\nu)}(p,q)$ if

$$\omega \in A(p,q) \quad \text{and} \quad \nu\omega \in A(p,q).$$

Let $p \geq 1$, we shall say that ω belongs to the class $A_p^{(\nu)}$ if $\omega \in A_p$ and $\nu^p\omega \in A_p$.

Observe that $1 = \lambda'(\frac{1}{p'} + \frac{1}{q})$, then it is clear that $\omega \in A^{(\nu)}(p,q)$ if and only if $\omega^{-p'} \in A^{(\nu^{-\lambda'})}_{1+p'/q}$ and if and only if $\omega^q \in A^{(\nu^{\lambda'})}_{1+q/p'}$; therefore by the properties of the class $A_r^{(\nu)}$, see [**7**], the class $A^{(\nu)}(p,q)$ is not empty if and only if $\nu^{\lambda'} \in A_2$.

We shall use the following lemma, due to Rubio de Francia for the classes A_p, whose proof for the classes $A_p^{(\nu)}$ can be found in [**7**].

(1.1) Lemma. *Assume $\nu \in A_2$, let $1 < r < \infty$ and $\omega \in A_r^{(\nu)}$. Then, for any positive u with $u \in L^{r'}(\omega)$ there exists $U \in L^{r'}(\omega)$ such that*

(a) $u \leq U$ *a.e.*

(b) $\|U\|_{L^{r'}(\omega)} \leq C\|u\|_{L^{r'}(\omega)}$ *and*

(c) $U\omega \in A_1^{(\nu)}$.

Now we state the main results of this paragraph.

(1.2) Theorem. *Let T be a sublinear operator defined on C_0^∞ and satisfying*

$$\|\omega T f\|_\infty \leq C_\omega \|\omega f\|_\infty ,$$

for every ω such that $\omega^{-1} \in A_1$ and $(\nu\omega)^{-1} \in A_1$. Then

$$\|Tf\|_{L^p(\omega)} \le C_\omega \|f\|_{L^p(\omega)}$$

holds for every $\omega \in A_p^{(\nu)}$ and $p > 1$.

(1.3) Theorem. *Let $1 < \lambda \le \infty$ and T be a sublinear operator defined on C_0^∞ and satisfying*

$$\|\omega Tf\|_\infty \le C_\omega \|f\|_{L^\lambda(\omega^\lambda)}\ ,$$

for every ω such that $\omega^{-\lambda'} \in A_1$ and $(\nu\omega)^{-\lambda'} \in A_1$. Then if $1 < p < \lambda$, $\frac{1}{p} - \frac{1}{q} = \frac{1}{\lambda}$ the inequality

$$\|Tf\|_{L^q(\omega^q)} \le C_\omega \|f|_{L^p(\omega^p)}$$

holds for every $\omega \in A^{(\nu)}(p,q)$.

The proof of Theorem (1.2) can be found in [**7**], we shall reproduce it here for the sake of completeness.

Let $f \in L^p(\omega)$, $\omega \in A_p^{(\nu)}$, $1 < p$. We define

$$g = \omega^{1/p(p-1)} |f| \omega^{1/p} / (\int |f|^p \omega)^{1/p} \qquad if \quad |f|\omega^{1/p} \ne 0$$

and

$$g = \omega^{1/p(p-1)} e^{-\pi|x|^2/p} \qquad if \quad |f|\omega^{1/p} = 0.$$

Then, $g > 0$ *a.e.*, $\int g^p \omega^{-p'/p} \le 2$, and

$$\|f\omega^{p'/p} g^{-1}\|_\infty = (\int |f|^p \omega dx)^{1/p}.$$

Now, by the properties of the classes $A_r^{(\nu)}$, see [**7**], $\omega^{-p'/p} \in A_p^{(\nu)}$, therefore we can apply lemma (1.1) and we obtain a function $G \ge g$ *a.e.*, $G\omega^{1-p'} \in A_{p'}^{(\nu)}$, and satisfying

$$\int G^p \omega^{1-p'} dx \le c \int g^p \omega^{1-p'} dx \le 2c.$$

Then,

$$(\int |f|^p \omega dx)^{1/p} \ge C \|\omega^{p'-1} G^{-1} f\|_\infty.$$

Since $(\omega^{p'-1}G^{-1})^{-1}$ and $(\nu\omega^{p'-1}G^{-1})^{-1}$ belong to A_1, we get

$$\begin{aligned}(\int |f|^p \omega dx)^{1/p} &\ge c\|\omega^{p'-1} G^{-1} Tf\|_\infty \\ &\ge c' \|\omega^{p'-1} G^{-1} Tf\|_\infty (\int G^p \omega^{1-p'} dx)^{1/p} \\ &\ge c' (\int |Tf|^p \omega dx)^{1/p},\end{aligned}$$

as we wanted to show. ■

Proof of Theorem (1.3): Let $\omega \in A^{(\nu)}(p,q)$, $\frac{1}{p}-\frac{1}{q}=\frac{1}{\lambda}$ and $f \in L^p(\omega^p)$. Since

$$(\int |f|^p \omega^p dx)^{1/p} = (\int (|f\omega^{p'}|^\lambda)^{p/\lambda}\omega^{-p'}dx)^{(\lambda/p)(1/\lambda)},$$

there exists $g \geq 0$ such that

$$\int g^{(p/\lambda)'}\omega^{-p'}dx = 1 \tag{1.4}$$

and

$$(\int |f|^p\omega^p dx)^{1/p} = (\int |f\omega^{p'}|^\lambda g\omega^{-p'}dx)^{1/\lambda}. \tag{1.5}$$

Let $h = g^{\lambda'/\lambda}$. Then (1.4) is equivalent to

$$\int h^{q/\lambda'}\omega^{-p'}dx = 1$$

Since $\omega \in A^{(\nu)}(p,q)$ we have $\omega^{-p'} \in A^{(\nu^{-\lambda'})}_{1+p'/q}$; setting $r = 1+\frac{p'}{q}$, we can apply lemma (1.1), observing that $r' = \frac{q}{\lambda'}$, to obtain a function $H \geq h$ such that

$$\int H^{q/\lambda'}\omega^{-p'}dx \leq c \qquad \textit{and} \quad H\omega^{-p'} \in A_1^{(\nu^{-\lambda'})}. \tag{1.6}$$

Therefore the weight $v = H^{-1/\lambda'}\omega^{p'/\lambda'}$ is such that $v^{-\lambda'} \in A_1$ and $(\nu v)^{-\lambda'} \in A_1$.

Then, returning to (1.5) and using the hypothesis we have

$$\begin{aligned}(\int |f|^p\omega^p dx)^{1/p} &\geq (\int |f|^\lambda (h^{-1/\lambda'}\omega^{p'/\lambda'})^\lambda dx)^{1/\lambda}\\ &\geq (\int |f|^\lambda (H^{-1/\lambda'}\omega^{p'/\lambda'})^\lambda dx)^{1/\lambda} \geq c\|(Tf)H^{-1/\lambda'}\omega^{p'/\lambda'}\|_\infty.\end{aligned}$$

Taking (1.6) into account, this is bigger than

$$c'\|(Tf)H^{-1/\lambda'}\omega^{p'/\lambda'}\|_\infty(\int H^{q/\lambda'}\omega^{-p'}dx)^{1/q} \geq c(\int |Tf|^q\omega^q dx)^{1/q}. \quad ■$$

Note. The theorems of this section are heavily inspired in [**10**].

2. Commutators for fractional and singular integrals

(2.1) Definition: we shall denote by $BMO_E(\alpha)$ the space of strongly measurable functions b such that

$$\int_Q \|b(x) - b_Q\|_E dx \le C \int_Q \alpha(x) dx,$$

where

$$b_Q = |Q|^{-1} \int_Q b(x) dx.$$

(2.2) Definition: We shall say that a positive function α belongs to $A(p,q)$ if

$$\left(\frac{1}{|Q|}\int_Q \alpha^{-p'}(x)dx\right)^{1/p'}\left(\frac{1}{|Q|}\int_Q \alpha^{q}(x)dx\right)^{1/q} \le C$$

holds for any cube $Q \subset \mathbf{R}^n$ and $p' + p = p'p$, the constant C not depending on Q.

Now we state the theorems of this section.

(2.3) Theorem. *Let E, F be Banach spaces. Let T be a bounded linear operator from $L^p_E(\mathbf{R}^n)$ into $L^q_F(\mathbf{R}^n)$ for $1 < p \le q < \infty$, $0 \le \gamma < n$ and $\frac{1}{p} - \frac{1}{q} = \frac{\gamma}{n}$. Assume that there exists an $\mathcal{L}(E,F)$-valued kernel satisfying:*

(K.1) *for any compactly supported f,*

$$Tf(x) = \int k(x,y) f(y) dy, \qquad \text{for} \quad x \notin \operatorname{supp} f\ ,$$

(K.2) *if $|x-y| > 2|x-x'|$ then*

$$\|k(x,y) - k(x',y)\| \le \frac{C|x-x'|}{|x-y|^{n+1-\gamma}},$$

let $\ell \to \tilde{\ell}$ be a bounded linear operator from $\mathcal{L}(E,E)$ into $\mathcal{L}(F,F)$ such that

$$\tilde{\ell} Tf(x) = T(\ell f)(x) \qquad \text{and}$$

$$k(x,y)\ell = \tilde{\ell} k(x,y)\ .$$

(2.4) *Given $\alpha, \beta \in A(p,q)$, $\nu = \alpha\beta^{-1}$ and b an $\mathcal{L}(E,E)$-valued function such that $b \in BMO_{\mathcal{L}(E,E)}(\nu)$ and $\tilde{b} \in BMO_{\mathcal{L}(F,F)}(\nu)$, then, the operator C_b defined by*

$$C_b f(x) = \tilde{b}(x) Tf(x) - T(bf)(x)\ ,$$

is bounded from $L^p_E(\alpha^p)$ into $L^q_F(\beta^q)$ for $1 < p \le q < \infty$ and $\frac{1}{p} - \frac{1}{q} = \frac{\gamma}{n}$.

(2.5) *Given $\alpha, \beta \in A(p,q)$, $\mu^2 = \alpha\beta^{-1}$, a and b $\mathcal{L}(E,E)$-valued functions such that $a, b \in BMO_{\mathcal{L}(E,E)}(\mu)$ and $\tilde{a}, \tilde{b} \in BMO_{\mathcal{L}(F,F)}(\mu)$, moreover, for every x and y, $a(x)b(y) = b(y)x(a)$ and $\tilde{a}(x)\tilde{b}(x) = \tilde{b}(x)\tilde{a}(x)$. Then, the operator $C_{a,b}$ defined by*

$$C_{a,b}f(x) = \tilde{b}(x)C_a f(x) - C_a(bf)(x) \,,$$

is bounded from $L^p_E(\alpha^p)$ into $L^q_F(\beta^q)$ for $1 < p \leq q < \infty$ and $\frac{1}{p} - \frac{1}{q} = \frac{\gamma}{n}$.

(2.6) Theorem. *Let F be a Banach lattice and V a bounded linear operator from $L^p(\mathbf{R}^n)$ into $L^q_F(\mathbf{R}^n)$ for $1 < p \leq q < \infty$, $0 \leq \gamma < n$ and $\frac{1}{p} - \frac{1}{q} = \frac{\gamma}{n}$. Assume that there exists an F-valued kernel $W(x,y)$ satisfying*

(W.1) *$W(x,y)$ is positive for every x and y,*

(W.2) *for any f with compact support*

$$Vf(x) = \int W(x,y)f(y)dy, \text{ and}$$

(W.3) *if $|x-y| > 2|x-x'|$, then*

$$\|W(x,y) - W(x',y)\| \leq \frac{C|x-x'|}{|x-y|^{n+1-\gamma}} \,.$$

(2.7) *Given $\alpha, \beta \in A(p,q)$, $\nu = \alpha\beta^{-1}$ and $b \in BMO(\nu)$, then the operator V_b^+ defined by*

$$V_b^+ f(x) = \int |b(x) - b(y)| W(x,y) f(y) dy,$$

is bounded from $L^p(\alpha^p)$ into $L^q_F(\beta^q)$ for $1 < p \leq q < \infty$ and $\frac{1}{p} - \frac{1}{q} = \frac{\gamma}{n}$.

(2.8) *Given $\alpha, \beta \in A(p,q)$, $\mu^2 = \alpha\beta^{-1}$, a and b functions in $BMO(\mu)$, then, the operator $V_{a,b}^+$ defined by*

$$V_{a,b}^+ f(x) = \int |a(x) - a(y)||b(x) - b(y)| W(x,y) f(y) dy,$$

is bounded from $L^p(\alpha^p)$ into $L^q_F(\beta^q)$ for $1 < p \leq q < \infty$ and $\frac{1}{p} - \frac{1}{q} = \frac{\gamma}{n}$.

(2.9) Remark. If $\nu^2 \in A_2$ then $b \in BMO(\nu)$ if and only if

$$\left(\frac{1}{|Q|}\int_Q |b(x) - b_Q|^2 dx\right)^{1/2} \leq \frac{C\nu(Q)}{|Q|}.$$

To see this it is enough to observe that if $\nu^2 \in A_2$ then ν satisfies a reverse Holder condition with exponent 2, see **[9]**.

(2.10) Remark. The theory of vector-valued Calderón-Zygmund operators, see [5], and potential operators, see [6], can be applied in both theorems despite of the fact that smoothness is required only on the first variable of the kernel. Thus the operator T (respectively V) turns out to be a bounded operator from $L^p_E(\alpha^p)$ into $L^q_F(\alpha^q)$ (respectively from $L^p(\alpha^p)$ into $L^q_F(\alpha^q)$) for $\alpha \in A(p,q)$, $\frac{1}{p} - \frac{1}{q} = \frac{\gamma}{n}$.

(2.11) Remark. Let $\nu^2 \in A_2$, and α, β such that $\alpha\beta^{-1} = \nu^2$. It is easy to check that if $\delta = \alpha^{1/2}\beta^{1/2}$, then δ^{-1} belongs to A_1 if α^{-1} and β^{-1} belongs to A_1 and $\delta \in A(p,q)$ if α and β belong to $A(p,q)$.

3. Applications

A. Let $0 \le \gamma < n$. Let T be a bounded linear operator from $L^p(\mathbf{R}^n)$ into $L^q(\mathbf{R}^n)$ for $\frac{1}{p} - \frac{1}{q} = \frac{\gamma}{n}$.

Assume that there exists a kernel $k(x,y)$ that satisfies

(i) for any compactly supported f,

$$Tf(x) = \int k(x,y)f(y)dy \qquad if \quad x \notin supp\, f, \quad and$$

(ii) if $|x-y| > 2|x-x'|$, then

$$|k(x,y) - k(x',y)| \le C\frac{|x-x'|}{|x-y|^{n+1-\gamma}}.$$

Given $\frac{1}{p} - \frac{1}{q} = \frac{\gamma}{n}$, a and b in $BMO(\nu)$, we have,

(3.1) for any pair $\alpha, \beta \in A(p,q)$, $\nu = \alpha\beta^{-1}$, the commutator

$$[T, M_b]f(x) = b(x)Tf(x) - T(bf)(x)$$

is bounded from $L^p(\alpha^p)$ into $L^q(\beta^q)$,

(3.2) for any pair $\alpha, \beta \in A(p,q)$, $\nu^2 = \alpha\beta^{-1}$, the commutator

$$[[T, M_b], M_a]f(x) = a(x)[T, M_b]f(x) - [T, M_b](af)(x)$$

is bounded from $L^p(\alpha^p)$ into $L^q(\beta^q)$.

In particular, the commutator of any Calderón-Zygmund operator with standard kernel will be bounded from $L^p(\alpha)$ into $L^p(\beta)$ for $\alpha, \beta \in A_p$ and $\alpha\beta^{-1} = \nu^p$. Also the commutator of the fractional integral of order γ will be bounded from $L^p(\alpha^p)$ into $L^q(\beta^q)$, $\frac{1}{p} - \frac{1}{q} = \frac{\gamma}{n}$, $\alpha, \beta \in A(p,q)$ and $\alpha\beta^{-1} = \nu$. Analogous results are true for the second commutator assuming $\nu = \mu^2$. For the case of the Hilbert transform see [1], for the case of singular integrals with unbounded kernel see [7], and for the case of fractional integrals, see [3] for an unweighted version.

B. Let $T, k, \alpha, \beta, \nu, a$ and b as in application A; and assume that in addition k satisfies

(iii) $$|k(x,y)| \leq \frac{c}{|x-y|^{n-\gamma}} \qquad \text{and}$$

(iv) $$p.v. \int k(x,y)f(y)dy \qquad \text{exists a.e.}$$

We define

$$T_\varepsilon f(x) = \int_{|x-y|>\varepsilon} k(x,y)f(y)dy\ ,$$

$$C_b^* f(x) = \sup_\varepsilon |b(x)T_\varepsilon f(x) - T_\varepsilon(bf)(x)|$$

and

$$C_{a,b}^* f(x) = \sup_\varepsilon |a(x)b(x)T_\varepsilon f(x) - a(x)T_\varepsilon(bf)(x) - b(x)T_\varepsilon(af)(x) + T_\varepsilon(abf)(x)|.$$

Then

(3.3) for any pair $\alpha, \beta \in A(p,q)$, $\nu = \alpha\beta^{-1}$, the operator C_b^* is bounded from $L^p(\alpha^p)$ into $L^q(\beta^q)$, and the operator

$$p.v. \int (b(x) - b(y))k(x,y)f(y)dy,$$

exists a.e. for $f \in L^p(\alpha^p)$ and it is bounded from $L^p(\alpha^p)$ into $L^q(\beta^q)$,

(3.4) for any pair $\alpha, \beta \in A(p,q)$, $\nu^2 = \alpha\beta^{-1}$, the operator $C_{a,b}^*$ is bounded from $L^p(\alpha^p)$ into $L^q(\alpha^q)$, and the operator

$$p.v. \int (a(x) - a(y))(b(x) - b(y))k(x,y)f(y)dy,$$

exists a.e. for $f \in L^p(\alpha^p)$ and it is bounded from $L^p(\alpha^p)$ into $L^q(\beta^q)$.

The proof of (3.3) in the case $p = q$ can be found in [8]; here we shall give a sketch for the case (3.4).

Let $\phi, \psi \in C^\infty([0,\infty))$ such that, $|\phi'(t)| \leq Ct^{-1}$, $|\psi'(t)| \leq Ct^{-1}$ and

$$\chi_{[2,\infty)} \leq \phi \leq \chi_{[1,\infty)},\ \chi_{[1,2]} \leq \psi \leq \chi_{[1/2,3]}.$$

We consider the operators

$$\Phi f(x) = \{\phi_\varepsilon f(x)\}_{\varepsilon>0} = \left\{ \int k(x,y)\phi(\frac{|x-y|}{\varepsilon})f(y)dy \right\}_{\varepsilon>0}$$

and

$$\Psi f(x) = \{\psi_\varepsilon f(x)\}_{\varepsilon>0} = \left\{ \int |k(x,y)|\psi(\frac{|x-y|}{\varepsilon})f(y)dy \right\}_{\varepsilon>0},$$

with kernels given by

$$\{\phi_\varepsilon(x,y)\}_\varepsilon = \left\{ k(x,y)\phi(\frac{|x-y|}{\varepsilon}) \right\}_{\varepsilon>0}$$

and

$$\{\psi_\varepsilon(x,y)\}_\varepsilon = \left\{ |k(x,y)|\psi(\frac{|x-y|}{\varepsilon}) \right\}_{\varepsilon>0}.$$

The kernel of Φ as $\ell^\infty(\mathbf{R})$ -valued function satisfies (K.2) of Theorem (2.3). Analogously, it can be shown that the kernel of Ψ satisfies (W.3) of Theorem (2.6).

By the vector valued Calderón-Zygmund theory, see [5] and [6], Φ and Ψ are bounded linear operators for L^p into $L^q_{\ell^\infty}$, $\frac{1}{p} - \frac{1}{q} = \frac{\gamma}{n}$. Therefore Φ satisfies the hypotheses of Theorem (2.3) and Ψ the hypotheses of Theorem (2.6).

Let $\tilde{b}(x) = (b(x), b(x), \dots, b(x), \dots)$, it is clear that $\tilde{b} \in BMO_{\ell^\infty}(\nu)$, and therefore by Theorem (2.3) and Theorem (2.6) the operators

$$\Phi_{a,b}f(x) = \\ \{a(x)b(x)\phi_\varepsilon f(x) - a(x)\phi_\varepsilon(bf)(x) - b(x)\phi_\varepsilon(af)(x) + \phi_\varepsilon(abf)(x)\}_{\varepsilon>0}$$

and

$$\Psi^+_{a,b}f(x) = \left\{ \int |a(x)-a(y)||b(x)-b(y)|\psi_\varepsilon(x,y)f(y)dy \right\}_{\varepsilon>0}$$

are bounded from $L^p(\alpha^p)$ into $L^q_{\ell^\infty}(\beta^q)$ for $\alpha, \beta \in A(p,q)$ and $\alpha\beta^{-1} = \nu^2$.

Now, we consider the operator

$$\tilde{T}_{a,b}f(x) = \\ \{a(x)b(x)T_\varepsilon f(x) - a(x)T_\varepsilon(bf)(x) - b(x)T_\varepsilon(af)(x) + T_\varepsilon(abf)(x)\}_{\varepsilon>0}$$

The difference operator

$$U_{a,b}f(x) = \Phi_{a,b}f(x) - \tilde{T}_{a,b}f(x) = \\ \left\{ \int (a(x)-a(y))(b(x)-b(y)) \left[\phi(\frac{|x-y|}{\varepsilon}) - \chi_{[1,\infty)}(\frac{|x-y|}{\varepsilon}) \right] k(x,y)f(y)dy \right\}_{\varepsilon>0}$$

satisfies, for a certain ψ as above, that

$$\|U_{a,b}f(x)\|_{\ell^\infty} \leq \\ \sup_{\varepsilon>0} \int |a(x)-a(y)||b(x)-b(y)||k(x,y)|\psi(\frac{|x-y|}{\varepsilon})|f(y)|dy = \|\Psi^+_{a,b}f(x)\|_{\ell^\infty}$$

and therefore $U_{a,b}$ is bounded from $L^p(\alpha^p)$ into $L^q_{\ell^\infty}(\beta^q)$ and, consequently, $\tilde{T}_{a,b}$ is bounded from $L^p(\alpha^p)$ into $L^q_{\ell^\infty}(\beta^q)$, that is to say $\tilde{C}^*_{a,b}$ is bounded from $L^p(\alpha^p)$ into $L^q(\beta^q)$.

C. Let $p, 1 < p < \infty$; α and $\beta \in A_p$, $\nu^{2p} = \alpha\beta^{-1}$, $a, b \in BMO(\nu)$, then the operator

$$S^*_{a,b}f(x) = \sup_r \left| \int_{\mathbf{R}} \frac{(b(x)-b(y))(a(x)-a(y))}{x-y} e^{-iry} f(y)dy \right|,$$

is bounded from $L^p(\alpha)$ into $L^p(\beta)$.

To prove this it is enough to observe that, by the Carleson-Hunt theorem, see [**4**], the operator

$$Sf(x) = \left\{ \int_{\mathbf{R}} \frac{e^{-iry}}{x-y} f(y)dy \right\}_r$$

is bounded from $L^p(\mathbf{R})$ into $L^p_{\ell^\infty}(\mathbf{R})$, for any $p, 1 < p < \infty$. The kernel of this operator satisfies (K.1) and (K.2), see [**5**], therefore it is enough to apply theorem 1 with $\tilde{b}(x) = (b(x), b(x), \dots, b(x), \dots)$.

D. Let H be the Hilbert transform

$$Hf(x) = p.v. \int \frac{f(y)}{x-y} dy,$$

and let E be a U.M.D. Banach space, see [**2**]. Let p, $1 < p < \infty$, α and $\beta \in A_p$, $\nu^2 = (\alpha\beta^{-1})^{1/p}$ and $a, b \in BMO_{\mathcal{L}(E.E.)}(\nu)$. Moreover, we assume that $a(x)b(y) = b(y)a(x)$ holds for every $x, y \in \mathbf{R}^n$. Then the operator

$$p.v. \int \frac{(a(x)-a(y))(b(x)-b(y))}{x-y} f(y)dy,$$

is bounded from $L^p_E(\alpha)$ into $L^p_E(\beta)$.

E. Let I_γ be the fractional integral, of order γ,

$$I_\gamma f(x) = \int \frac{f(y)}{|x-y|^{n-\gamma}} dy, \quad ,0 < \gamma < n.$$

It is known, see [**6**], that I_α is bounded from $L^p_E(\mathbf{R}^n)$ into $L^q_E(\mathbf{R}^n)$, for any Banach space E and $\frac{1}{p} - \frac{1}{q} = \frac{\gamma}{n}$. Let $\alpha, \beta \in A(p,q)$, $\nu^2 = \alpha\beta^{-1}$ and $a, b \in BMO_{\mathcal{L}(E,E)}(\nu)$. Moreover, we assume that $a(x)b(y) = b(y)a(x)$ holds for every $x, y \in \mathbf{R}^n$. Then the operators

$$[[I_\gamma, b]a]f(x) = \int \frac{(a(x)-a(y))(b(x)-b(y))}{|x-y|^{n-\gamma}} f(y)dy$$

and

$$I^+_{\gamma,a,b}f(x) = \int \frac{|a(x)-a(y)||b(x)-b(y)|}{|x-y|^{n-\gamma}} f(y)dy,$$

are bounded from $L^p_E(\alpha^p)$ into $L^q_E(\beta^q)$ for any Banach space E.

F. Maximal operators. Let $0 \le \gamma < n$. Suppose that $\phi \in L^{\frac{n}{n-\gamma}}(\mathbf{R}^n)$ and verifies

$$|\phi(x-y)-\phi(x)| \le C|y||x|^{-n-1+\gamma}, \text{ when } |x| > 2|y|.$$

Set $\phi_\varepsilon(x) = \varepsilon^{-n+\gamma}\phi(\varepsilon^{-1}x)$. Then the operator

$$M_\phi f(x)) = \{f * \phi_\varepsilon(x)\}_{\epsilon>0},$$

can be viewed as a vector-valued Calderón-Zygmund operator, bounded from $L^p(\mathbf{R}^n)$ into $L^q_{\ell^\infty}(\mathbf{R}^n)$, $\frac{1}{p}-\frac{1}{q}=\frac{\gamma}{n}$, see [5] and [6]. Therefore proceeding as in application C we have that the operators

$$M_{\phi,b}f(x) = \{b(x)f * \phi_\varepsilon(x) - (bf) * \phi_\varepsilon(x)\}_{\epsilon>0}$$

and

$$M^+_{\phi,b}f(x) = \{\int |b(x)-b(y)|\phi_\varepsilon(x-y)f(y)dy\}_{\epsilon>0},$$

are bounded from $L^p(\alpha^p)$ into $L^q_{\ell^\infty}(\beta^q)$, $\frac{1}{p}-\frac{1}{q}=\frac{\gamma}{n}$, where α and $\beta \in A(p,q)$, $\alpha\beta^{-1} = \nu$ and $b \in B.M.O(\nu)$; also we have that the operators

$$\begin{aligned} M_{\phi,a,b}f(x) = &\{a(x)b(x)[f * \phi_\varepsilon](x) - a(x)[(bf) * \phi_\varepsilon](x) \\ &- b(x)[(af) * \phi_\varepsilon](x) + [(abf) * \phi_\varepsilon](x)\} \end{aligned}$$

and

$$M^+_{\phi,a,b}f(x) = \{\int |b(x)-b(y)||a(x)-a(y)|\phi_\varepsilon(x-y)f(y)dy\}_{\epsilon>0}$$

are bounded from $L^p(\alpha^p)$ into $L^q_{\ell^\infty}(\beta^q)$, $\frac{1}{p}-\frac{1}{q}=\frac{\gamma}{n}$, where α and $\beta \in A(p,q)$, $\alpha\beta^{-1} = \nu^2$, b and a belong to $BMO(\nu)$.

It is clear that choosing ϕ as above and such that $\chi_{[-1,1]} \le \phi$ we can deduce that the operators

$$\begin{aligned} S^+_b f(x) &= \sup_{x\in Q} \frac{1}{|Q|^{1-\frac{\gamma}{n}}} \int_Q |b(x)-b(y)|f(y)dy, \\ S_b f(x) &= \sup_{x\in Q} \frac{1}{|Q|^{1-\frac{\gamma}{n}}} \int_Q (b(x)-b(y))f(y)dy, \\ S^+_{a,b} f(x) &= \sup_{x\in Q} \frac{1}{|Q|^{1-\frac{\gamma}{n}}} \int_Q |a(x)-a(y)||b(x)-b(y)|f(y)dy, \text{and} \\ S_{a,b} f(x) &= \sup_{x\in Q} \frac{1}{|Q|^{1-\frac{\gamma}{n}}} \int_Q (a(x)-a(y))(b(x)-b(y))f(y)dy, \end{aligned}$$

satisfy the analogous boundedness properties. In fact we have the following theorem

(3.5) Theorem. *Let ν be a weight in A_2 such that $\nu^{\frac{n}{n-\gamma}} \in A_2$. Then the following conditions are equivalent*

(a) *For $\frac{1}{p} - \frac{1}{q} = \frac{\gamma}{n}$, α and $\beta \in A(p,q)$ and $\nu = \alpha\beta^{-1}$, the operator S_b^+ maps $L^p(\alpha^p)$ into $L^q(\beta^q)$.*

(b) *For $\frac{1}{p} - \frac{1}{q} = \frac{\gamma}{n}$, α and $\beta \in A(p,q)$ and $\nu = \alpha\beta^{-1}$, the operator S_b maps $L^p(\alpha^p)$ into $L^q(\beta^q)$.*

(c) *If $\frac{n}{n-\gamma} = \frac{q_0}{2}$, $\nu^{q_0/2} = \nu_0\nu_1^{-1}$ with ν_0 and $\nu_1 \in A_1$, then S_b maps $L^{p_0}((\nu_0\nu_1^{-1})^{p_0/q_0})$ into $L^{q_0}(\nu_0^{-1}\nu_1)$ for $\frac{1}{p_0} - \frac{1}{q_0} = \frac{\gamma}{n}$.*

(d) *b belongs to $B.M.O.(\nu)$.*

Proof: We have seen that $(d) \Rightarrow (a)$ and it is obvious that $(a) \Rightarrow (b)$.

To see that $(b) \Rightarrow (c)$ observe that with this election of q_0 we have $p_0' = q_0$ and $\beta = (\nu_0^{-1}\nu_1)^{1/q_0} \in A(p,q)$, $\alpha = (\nu_0\nu_1^{-1})^{1/q_0} \in A(p,q)$, and $\alpha\beta^{-1} = \nu$.

Now we prove $(c) \Rightarrow (d)$.

$$\int_Q |b(x) - b_Q|dx = |Q|^{-\gamma/n} \int_Q \left| \frac{1}{|Q|^{1-\gamma/n}} \int_Q (b(x) - b(y))dy \right| dx$$

$$\leq |Q|^{-\gamma/n} \left(\int_Q \left| \frac{1}{|Q|^{1-\gamma/n}} \int_Q (b(x) - b(y))dy \right|^{q_0} \nu_0^{-1}(x)\nu_1(x)dx \right)^{1/q_0}$$

$$\cdot \left(\int_Q (\nu_0(x)\nu_1^{-1}(x))^{q_0'/q_0} \right)^{1/q_0'}$$

$$\leq C|Q|^{-\gamma/n} \left(\int_Q (\nu_0(x)\nu_1^{-1}(x))^{p_0/q_0} dx \right)^{1/p_0} \left(\int_Q (\nu_0(x)\nu_1^{-1}(x))^{q_0'/q_0} dx \right)^{1/q_0'}$$

$$= C|Q|^{-\gamma/n} \left(\int_Q (\nu(x)^{q_0/2})^{p_0/q_0} dx \right)^{2/p_0} \leq C|Q|^{-\gamma/n} \left(\int_Q \nu(x)^{p_0/2} \right)^{2/p_0}$$

$$\leq C|Q|^{-\gamma/n} \left(\int_Q \nu(x)dx \right) |Q|^{(2/p_0)/(2/p_0)'} = C \left(\int_Q \nu(x)dx \right) . \blacksquare$$

4. Proofs of the commutator theorems

(4.1) Definition . Let $1 \leq s < \infty$, E be a Banach space, $\nu \in A_2$, α, β positive functions, a and b functions belonging to $BMO_{\mathcal{L}(E,E)}(\nu)$ and f be an E-valued function. We define the following maximal functions.

$$M_1 f(x) = sup \frac{1}{|Q|} \int_Q \|(b(y) - b_Q)f(y)\| dy \,, \tag{4.2}$$

$$M_2^s f(x) = sup|Q|^{\gamma/n} \left(\frac{1}{|Q|} \int_Q \|(b(y) - b_Q)f(y)\|^s dy \right)^{1/s} , \tag{4.3}$$

$$M_3 f(x) = sup \left(\inf_{y\in Q} (\nu \mathcal{X}_Q)^*(y) \right) \left(|Q|^{\gamma/n} \frac{1}{|Q|} \int_Q \|f(y)\| dy \right) , \tag{4.4}$$

$$M_4 f(x) = sup \frac{1}{|Q|} \int_Q \|(a(y) - a_Q) f(y)\| dy , \tag{4.5}$$

$$M_5^s f(x) = sup |Q|^{\gamma/n} \left[\frac{1}{|Q|} \int_Q \|(a(y) - a_Q) f(y)\|^s \alpha^{s/2}(y) dy \right]^{1/s} \left[\frac{1}{|Q|} \int_Q \|b(y) - b_Q\|^s \alpha^{-s/2}(y) dy \right]^{1/s} , \tag{4.6}$$

$$M_6^s f(x) = sup |Q|^{\gamma/n} \left[\frac{1}{|Q|} \int_Q \|(a(y) - a_Q)(b(y) - b_Q) f(y)\|^s dy \right]^{1/s} , \tag{4.7}$$

$$M_7 f(x) = sup \left(\inf_{y\in Q} (\nu \mathcal{X}_Q)^*(y) \right) \left(\frac{|Q|^{\gamma/n}}{|Q|} \int_Q \|(b(y) - b_Q) f(y)\| dy \right) , \tag{4.8}$$

$$M_8 f(x) = sup \left(\inf_{y\in Q} (\nu \mathcal{X}_Q)^*(y) \right) \left(\frac{|Q|^{\gamma/n}}{|Q|} \int_Q \|(a(y) - a_Q) f(y)\| dy \right) , \tag{4.9}$$

$$M_9 f(x) = sup \left(\inf_{y\in Q} (\nu \mathcal{X}_Q)^*(y) \right)^2 \left(\frac{|Q|^{\gamma/n}}{|Q|} \int_Q \|f(y)\| dy \right) , \tag{4.10}$$

$$M_{10}^j f(x) = \sup \left(\frac{1}{|Q|} \int_Q \|a(y) - a_Q\| dy \right) \left(\frac{1}{|Q|} \int_Q \|b(y) - b_Q\| dy \right) \left(\frac{|2^j Q|^{\gamma/n}}{|2^j Q|} \int_{2^j Q} \|f(y)\| dy \right) , \tag{4.11}$$

$$M_{11}^j f(x) = \sup \left(\inf_{y\in 2^j Q} (\nu \mathcal{X}_{2^j Q})^*(y) \right) \left(\frac{1}{|Q|} \int_Q \|b(y) - b_Q\| dy \right) \left(\frac{|2^j Q|^{\gamma/n}}{|2^j Q|} \int_{2^j Q} \|f(y)\| dy \right) , \tag{4.12}$$

(4.13) $M_{12}^j f(x) =$

$$sup\left(\frac{1}{|Q|}\int_Q \|b(y)-b_Q\|dy\right)\left(\frac{|2^jQ|^{\gamma/n}}{|2^jQ|}\int_{2^jQ}\|(a_{2^jQ}-a(y))f(y)\|dy\right),$$

(4.14) $$M_{13}f(x) = sup\left(\frac{1}{|Q|}\int_Q \|b(y)-b_Q\|dy\right)\left(\frac{1}{|Q|}\int_Q \|f(y)\|dy\right),$$

(4.15)

$$M_{14}^s f(x) = \sup\left(\frac{1}{|Q|}\int_Q \|b(y)-b_Q\|dy\right)|Q|^{\gamma/n}\left(\frac{1}{|Q|}\int_Q \|f(y)\|^s\alpha^{s/2}(y)dy\right)^{1/s}$$
$$\left(\frac{1}{|Q|}\int_Q \beta^{-s/2}(y)dy\right)^{1/s}.$$

In all the cases the supremum is taken over all cubes in $\mathbf{R}^n$ with sides paralell to the axes and centered in x. $(\nu\mathcal{X}_Q)^*$ stands for the Hardy-Littlewood maximal function of $\nu\mathcal{X}_Q$.

(4.16) Proposition. *Let E be a Banach space. Let $0 \le \gamma < n$, assume $\alpha^{-n/(n-\gamma)}$ and $\beta^{-n/(n-\gamma)} \in A_1$, $\nu = \alpha\beta^{-1}$ and $b \in BMO_{\mathcal{L}(E,E)}(\nu)$. Then*

(4.17) $$\|\beta M_1 f\|_{L^\infty} \le C\|f\alpha\|_{L^\infty_E},$$

(4.18) *There exists $\varepsilon > 0$ such that if $1 \le s < (1+\varepsilon)$ then*

$$\|\beta M_2^s f\|_{L^\infty} \le C\|f\alpha\|_{L_E^{n/\gamma}} \quad \textit{and}$$

(4.19) $$\|\beta M_3 f\|_{L^\infty} \le C\|f\alpha\|_{L_E^{n/\gamma}}.$$

(4.20) Proposition. *Let E be a Banach space. Let $0 \le \gamma < n$, assume $\alpha^{-n/(n-\gamma)}$, $\delta^{-n/n-\gamma}$, and $\beta^{-n/(n-\gamma)} \in A_1$, $\nu^2 = \alpha\beta^{-1}$, $\nu = \alpha\delta^{-1} = \delta\beta^{-1}$, and $a, b \in BMO_{\mathcal{L}(E,E)}$. Then*

(4.21) $$\|\beta M_i f\|_{L^\infty} \le C\|f\alpha\|_{L_E^{n/\gamma}}, \quad i = 7, 8, 9, \quad \textit{and}$$

(4.22) $$\|\beta M_i^j f\|_{L^\infty} \le C\|f\alpha\|_{L_E^{n/\gamma}}, \quad i = 10, 11, 12. \quad j \ge 1.$$

(4.23) *If* $u = \frac{2n}{n+\gamma}$

$$\|\beta M_5^u f\|_{L^\infty} \le C\|f\alpha\|_{L_E^{n/\gamma}}.$$

(4.24) *There exists* $\varepsilon > 0$ *such that if* $1 \le s < (1+\varepsilon)$ *then*

$$\|\beta M_6^s f\|_{L^\infty} \le C\|f\alpha\|_{L_E^{n/\gamma}} \qquad \text{and}$$

(4.25)
$$\|\beta M_i f\|_{L^\infty} \le C\|f\delta\|_{L_E^\infty}, i = 4, 13.$$

(4.26) *If* $u = \frac{2n}{n+\gamma}$

$$\|\beta M_{14}^u f\|_{L^\infty} \le C\|f\alpha\|_{L_E^{n/\gamma}}.$$

We postpone the proofs of these Propositions. Now we state and prove the following Corollaries.

(4.27) Corollary. *Let* $\nu^{-\frac{n}{n-\gamma}} \in A_2$, $0 \le \gamma < n$. *Then in the hypothesis of Proposition (4.16) we have that*

(4.28) *if* $\alpha, \beta \in A_p$, $1 < p < \infty$, *and* $\alpha\beta^{-1} = \nu^p$ *then*

$$\|M_1 f\|_{L^p(\beta)} \le C\|f\|_{L_E^p(\alpha)},$$

(4.29) *if* $\alpha, \beta \in A(p,q)$, $\frac{1}{p} - \frac{1}{q} = \frac{\gamma}{n}$ *and* $\alpha\beta^{-1} = \nu$ *then*

$$\|M_2^s f\|_{L^q(\beta^q)} \le C\|f\|_{L_E^p(\alpha^p)},\ 1 \le s < (1+\varepsilon)$$

and

$$\|M_3 f\|_{L^q(\beta^q)} \le C\|f\|_{L_E^p(\alpha^p)}.$$

(4.30) Corollary. *Let* $\nu^{-\frac{2n}{n-\gamma}} \in A_2$, $0 \le \gamma < n$. *Then in the hypothesis of Proposition (4.20) we have that*

(4.31) *if* $\alpha, \beta \in A(p,q)$, $\frac{1}{p} - \frac{1}{q} = \frac{\gamma}{n}$, *and* $\alpha\beta^{-1} = \nu^2$, *then*

$$\|M_i f\|_{L^q(\beta^q)} \le C\|f\|_{L_E^p(\alpha^p)}\ i = 7, 8, 9,$$

$$\|M_i^j f\|_{L^q(\beta^q)} \le C\|f\|_{L_E^p(\alpha^p)}\ i = 10, 11, 12.,\ j \ge 1,$$

$$\|M_i^u f\|_{L^q(\beta^q)} \le C\|f\|_{L_E^p(\alpha^p)},\ u = \frac{2n}{n+\gamma},\ i = 5, 14$$

and

$$\|M_6^s f\|_{L^q(\beta^q)} \le C\|f\|_{L_E^p(\alpha^p)},\ 1 \le s < (1+\varepsilon),$$

(4.32) *if* $\delta, \beta \in A_p$, $1 < p < \infty$ *and* $\delta\beta^{-1} = \nu^p$, *then*

$$\|M_i f\|_{L^p(\beta)} \leq C\|f\|_{L^p_E(\delta)},\ i = 4, 13.$$

For the proof of these Corollaries it is enough to observe that for a sublinear operator S, the inequality

$$\|Sf\|_{L^p(\beta)} \leq C\|f\|_{L^p(\alpha)}\ \alpha, \beta \in A_p \quad and \quad \alpha\beta^{-1} = \nu^p$$

is equivalent to the inequality

$$\|U(g)\|_{L^p(w)} \leq C\|g\|_{L^p(w)},\ w \in A_p^{(\nu)}\ ,$$

U being the operator $U(g) = S(g\nu^{-1})$.

Analogously, observe that the inequality

$$\|Sf\|_{L^q(\beta^q)} \leq C\|f\|_{L^p(\alpha^p)}\ ,\ \alpha, \beta \in A(p,q) \quad and \quad \alpha\beta^{-1} = \nu\ ,$$

is equivalent to the inequality

$$\|U(g)\|_{L^q(w^q)} \leq C\|g\|_{L^p(w^p)},\ w \in A^{(\nu)}(p,q),$$

U being the operator $U(g) = S\left(g\nu^{-1}\right)$.

With these two observations the corollaries (4.27) and (4.30) are direct consequences of Theorems (1.2) and (1.3).

(4.33) Proposition. *There exists* $\varepsilon > 0$, *such that if*

$$1 \leq s < (1+\varepsilon),\ and\, u = \frac{2n}{n+\gamma},$$

then the operators considered in Theorem (2.3) and in Theorem (2.6) satisfy the following inequalities

$$(C_b f)^{\#}(x) \leq C\left\{M_1(Tf)(x) + M_2^s f(x) + M_3 f(x)\right\}\ , \tag{4.34}$$

$$(V_b^{+} f)^{\#}(x) \leq C\left\{M_1(Vf)(x) + M_2^s f(x) + M_3 f(x)\right\}\ , \tag{4.35}$$

(4.36)

$$\begin{aligned}(C_{a,b} f)^{\#}(x) \leq C\{&M_4(C_b f)(x) + M_{13}(C_a f)(x) \\ &+ M_5^u f(x) + M_6^s f(x) + M_7 f(x) + M_8 f(x) + M_9 f(x) \\ &+ \sum_{j=1}^{\infty} 2^{-j}(M_{10}^j f(x) + M_{12}^j f(x)) + \sum_{j=1}^{\infty} j2^{-j} M_{11}^j f(x) \\ &+ M_{14}^u f(x)\} \qquad and\end{aligned}$$

(4.37)

$$\begin{aligned}(V^+_{a,b}f)^{\#}(x) \le C\,\{&M_4(V^+_b f)(x) + M_{13}(V^+_a f)(x)\\ &+ M^u_5 f(x) + M^s_6 f(x) + M_7 f(x) + M_8 f(x) + M_9 f(x)\\ &+ \sum_{j=1}^{\infty} 2^{-j}(M^j_{10} f(x) + M^j_{12} f(x)) + \sum_{j=1}^{\infty} j2^{-j} M^j_{11} f(x)\\ &+ M^u_{14} f(x)\}\,.\end{aligned}$$

Assuming this Proposition (4.33) we can give the proof of Theorem (2.3) and Theorem (2.6). We prove Theorem (2.3) only, since the proof of Theorem (2.6) is similar.

In fact, we shall give only the proof of (2.5) assuming that (2.4) is true. The proof of (2.4) is similar using remark (2.10).

By (4.36) and Corollary (4.30) we have

$$\left(\int (C_{a,b}f)^{\#}(x)^q \beta^q(x)dx\right)^{1/q} \le C\left\{\left(\int \|C_b f(x)\|^q \delta^q(x)dx\right)^{1/q}\right.$$
$$\left. + \left(\int \|C_a f(x)\|^q \delta^q(x)dx\right)^{1/q} + \left(\int \|f(x)\|^p \alpha^p(x)dx\right)^{1/p}\right\}.$$

Then by (2.4) and the vector-valued version of the sharp function theorem, see [5], we have

$$\begin{aligned}\left(\int \|C_{a,b}f(x)\|^q \beta^q(x)dx\right)^{1/q} &\le C\left(\int (C_{a,b}f)^{\#}(x)^q \beta^q(x)dx\right)^{1/q}\\ &\le C\left(\int \|f(x)\|^p \alpha^p(x)dx\right)^{1/p}.\end{aligned}$$

This ends the proof of section (2.5) in Theorem (2.3).

Now we give the proofs of the technical propositions (4.16), (4.20) and (4.33). We shall need the following lemmas.

(4.38) Lemma. *Let E be a Banach space. Let Q be a cube and $Q_k = 2^k Q$. Then if $b \in BMO_E(\nu)$, $\nu \in A_2$, it follows that*

$$\|b_Q - b_{Q_k}\| \le Ck\nu_{Q_{i(k)}} \le kC \inf_{y\in Q_k} (\nu\mathcal{X}_{Q_k})^*(y)\,,$$

where $Q_{i(k)}$ is the cube such that $\nu_{Q_{i(k)}} = \max_{1\le j\le k} \nu_{Q_j}$ and $(\nu\mathcal{X}_{Q_k})^*$ is the Hardy-Littlewood maximal function of $\nu\mathcal{X}_{Q_k}$.

(4.39) Lemma. *If $w^{-t} \in A_1$, there exists $\varepsilon > 0$ such that for every $1 \le r \le t(1+\varepsilon)$, $w^{-r} \in A_1$.*

(4.40) Lemma. *If $w^{-t} \in A_1$, there exists $\varepsilon > 0$ such that $w^{r'} \in A_{r'/t}$ for every $1 < r \leq (1+\varepsilon)$.*

(4.41) Lemma. *Let E be a Banach space, if $b \in BMO_E(\nu)$ and $\nu^l = \alpha\beta^{-1}$, $\alpha^{-t} \in A_1$, $\beta^{-t} \in A_1$ then there exists $\varepsilon > 0$ such that*

$$\left(\frac{1}{|Q|}\int_Q \|b(x) - b_Q\|^{lr}\alpha^{-r}(x)dx\right)^{1/r} \leq C\beta(x_0)^{-1}$$

holds for $1 < r < t(1+\varepsilon)$ and $x_0 \in Q$, $l = 1, 2$.

The proof of these lemmas can be found in [1].

(4.42) Lemma. *Let E be a Banach space; $0 \leq \gamma < n$, $\alpha^{-\frac{n}{n-\gamma}}, \beta^{-\frac{n}{n-\gamma}} \in A_1$, $\nu = \alpha\beta^{-1}$ and $b \in BMO_{\mathcal{L}(E,E)}(\nu)$. Then for any function f we have,*

$$(4.43) \qquad if \quad 1 \leq p < \frac{n}{\gamma}\ then, \left(\frac{1}{|Q|}\int_Q \|f\alpha\|^p dx\right)^{1/p} \leq \|f\alpha\|_{n/\gamma}|Q|^{-\gamma/n},$$

(4.44) *there exists $\varepsilon > 0$ such that if $1 \leq s < (1+\varepsilon)$, then,*

$$\left(\frac{1}{|Q|}\int_Q \|(b - b_Q)f\|^s dx\right)^{1/s} \leq C\|f\alpha\|_{n/\gamma}|Q|^{-\gamma/n}(inf_{x\in Q}\beta^{-1}(x)),$$

(4.45) *there exists $\varepsilon > 0$ such that if $1 \leq s < \frac{n}{n-\gamma}(1+\varepsilon)$, then,*

$$\begin{aligned}\frac{1}{|Q|}\int_Q \|(b - b_Q)f\|dx &\leq \left(\frac{1}{|Q|}\int_Q \|b - b_Q\|^s\alpha^{-s}dx\right)^{1/s}\|f\alpha\|_\infty \\ &\leq C\left(\inf_{x\in Q}\beta^{-1}(x)\right)\|f\alpha\|_\infty \qquad and\end{aligned}$$

$$(4.46) \qquad \left\|\frac{1}{|Q|}\int_Q f dx\right\| \leq C\|f\alpha\|_{n/\gamma}|Q|^{-\gamma/n}\left(inf_{x\in Q}\alpha^{-1}(x)\right).$$

Proof: (4.43) is obvious by using Hölder's inequality. Lemma (4.41) and Hölder's inequality give (4.45). In order to prove (4.46) observe that

$$\left\|\frac{1}{|Q|}\int_Q f dx\right\| \leq \left(\frac{1}{|Q|}\int_Q \|f\alpha\|^p dx\right)^{1/p}\left(\frac{1}{|Q|}\int_Q \alpha^{-p'}\right)^{1/p'}.$$

Choosing p, $1 \le p < \frac{n}{\gamma}$, such that $p' < \left(\frac{n}{n-\gamma}\right)(1+\varepsilon)$ and $\alpha^{-p'} \in A_1$, then (4.43) gives the result. Finally by Hölder's inequality we have in (4.44) that

$$\left(\frac{1}{|Q|}\int_Q \|(b-b_Q)f\|^s dx\right)^{1/s}$$
$$\le \left(\frac{1}{|Q|}\int_Q \|(b-b_Q)\|^{st}\alpha^{-st}dx\right)^{1/st}\left(\frac{1}{|Q|}\int_Q \|f\alpha\|^{st'}dx\right)^{1/st'}.$$

Now if we choose t such that $st < \frac{n}{n-\gamma}(1+\varepsilon)$ and $st' < \frac{n}{\gamma}$, where ε is the one which appears in lemma (4.41), we get that the last product is less than

$$C\|f\alpha\|_{\frac{n}{\gamma}}|Q|^{-\gamma/n} inf_{x\in Q}\beta^{-1}(x). \quad \blacksquare$$

(4.47) Lemma. *Let $t \ge 1$, and $\omega^{-t} \in A_1$, then $\omega^{1/2} \in A((2t)', 2t)$.*

Proof of Proposition (4.16): Through this proof "sup "always shall mean the supremum over the cubes centered at x. The proof of (4.17) and (4.18) are direct applications of (4.45) and (4.44). ■

To show (4.19), choose r such that $\frac{n}{n-\gamma} < r < \frac{n}{(n-\gamma)}(1+\varepsilon)$, $r' < \frac{n}{\gamma}$ and $\alpha^{-r} \in A_1$ then by (4.43), $M_3 f(x)$ is less than

$$\sup\left(\inf_{y\in Q}(\nu\mathcal{X}_Q)^*(y)\right)|Q|^{\gamma/n}\left(\frac{1}{|Q|}\int_Q \|f(y)\alpha(y)\|^{r'}dy\right)^{1/r'}\left(\frac{1}{|Q|}\int_Q \alpha^{-r}(y)dy\right)^{1/r}$$
$$\le C\sup\left(\inf_{y\in Q}(\nu\mathcal{X}_Q)^*(y)\right)\|f\alpha\|_{\frac{n}{\gamma}}\left(inf_{y\in Q}\alpha^{-1}(y)\right)$$
$$\le C\sup\|f\alpha\|_{\frac{n}{\gamma}}\inf_{y\in Q}\left(\left(\inf_{z\in Q}\alpha^{-1}(z)\right)\cdot(\nu\mathcal{X}_Q)^*(y)\right) \le C\|f\alpha\|_{\frac{n}{\gamma}}\beta^{-1}(x).$$

Proof of Proposition (4.20): Through this proof the word "sup "always shall mean the supremun over the cubes centered at x. Let $\alpha\delta^{-1} = \nu = \delta\beta^{-1}$.

If $u = \frac{2n}{n+\gamma}$, we have that $u\left(\frac{n/\gamma}{u}\right)' = \frac{2n}{n-\gamma}$, then by Hölder's inequality, we have

$$M_5^u f(x) \le \sup\|f\alpha\|_{n/\gamma}\left(\frac{1}{|Q|}\int_Q \|a(y)-a_Q\|^{2n/n-\gamma}\alpha^{-n/n-\gamma}(y)dy\right)^{(n-\gamma)/2n}$$
$$\left(\frac{1}{|Q|}\int_Q \|b(y)-b_Q\|^u\alpha^{-u/2}(y)dy\right)^{1/u}.$$

Observe that $\frac{u}{2} < \frac{n}{n-\gamma}$, then applying lemma (4.41) we get

$$M_5^u f(x) \leq C\|f\alpha\|_{n/\gamma}\beta^{-1/2}(x)\beta^{-1/2}(x) = C\|f\alpha\|_{n/\gamma}\beta^{-1}(x).$$

If $1 \leq s < (1+\varepsilon)$, then by Hölder's inequality, we have

$$M_6^s f(x) \leq \sup |Q|^{\gamma/n}\left(\frac{1}{|Q|}\int_Q \|(a(y)-a_Q)(b(y)-b_q)\|^{st'}\alpha^{-st'}(y)dy\right)^{1/st'}$$
$$\left(\frac{1}{|Q|}\int_Q \|f(y)\|^{st}\alpha^{st}(y)dy\right)^{1/st}.$$

Now if we choose t such that $st' < \frac{n}{n-\gamma}(1+\varepsilon)$ and $st < \frac{n}{\gamma}$, where ε is the one which appears in lemma (4.41) we get

$$M_6^s f(x) \leq \sup \int_Q \left(\|(a(y)-a_Q)(b(y)-b_Q)\|^{st'}\alpha^{-st'}(y)dy\right)^{1/st'} \|f\alpha\|_{n/\gamma}.$$

Now by Hölder's inequality and lemma (4.41), we have,

$$M_6^s f(x) \leq \sup \left(\frac{1}{|Q|}\int_Q \|a(y)-a_Q\|^{2st'}\alpha^{-st'}(y)dy\right)^{1/2st'}$$
$$\left(\frac{1}{|Q|}\int_Q \|b(y)-b_Q\|^{2st'}\alpha^{-st'}(y)dy\right)^{1/2st'} \|f\alpha\|_{n/\gamma} \leq C\beta^{-1}(x)\|f\alpha\|_{n/\gamma}.$$

Using (4.44) we get,

$$\begin{aligned} M_7 f(x) &\leq C \sup\left(\inf_{y\in Q}(\nu\mathcal{X}_Q)^*(y)\right)\|f\alpha\|_{\frac{n}{\gamma}} inf_{y\in Q}\delta^{-1}(y) \\ &\leq C\sup\left(\inf_{y\in Q}(\inf_{z\in Q}\delta^{-1}(z))(\nu\mathcal{X}_Q)^*(y)\right)\|f\alpha\|_{\frac{n}{\gamma}} \\ &\leq C\sup\left(\inf_{y\in Q}\beta^{-1}(y)\right)\|f\alpha\|_{\frac{n}{\gamma}}. \ \blacksquare \end{aligned}$$

The proof for $M_9 f$ is pararell to the proof for M_3.

For M_{10}^j we use Hölder's inequality with $r' < \frac{n}{\gamma}$ and $r < \frac{n}{n-\gamma}(1+\varepsilon)$ such that $\alpha^{-r} \in A_1$ getting

$$M_{10}^j f(x) \leq \sup\left(\frac{1}{|Q|}\int_Q \|b(y)-b_Q\|dy\right)\left(\frac{1}{|Q|}\int_Q \|a(y)-a_Q\|dy\right)|2^jQ|^{\gamma/n}$$
$$\left(\frac{1}{|2^jQ|}\int_{2^jQ}\|f(y)\alpha(y)\|^{r'}dy\right)^{1/r'}\left(\frac{1}{|2^jQ|}\int_{2^jQ}\alpha^{-r}(y)dy\right)^{1/r}$$
$$\leq C\|f\alpha\|_{\frac{n}{\gamma}} sup\left(\frac{1}{|Q|}\int_Q \|b(y)-b_Q\|dy\right)\left(\frac{1}{|Q|}\int_Q \|a(y)-a_Q\|dy\right)\left(inf_{y\in 2^jQ}\alpha^{-1}(y)\right)$$
$$\leq C\|f\alpha\|_{\frac{n}{\gamma}} sup\left(\frac{1}{|Q|}\int_Q \|b(y)-b_Q\|dy\right)\left(\frac{1}{|Q|}\int_Q \|a(y)-a_Q\|\alpha^{-1}(y)dy\right).$$

Now applying Remark (2.11) and Lemma (4.41) twice we obtain the desired result for M^j_{10}. We don't give the proof for M^j_{11} which is a mixture of the proofs for M^j_{10} and for M_3. Analogously the proof of M^j_{12} is a mixture of the proofs for M_7 and M^j_{10}.

Since $\delta^{-1} \in A_1$ we have by Lemma (4.41)

$$\begin{aligned} M_{13}f(x) &\leq C\|f\delta\|_\infty \sup\left(\frac{1}{|Q|}\int_Q \|b(y)-b_Q\|dy\right)\left(\inf_{y\in Q}\delta^{-1}(y)\right) \\ &\leq C\|f\delta\|_\infty \cdot \beta^{-1}(x). \end{aligned}$$

Finally, if $u = \frac{2n}{n+\gamma}$, then by Hölder's inequality and lemma (4.43), we have,

$$\begin{aligned} &M^u_{14}f(x) \leq \sup\left(\frac{1}{|Q|}\int_Q \|b(y)-b_Q\|dy\right)\|f\alpha\|_{n/\gamma} \\ &\left(\frac{1}{|Q|}\int_Q \alpha^{-n/(n-\gamma)}(y)dy\right)^{(n-\gamma)/2n}\left(\frac{1}{|Q|}\int_Q \beta^{-u/2}(y)dy\right)^{1/u} \\ &\leq \sup\left(\frac{1}{|Q|}\int_Q \|b(y)-b_Q\|\alpha^{-1/2}(y)dy\right)\|f\alpha\|_{n/\gamma}\left(\frac{1}{|Q|}\int_Q \beta^{-u/2}(y)dy\right)^{1/u}. \end{aligned}$$

Then, applying lemma (4.41) to $\nu = \alpha^{1/2}\beta^{-1/2}$, we have,

$$M^u_{14}f(x) \leq \sup \beta^{-1/2}(x)\|f\alpha\|_{n/\gamma}\left(\frac{1}{|Q|}\int_Q \beta^{-u/2}(y)dy\right)^{1/u}.$$

Since $\frac{u}{2} < \frac{n}{n-\gamma}$ then $\beta^{-u/2} \in A_1$ and then we get the desired result. ■

Proof of Proposition (4.33):

We shall prove (4.34) and (4.37), the other cases can be proved analogously.

Let Q be a cube in $\mathbf{R}^n$ with center ar x_0. Given a function f with compact support, we define

$$f_1(x) = f(x)\mathcal{X}_{2Q}(x), \quad f_2(x) = f(x) - f_1(x).$$

Let

$$c_Q = T((b_Q - b)f_2)(x_0).$$

Then if $x \in Q$, we have

$$\begin{aligned} C_b f(x) &= \tilde{b}(x)Tf(x) - T(bf)(x) = (\tilde{b}(x) - \tilde{b}_Q)Tf(x) \\ &\quad + T(b_Q f)(x) - T(bf)(x) = (\tilde{b}(x) - \tilde{b}_Q)Tf(x) \\ &\quad + T((b_Q - b)f)(x). \end{aligned}$$

Therefore, for $x \in Q$, we have

$$\begin{aligned}\|C_b f(x) - c_Q\|_F &\le \|(\tilde{b}(x) - \tilde{b}_Q)Tf(x)\|_F \\ &\quad + \|T((b_Q - b)f_1)(x)\|_F \\ &\quad + \|T((b_Q - b)f_2)(x) - T((b_Q - b)f_2)(x_0)\|_F \\ &= \sigma_1(x) + \sigma_2(x) + \sigma_3(x)\,.\end{aligned}$$

We shall estimate $(C_b f)^{\#}(x_0)$ in terms of the $\sigma_1(x)$. Obviously

$$\frac{1}{|Q|}\int_Q \sigma_1(x)dx \le M_1(Tf)(x_0)\,.$$

Now, for $\sigma_2(x)$ choose r such that $\frac{1}{s} - \frac{1}{r'} = \frac{\gamma}{n}$ and $s < (1+\varepsilon)$. Then using the boundedness properties of T, we have,

$$\begin{aligned}&\frac{1}{|Q|}\int_Q \sigma_2(x)dx \le \left(\frac{1}{|Q|}\int_Q \|T((b-b_Q)f_1)(x)\|^{r'}dx\right)^{1/r'} \\ &\le C\left(\frac{1}{|Q|}\int_Q \|(b(x)-b_Q)f(x)\|^{s}dx\right)^{1/s}|Q|^{\gamma/n} \le CM_2^s f(x_0).\end{aligned}$$

On the other hand, by using hypotheses (K.1) and (K.2), we have,

$$\begin{aligned}\sigma_3(x) &\le \int \|b(y) - b_Q\|\|f_2(y)\|\|K(x,y) - K(x_0,y)\|dy \\ &\le C\sum_{j=1}^{\infty}\frac{|Q|^{1/n}}{|2^jQ|^{\frac{1}{n}+1-\frac{\gamma}{n}}}\int_{2^{j+1}Q\setminus 2^jQ}\|b(y)-b_Q\|\|f(y)\|dy \\ &\le C'\sum_{j=1}^{\infty}\frac{1}{2^j}\left\{\frac{1}{|2^jQ|^{1-\gamma/n}}\int_{2^jQ}\|b(y)-b_{2^jQ}\|\|f(y)\|dy\right. \\ &\quad\left.+\frac{1}{|2^jQ|^{1-\gamma/n}}\int_{2^jQ}\|b_{2^jQ}-b_Q\|\|f(y)\|dy\right\} \\ &\le C\sum_j\frac{1}{2^j}\left\{M_2f(x_0)+\|b_{2^jQ}-b_Q\|\frac{1}{|2^jQ|^{1-\gamma/n}}\int_{2^jQ}\|f(y)\|dy\right\}.\end{aligned}$$

Using (4.38), we get

$$\begin{aligned}&\sigma_3(x) \le \\ &C\sum_j\frac{1}{2^j}\left\{M_2f(x_0)+j\left(\inf_{y\in 2^jQ}(\nu\mathcal{X}_{2^jQ})^*(y)\right)\left(\frac{1}{|2^jQ|^{1-\gamma/n}}\int_{2^jQ}\|f(y)\|dy\right)\right\} \\ &\qquad\qquad \le C\sum_j\frac{1}{2^j}\left\{M_2f(x_0)+jM_3f(x_0)\right\}.\end{aligned}$$

This finishes the proof of (4.34). ∎

In order to prove (4.37), given a cube Q and a positive compactly supported function f, we decompose f into f_1 and f_2 as before and we consider

$$w_Q = \int |a_Q - a(y)||b_Q - b(y)|W(x_0, y) f_2(y) dy.$$

We observe that w_Q is finite since $b_Q - b$ and $a_Q - a$ belongs to $L^2(Q)$. If $x \in Q$ and $|\cdot|$ is the absolute value in F, standard computations give

$$\begin{aligned}
|V^{+}_{a,b} f(x) - w_Q| &\leq \int |(a(x) - a(y))(b(x) - b(y))W(x, y) f(y) \\
&\quad - (a_Q - a(y))(b_Q - b(y))W(x_0, y) f_2(y)| dy \\
&\leq |a(x) - a_Q| \int |b(x) - b(y)| W(x, y) f(y) dy \\
&\quad + |b(x) - b_Q| \int |a_Q - a(y)| W(x, y) f_1(y) dy \\
&\quad + |b(x) - b_Q| \int |a_Q - a(y)| W(x, y) f_2(y) dy \\
&\quad + \int |a_Q - a(y)||b_Q - b(y)| W(x, y) f_1(y) dy \\
&\quad + \int |a_Q - a(y)||b_Q - b(y)||W(x, y) - W(x_0, y)| f_2(y) dy \\
&= \lambda_1(x) + \lambda_2(x) + \lambda_3(x) + \lambda_4(x) + \lambda_5(x).
\end{aligned}$$

For λ_3, and since $a_Q = \frac{1}{|Q|}\int_Q a(z)dz$ we have

$$\lambda_3(x) = |b(x) - b_Q| \int |\frac{1}{Q} \int_Q (a(z) - a(y)) W(x, y) f_2(y) dz | dy \, .$$

Then,

$$\begin{aligned}
\lambda_3(x) &\leq |b(x) - b_Q| \frac{1}{|Q|} \int_Q \int |a(z) - a(y)||W(x, y) - W(z, y)| f_2(y) dy dz \\
&\quad + |b(x) - b_Q| \left(\frac{1}{|Q|} \int_Q V_a^{+}(f)(z) dz \right) + |b(x) - b_Q| \left(\frac{1}{|Q|} \int_Q V_a^{+}(f_1)(z) dz \right) \\
&= \lambda_{3,1}(x) + \lambda_{3,2}(x) + \lambda_{3,3}(x) \, .
\end{aligned}$$

It is clear that

$$\frac{1}{|Q|} \int_Q \|\lambda_1(x)\| dx \leq M_4(V_b^{+} f)(x_0) \, .$$

Choose $u = \frac{2n}{n+\gamma}$, then $\frac{1}{u} - \frac{1}{u'} = \frac{\gamma}{n}$. Therefore by the hypothesis on V, we have,

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q \|\lambda_2(x)\|dx \\
&\quad\leq \left(\frac{1}{|Q|}\int_Q |b(x)-b_Q|^u dx\right)^{1/u}\left(\frac{1}{|Q|}\int \|V(|a_Q - a|f_1)(x)\|^{u'}dx\right)^{1/u'} \\
&\leq C\left(\frac{1}{|Q|}\int_Q |b(x)-b_Q|^u dx\right)^{1/u}|Q|^{\gamma/n-1/u}\left(\int_Q (|a_Q - a(x)|f(x))^u dx\right)^{1/u} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\leq CM_5^u f(x_0).
\end{aligned}$$

In order to handle $\lambda_{3,1}(x)$ we observe that

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q (|a(z)-a(y)||W(x,y)-W(z,y)|f_2(y)dy)\,dz \\
&\leq \frac{1}{|Q|}\int_Q\left(\sum_j \frac{1}{2^j|2^jQ|^{1-\gamma/n}}\int_{2^jQ}|a(z)-a(y)|f(y)dy\right)dz \\
&\leq \sum_j \frac{|2^jQ|}{2^j}\frac{1}{|Q|}\int_Q\left(\frac{1}{|2^jQ|}\int_{2^jQ}|a(z)-a_Q+a_Q-a_{2^jQ}+a_{2^jQ}-a(y)|f(y)\right)dz \\
&\leq \sum_j \frac{|2^jQ|^{\gamma/n}}{2^j}\frac{1}{|Q|}\int_{2^jQ}\frac{1}{|2^jQ|}\int_{2^jQ}|a(z)-a_Q|f(y)dydz \\
&\quad+\sum_j \frac{|2^jQ|^{\gamma/n}}{2^j}\frac{1}{|Q|}\int_Q\frac{1}{|2^jQ|}\int_{2^jQ}|a_Q-a_{2^jQ}|f(y)dydz \\
&\quad+\sum_j \frac{|2^jQ|^{\gamma/n}}{2^j}\frac{1}{|Q|}\int_Q\frac{1}{|2^jQ|}\int_{2^jQ}|a_{2^jQ}-a(y)|f(y)dydz.
\end{aligned}$$

By Lemma (4.38) this is less than or equal to

$$\begin{aligned}
&\sum_j \frac{|2^jQ|^{\gamma/n}}{2^j}\left(\frac{1}{|Q|}\int_Q |a(z)-a_Q|dz\right)\left(\frac{1}{|2^jQ|}\int_{2^jQ}f(y)dy\right) \\
&\quad+C\sum_j |2^jQ|^{\gamma/n}\frac{j}{2^j}\left(\inf_{y\in 2^jQ}(\nu\mathcal{X}_Q)^*(y)\right)\left(\frac{1}{|2^jQ|}\int_{2^jQ}f(y)dy\right) \\
&\quad+\sum_j \frac{|2^jQ|}{2^j}\left(\frac{1}{|2^jQ|}\int_{2^jQ}|a_{2^jQ}-a(y)|f(y)dy\right).
\end{aligned}$$

Therefore,

$$\sup \frac{1}{|Q|}\int_Q \|\lambda_{3,1}(x)\|_F dx \le C\sum_j \Big\{2^{-j}M_{10}^j f(x_0) + j2^{-j}M_{11}^j f(x_0) + 2^{-j}M_{12}^j f(x_0)\Big\}.$$

It is clear that

$$\frac{1}{|Q|}\int_Q \|\lambda_{3,2}(x)\|dx \le CM_{13}(V_a^+ f)(x_0).$$

On the other hand if $u = \frac{2n}{n+\gamma}$, we have $\frac{1}{u} - \frac{1}{u'} = \frac{\gamma}{n}$, then by lemma (4.47) and Theorem (2.7) for the case $\alpha^{1/2}\beta^{-1/2} = \nu$, we have,

$$\frac{1}{|Q|}\int_Q \|\lambda_{33}(x)\|dx \le \left(\frac{1}{|Q|}\int_Q |b(x)-b_Q|dx\right) \left(\frac{1}{|Q|}\int_Q \|V_a^+ f_1(z)\|^{u'}\beta^{u'/2}(z)dz\right)^{1/u'}\left(\frac{1}{|Q|}\int_Q \beta^{-u/2}(z)dz\right)^{1/u}$$
$$\le \left(\frac{1}{|Q|}\int_Q |b(x)-b_Q|dx\right)|Q|^{\gamma/n}\left(\frac{1}{|Q|}\int_Q f(z)^u\alpha^{u/2}(z)dz\right)^{1/u} \left(\frac{1}{|Q|}\int_Q \beta^{-u/2}(z)dz\right)^{1/u} \le CM_{14}^u f(x_0).$$

We handle $\lambda_4(x)$ as follows. Choose $\frac{1}{s} - \frac{1}{r'} = \frac{\gamma}{n}$ and $1 \le s < (1+\varepsilon)$, then by the hypotheses on V, we have,

$$\frac{1}{|Q|}\int_Q \|\lambda_4(x)\|dx \le \left(\frac{1}{|Q|}\int_Q \|V(|a_Q-a||b_Q-b|f_1)(x)\|^{r'}dx\right)^{1/r'}$$
$$\le C|Q|^{\gamma/n}\left(\frac{1}{|Q|}\int_Q (|a_Q-a(x)||b_Q-b(x)|f(x))^s dx\right)^{1/s} \le CM_6^s f(x_0).$$

Finally we observe that by (W.3), we have,

$$\|\lambda_5(x)\| \le C\sum_j \frac{|2^jQ|^{\gamma/n}}{2^j|2^jQ|}\int_{2^jQ}|a_Q-a(y)||b_Q-b(y)|f(y)dy, \qquad \text{and}$$

$$\int_{2^jQ}|a_Q-a(y)||b_Q-b(y)|f(y)dy$$
$$\le \int_{2^jQ}|a_{2^jQ}-a(y)||b_{2^jQ}-b(y)|f(y)dy + |a_Q-a_{2^jQ}|\int_{2^jQ}|b_{2^jQ}-b(y)|f(y)dy$$
$$+ |b_Q-b_{2^jQ}|\int_{2^jQ}|a_{2^jQ}-a(y)|f(y)dy + |b_Q-b_{2^jQ}||a_Q-a_{2^jQ}|\int_{2^jQ}f(y)dy.$$

By lemma (4.38), this is less than

$$\int_{2^jQ}|a_{2^jQ}-a(y)||b_{2^jQ}-b(y)|f(y)dy$$
$$+j\left(\inf_{y\in 2^jQ}(\nu\mathcal{X}_{2^jQ})^*(y)\right)\int_{2^jQ}|b_{2^jQ}-b(y)|f(y)dy$$
$$+j\left(\inf_{y\in 2^jQ}(\nu\mathcal{X}_{2^jQ})^*(y)\right)\left(\int_{2^jQ}|a_{2^jQ}-a(y)|f(y)dy\right)$$
$$+j^2\left(\inf_{y\in 2^jQ}(\nu\mathcal{X}_{2^jQ})^*(y)\right)^2\left(\int_{2^jQ}f(y)dy\right).$$

Therefore

$$\|\lambda_5(x)\|\leq C\left\{\sum_{j=1}^{\infty}(2^{-j}M_6^1f(x_0)+j2^{-j}M_7f(x_0)\right.$$
$$\left.+j2^{-j}M_8f(x_0)+j^22^{-j}M_9f(x_0)\right\}.$$

Then, we have,

$$\|\lambda_5(x)\|\leq C\left\{M_6^1f(x_0)+M_7f(x_0)+M_8f(x_0)+M_9f(x_0)\right\},$$

ending the proof of (4.37). ■

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