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WEIGHTED INEQUALITIES THROUGH FACTORIZATION

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1. Introduction and results

In [4] P. Jones solved the question posed by B. Muckenhoupt in [7] concerning the factorization of A_p weights. We recall that a non-negative measurable function w on $\mathbf{R}^n$ is in the class A_p, $1 < p < \infty$ if and only if the Hardy-Littlewood maximal operator is bounded on $L^p(\mathbf{R}^n, w)$. In what follows $L^p(X, w)$ denotes the class of all measurable functions f defined on X for which $\|fw^{1/p}\|_{L^p(X)} < \infty$, where X is a measure space and w is a non-negative measurable function on X.

It has recently been proved that the factorization of A_p weights is a particular case of a general factorization theorem concerning positive sublinear operators. The case in which the operator is bounded from $L^p(X,v)$ to $L^p(Y,u), 1 < p < \infty$, for u and v non-negative measurable functions on X and Y respectively, is treated in [8]. The case in which the operator is bounded from $L^p(X,v)$ to $L^q(X,u), 1 < p < q < \infty$ is treated in [3].

Our first result is a factorization theorem for weights u and v associated to operators bounded from $L^p(X,v)$ to $L^q(Y,u)$, where X and Y are two, possibly different, measure spaces, and p and q are any index between 1 and ∞.

Let X and Y be two measure spaces and let $M(X), M(Y)$ be the class of measurable functions defined in X and Y respectively. An operator T defined on a subset of $M(X)$ with values in $M(Y)$ is called **sublinear** if $|T(f+g)| \leq |T(f)| + |T(g)|$ and is called **positive** if $|f| \leq g \mapsto |T(f)| \leq T(g)$, for all $f, g \in M(X)$ which belong to the domain of T.

Theorem 1 (Factorization). *Let T and T' be two positive sublinear operators defined on subsets of $M(X)$ and $M(Y)$ respectively. Let $v \in M(X)$ and $u \in M(Y)$ be non-negative functions and $1 < p,q < \infty$. Suppose that T is bounded from $L^p(X,v)$ to $L^q(Y,u)$ with norm $\|T\|$ and T' is bounded from $L^{q'}(Y,u^{-q'/q})$ to $L^{p'}(X,v^{-p'/p})$ with norm $\|T'\|$. Then there exist non-negative functions $u_0 \in M(X)$, $v_0 \in M(Y)$, $u_1 \in M(Y)$ and $v_1 \in M(X)$ such that $v = u_0^{-p/p'} v_1$, $u = v_0^{-q/q'} u_1$, $\|u_0 v_1\|_{L^1(X)} \leq 1$, $\|v_0 u_1\|_{L^1(Y)} \leq 1$, $T(u_0) \leq \|T\| v_0$ and $T'(u_1) \leq 2^{p/p'} \|T'\| v_1$.*

This theorem can be applied to a large class of operators to obtain the factorization of their associated weights. The reader can find several examples in [8] and [3].

For integral operators with non-negative kernel, the factorization theorem has a converse for some particular cases of p and q. Let $k(x,y)$ be a measurable non-negative function on $X \times Y$. Let us denote by K and K^* the transformations:

$$(Kf)(y) = \int_X k(x,y)f(x)dx, \qquad (K^*g)(x) = \int_Y k(x,y)g(y)dy,$$

the domain of K being the set of all functions $f \in M(X)$ such that the first integral exists and is finite for almost all y, and the domain of K^* being analogously defined.

Theorem 2. *Let $1 < q < p < \infty$ and $v \in M(X)$, $u \in M(Y)$ be non-negative. A necessary and sufficient condition for K to be bounded from $L^p(X,v)$ to $L^q(Y,u)$ is that there exist non-negative functions $u_0 \in M(X)$, $v_0 \in M(Y)$, $u_1 \in M(Y)$, $v_1 \in M(X)$ and finite constants C_0, C_1 such that $\|u_0 v_1\|_{L^1(X)} \le 1$, $v = u_0^{-p/p'} v_1$, $u = v_0^{-q/q'} u_1$, $K(u_0) \le C_0 v_0$ and $K^*(u_1) \le C_1 v_1$. Moreover $\|K\| \le C_0^{1/q'} C_1^{1/q}$.*

The case $p = q$ is simpler:

Theorem 3. *Let $v \in M(X)$ and $u \in M(Y)$ be non-negative. A necessary and sufficient condition for K to be bounded from $L^p(X,v)$ to $L^p(Y,u)$ is that there exist non-negative functions $u_0 \in M(X)$, $v_0 \in M(Y)$, $u_1 \in M(Y)$, $v_1 \in M(X)$ and finite constants C_0, C_1 such that $v = u_0^{-p/p'} v_1$, $u = v_0^{-p/p'} u_1$, $K(u_0) \le C_0 v_0$ and $K^*(u_1) \le C_1 v_1$. Moreover $\|K\| \le C_0{}^{1/p'} C_1^{1/p}$.*

The case $v \equiv u \equiv 1$ of theorems 2 and 3 is proved in [1]. Our proof of these theorems is an adaptation of the proof of the corresponding results in [1]. In the case $p < q$ the conditions of theorem 2 are not sufficient for the boundedness of K from $L^p(X,v)$ to $L^q(X,u)$ even in the case $v \equiv u \equiv 1$ (see [1]). Observe that in theorem 2 we only need the condition $\|u_0 v_1\|_{L^1(X)} \le 1$ while the "symmetric" condition $\|u_1 v_0\|_{L^1(Y)} \le 1$ is not needed. Neither of these is needed in theorem 3.

For some applications it is better to replace the sufficient condition of theorem 2 by the following one, whose statement is a generalization of the sufficient condition of theorem 3:

Theorem 4. *Let $1 < q < p < \infty$ and $v \in M(X)$, $u \in M(Y)$ be non-negative. Suppose that there exist non-negative mesurable functions u_0, v_0, u_1, v_1 such that $v = u_0{}^{-p/p'} v_1{}^{q'/p'}$, $u = u_1 v_0{}^{-p/q'}$, $K(u_0)v_0^{-1} \in L^r(u)$ (with $L^r(u)$ norm equal to C_0) and $K^*(u_1)v_1^{-1} \in L^r(v^{-p'/p})$ (with $L^r(v^{-p'/p})$ norm equal to C_1), where $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$. Then, K is a bounded operator from $L^p(X,v)$ to $L^q(Y,u)$ with norm less than or equal to $C_0{}^{r'/q'} C_1^{r'/p}$.*

For the cases $q = 1$ or $p = \infty$, which are not covered by the above theorems, we have the following satisfactory result:

Theorem 5. *(A) If $1 \leq p \leq \infty$, a necessary and sufficient condition for K to be bounded from $L^p(X, v)$ to $L^1(Y, u)$ with norm $\|K\|$ is*

$$\| \int_Y k(x,y)u(y)dy \|_{L^{p'}(X, v^{-p'/p})} \leq \|K\|$$

(B) If $1 \leq q \leq \infty$, a necessary and sufficient condition for K to be bounded from $\underline{L}^\infty(X, v)$ to $L^q(Y, u)$ with norm $\|K\|$ is

$$\| \int_X k(x,y)v^{-1}(y)dy \|_{L^q(Y,u)} \leq \|K\|$$

In this theorem $\underline{L}^\infty(X, v) = \{f \in M(X) : \|fv\|_\infty < \infty\}$.

Examples of operators to which these theorems can be applied are the following: the **Hardy operator**

$$Tf(x) = \int_0^x f(y)dy \qquad , x > 0,$$

and its dual

$$T^* f(x) = \int_x^\infty f(y)dy \qquad , x > 0;$$

the **fractional integral operator**

$$(I_\alpha f)(x) = \int_{\mathbf{R}^n} f(x-y)|y|^{\alpha - n} dy, \qquad x \in \mathbf{R}^n,\ 0 \leq \alpha < n$$

which is self-adjoint; the **Riemann-Liouville operator**

$$(T_\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(y)}{(x-y)^{1-\alpha}} dy, \qquad x \geq 0,\ \alpha \geq 0;$$

the **Laplace transform**

$$\mathcal{L}f(x) = \int_0^\infty e^{-xy} f(y)dy$$

and the **multidimensional Hardy operator**

$$T_n f(x_1, \ldots, x_n) = \int_0^{x_1} \cdots \int_0^{x_n} f(y_1, \ldots, y_n) dy_n \ldots dy_1.$$

The proofs of theorems 1 to 5 will be given in section 2. Applications will be given in section 3. These are concerned with weighted inequalities for some of the above operators.

I would like to thank B. Jawerth for calling my attention to [1], which turned out to be the starting point of this research.

2. Proofs of Theorems 1 to 5

To prove theorem 1 we need the following lemma which can be found in [1].

Lemma 2.1. *Let B be a Banach space and P a convex cone in B. By calling this cone "positive", B will be taken as an ordered Banach space. Let us suppose for B and P that every bounded increasing sequence in P converges, more precisely:*

$$\{f_n\} \subset P,\ f_{n+1} - f_n \in P,\ \|f_n\| \leq M < \infty \Rightarrow f_n \to f \in P$$

Let S be a transformation defined in B such that $S(P) \subset P$, S is nondecreasing (that is, $f, g, g - f \in P \Rightarrow Sg - Sf \in P$), S is continuous and $\|f\| \leq 1 \Rightarrow \|Sf\| \leq C_0 < \infty$.

Then there exists $\alpha \in P$, $\alpha \neq 0$, $\|\alpha\| \leq 1$ such that

$$2C_0\alpha - S\alpha \in P$$

To prove **theorem 1** we take $B = L^p(X)$, P such that $f \in P \Leftrightarrow f(x) \geq 0\ a.e.$ and

$$S(f) = \left[T'\left(\left(\frac{T(fv^{-1/p})}{\|T\|}\right)^{q/q'} u\right)\right]^{p'/p} v^{-p'/p^2}$$

and apply lemma 2.1. Observe that the boundedness of T and T' implies

$$\|Sf\|_{L^p(X)} \leq \|T'\|^{p'/p} \|f\|_{L^p(X)}^{p'q/q'p}$$

so that $\|f\| \leq 1 \Rightarrow \|Sf\| \leq \|T'\|^{p'/p}$. Hence there exists $\alpha \neq 0$, $\alpha \geq 0$, $\alpha \in L^p(X)$ with norm less than or equal to 1 and $S(\alpha) \leq 2\|T'\|^{p'/p}\alpha$. The proof of theorem 1 is finished by taking

$$u_0 = \alpha v^{-1/p},\ v_0 = T(u_0)/\|T\|,\ u_1 = (T(u_0)/\|T\|)^{q/q'} u$$

and $v_1 = u_0^{p/p'} v$.

Theorems 2 and 3 will be established once we prove the sufficiency of the conditions, since the necessity follows immediately from theorem 1. To prove theorem 2 take $f \in L^p(X, v)$, $g \in L^{q'}(Y, u^{-q'/q})$ and use Holder's inequality with r, p and q', where $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$ to obtain

(2.2)

$$\left|\int_X\int_Y k(x,y)f(x)g(y)dxdy\right| \leq \left(\int_X\int_Y k(x,y)u_1(y)u_0(x)dxdy\right)^{1/r}$$
$$\cdot\left(\int_X\int_Y k(x,y)u_1(y)u_0(x)^{-p/p'}|f(x)|^p dxdy\right)^{1/p}$$
$$\cdot\left(\int_X\int_Y k(x,y)u_0(x)u_1(y)^{-q'/q}|g(y)|^{q'} dxdy\right)^{1/q'}.$$

Using $K^*(u_1) \leq C_1v_1$ and $\|u_0v_1\|_{L^1(X)} \leq 1$ the first factor on the right hand side of the above inequality is bounded by $C_1^{1/r}$. Using $K^*(u_1) \leq C_1v_1$ and $v = u_0^{-p/p'}v_1$ we deduce that the second factor is bounded by $C_1^{1/p}\|f\|_{L^p(X,v)}$. Finally, using $K(u_0) \leq C_0v_0$ and $u = v_0^{-q/q'}u_1$ the third factor can be majorated by $C_0^{1/q'}\|g\|_{L^{q'}(Y,u^{-q'/q})}$. Putting these estimates together we obtain

$$\left|\int_X\int_Y k(x,y)f(x)g(y)dxdy\right| \leq C_0^{1/q'}C_1^{1/q}\|f\|_{L^p(X;v)}\|g\|^{q'}_{L(Y;u^{-q'/q})}.$$

From here the desired result follows.

The same argument applies for $p = q$, that is for theorem 3, except that in this case $\frac{1}{r} = \frac{1}{q} - \frac{1}{p} = 0$, and hence the first factor on the right hand side of (2.2) does not appear. Thus theorem 3 does not require the condition $\|u_0v_1\|_{L^1(X)} \leq 1$.

To prove **theorem 4** let $f \in L^p(X,v)$ and $g \in L^{q'}(Y,u^{-q'/q})$. From $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$ we deduce $\frac{1}{q'} + \frac{1}{p} = \frac{1}{r'}$ so that we can apply Holder's inequality with indices p/r' and q'/r' to obtain

$$\left|\int_X\int_Y k(x,y)f(x)g(y)dxdy\right|$$
$$\leq \left(\int_X\int_Y k(x,y)u_1(y)u_0(x)^{-p/q'}|f(x)|^{p/r'}dxdy\right)^{r'/p}$$
$$\cdot\left(\int_X\int_Y k(x,y)u_0(x)u_1(y)^{-q'/p}|g(y)|^{q'/r'}dxdy\right)^{r'/q'} = (I).(II).$$

Using Holder's inequality with index r, together with $v = u_0^{-p/p'}v_1^{q'/p'}$ and $K^*(u_1)v_1^{-1} \in L^r(v^{-p'/p})$ we obtain

$$(I) \leq \left(\int_X [K^*u_1)(x)]^r v_1^{-r}(x)v^{-p'/p}(x)dx\right)^{r'/rp}$$

$$\cdot\left(\int_X |f(x)|^p v_1(x)^{r'} v(x)^{p'r'/pr} u_0(x)^{-pr'/q'} dx\right)^{1/p}$$

$$= C_1^{r'/p}\left(\int_X |f(x)|^p v(x) dx\right)^{1/p}.$$

Using again Holder's inequality with the same index, together with $u = u_1 v_0^{-p/q'}$ and $K(u_0)v_0^{-1} \in L^r(u)$ we obtain

$$(II) \leq \left(\int [K(u_0)(y)]^r v_0(y)^{-r} u(y) dy\right)^{r'/rq'}$$

$$\cdot\left(\int_y |g(y)|^{q'} u_1(y)^{-q'r'/p} v_0(y)^{r'} u(y)^{-r'/r} dy\right)^{1/q'}$$

$$= C_0^{r'/q'}\left(\int_y |g(y)|^{q'} u(y)^{-q'/q} dy\right)^{1/q'}.$$

The desired result follows by putting these two estimates together. This finishes the proof of theorem 4.

We now prove **theorem 5** .

(A) **Sufficiency** . For $f \in L^p(X,v)$ and $g \in \underline{L}^\infty(Y,u^{-1})$ we have

$$\left|\int_X\int_Y k(x,y)f(x)g(y)dxdy\right| \leq \|g\|_{\underline{L}^\infty(Y,u^{-1})}\int_X\int_Y k(x,y)|f(x)|u(y)dxdy$$

$$\leq \|g\|_{\underline{L}^\infty(Y,u^{-1})}\left\{\int_X\left(\int_Y k(x,y)u(y)dy\right)^{p'} v(x)^{-p'/p}dx\right\}^{1/p'}$$

$$\left\{\int_X |f(x)|^p v(x) dx\right\}^{1/p} \leq \|g\|_{\underline{L}^\infty(Y,u^{-1})}\|K\|\|f\|_{L^p(X,v)}.$$

Necessity . The boundedness of K implies

$$\left|\int_X\int_Y k(x,y)f(x)g(y)dydx\right| \leq \|K\|\|f\|_{L^p(X,v)}\|g\|_{\underline{L}^\infty(Y,u^{-1})}$$

for all $f \in L^p(X,v)$, $g \in \underline{L}^\infty(Y,u^{-1})$. With $g \equiv u$ we obtain

$$\left|\int_X f(x)\left(\int_Y k(x,y)u(y)dy\right)dx\right| \leq \|K\|\|f\|_{L^p(X,v)}$$

for all $f \in L^p(X,v)$. Thus the result follows.

(B) The proof is analogous.

3. Applications

Consider the integral transformations

$$(3.1)\qquad (Kf)(x) = \int_{-\infty}^{x} k(x,y)f(y)dy,\ (K^{*}f)(x) = \int_{x}^{\infty} k(y,x)f(y)dy$$

defined on the real line, where $k(x,y)$ is a nonnegative measurable function defined on $\Delta = \{(x,y) \in \mathbb{R}^2 : y < x\}$. Given two non-negative measurable functions u and v defined on the real line, we write $(u,v) \in W_1(K,p,q)$, $1 \leq q \leq p < \infty$ if

$$(3.2)\quad B_1 = \left\{\int_{-\infty}^{\infty}\left[\left(\int_{y}^{\infty} k(x,y)u(x)dx\right)^{1/q}\left(\int_{-\infty}^{y} k(y,z)v(z)^{-p'/p}dz\right)^{1/q'}\right]^{r} v(y)^{-p'/p}dy\right\}^{1/r} < +\infty$$

and $(u,v) \in W_2(K,p,q)$, $1 < q \leq p \leq \infty$ if

$$(3.3)\quad B_2 = \left\{\int_{-\infty}^{\infty}\left[\left(\int_{y}^{\infty} k(x,y)u(x)dx\right)^{1/p}\left(\int_{-\infty}^{y} k(y,z)v(z)^{-p'/p}dz\right)^{1/p'}\right]^{r} u(y)dy\right\}^{1/r} < +\infty$$

where $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$. Observe that in the limiting case $p = q$, (3.2) and (3.3) become

$$(3.4)\quad B = \sup_{y>0}\left(\int_{y}^{\infty} k(x,y)u(x)dx\right)^{1/p}\left(\int_{-\infty}^{y} k(y,z)v(z)^{-p'/p}dz\right)^{1/p'} < +\infty$$

Proposition 3.1. *Let K be the integral transformation defined by (3.1), where $k(x,y) \geq 0$ is nondecreasing in y and nonincreasing in x. If $(u,v) \in W_j(K,p,q)$, $1 < q \leq p < \infty$, $j = 1,2$, then*

$$\left(\int_{-\infty}^{\infty} |(Kf)(x)|^{q}u(x)dx\right)^{1/q} \leq C\left(\int_{-\infty}^{\infty} |f(x)|^{p}v(x)dx\right)^{1/p}$$

where $C \le (qB_1)^{r'/p}(p'B_2)^{r'/q'}$.

Proof: Since $t \ge y$ implies $k(x,t) \ge k(x,y)$ the norm of the function

$$\varphi(y) = \left(\int_y^\infty k(t,y)\left(\int_t^\infty k(x,t)u(x)dx\right)^{-1/q'} u(t)dt\right)\left(\int_{-\infty}^y k(y,z)v(z)^{-p'/p}dz\right)^{1/q'}$$

in $L^r(v^{-p'/p})$ is bounded by the norm of the function

$$\tilde\varphi(y) = \left(\int_y^\infty k(t,y)\left(\int_t^\infty k(x,y)u(x)dx\right)^{-1/q'} u(t)dt\right)\left(\int_{-\infty}^y k(y,z)v(z)^{-p'/p}dz\right)^{1/q'}$$

in the same space. Integrating by parts we obtain

$$\|\varphi\|_{L^r(v^{-p'/p})} \le qB_1.$$

Taking

$$u_1(t) = \left(\int_t^\infty k(x,t)u(x)dx\right)^{-1/q'} u(t),$$
$$v_1(y) = \left(\int_{-\infty}^y k(y,z)v(z)^{-p'/p}dz\right)^{-1/q'}$$

the above inequality can be written as $K^*(u_1)v_1^{-1} \in L^r(v^{-p'/p})$ with norm not exceeding qB_1.

Since $t \le y$ implies $k(t,z) \ge k(y,z)$, the norm of the function

$$\phi(y) = \left(\int_y^\infty k(x,y)u(x)dx\right)^{1/p}\left(\int_{-\infty}^y k(y,t)v(t)^{-p'/p}\left(\int_{-\infty}^t k(t,z)v(z)^{-p'/p}dz\right)^{-1/p} dt\right)$$

in $L^r(u)$ is bounded by the norm of the function

$$\tilde\phi(y) = \left(\int_y^\infty k(x,y)u(x)dx\right)^{1/p}\int_{-\infty}^y k(y,t)v(t)^{-p'/p}\left(\int_{-\infty}^t k(y,z)v(z)^{-p'/p}dz\right)^{-1/p} dt$$

in the same space. Integrating by parts we obtain

$$\|\phi\|_{L^r(u)} \leq p' B_2.$$

Taking

$$u_0(t) = \left(\int_{-\infty}^{t} k(y,z) v(z)^{-p'/p} dz\right)^{-1/p} v(t)^{-p'/p},$$
$$v_0(t) = \left(\int_{t}^{\infty} k(x,t) u(x) dx\right)^{-1/p}$$

the above inequality can be written as $K(u_0)v_0^{-1} \in L^r(u)$ with norm not exceeding $p'B_2$. The proof of proposition 3.1 is finished by applying theorem 4 if $q < p$ and theorem 3 if $q = p$. ■

Remarks. 1. For the case $q = 1$, $(u,v) \in W_1(K,p,1)$ is an equivalent condition for the boundedness of the operator K defined by (3.1) from $L^p(X,v)$ to $L^1(Y,u)$. This follows from part (A) of theorem 5.

2. For the case $p = \infty$, $(u,v) \in W_2(K,\infty,q)$ is an equivalent condition for the boundedness of K from $\underline{L}^\infty(X,v)$ to $L^q(Y,u)$. This follows from part (B) of theorem 5.

3. A result similar to proposition 3.1. can be obtained for K^*. Details are left to the interested reader.

4. The operators defined by (3.1) have been studied in [2]; the conditions imposed on the weights u and v to obtain weighted inequalities for K are different from those used here.

For the particular case of the Hardy operator $Tf(x) = \int_0^x f(y)dy$, $W_1(T,p,q)$ becomes

$$(3.5) \quad B_1 = \left\{\int_0^\infty \left[\left(\int_y^\infty u\right)^{1/q} \left(\int_0^y v^{-p'/p}\right)^{1/q'}\right]^r v(y)^{-p'/p} dy\right\}^{1/r} < +\infty,$$

$W_2(T,p,q)$ becomes

$$(3.6) \qquad B_2 = \left\{\int_0^\infty \left[\left(\int_y^\infty u\right)^{1/p} \left(\int_0^y v^{-p'/p}\right)^{1/p'}\right]^r u(y) dy\right\}^{1/r} < +\infty,$$

and when $p = q$ we have

$$(3.7) \qquad B = \sup_{y>0} \left(\int_y^\infty u\right)^{1/p} \left(\int_0^y v^{-p'/p}\right)^{1/p'} < +\infty.$$

In this case we shall show that the condition $W_2(T,p,q)$ is implied by W_1 (T,p,q) and that this condition is also necessary for the boundedness of T.

Proposition 3.2. *Let $1 < q \le p < \infty$. A necessary and sufficient condition for the Hardy operator $Tf(x) = \int_0^x f(y)dy$ to be bounded from $L^p(v)$ to $L^q(u)$ with norm C is $(u,v) \in W_1(T,p,q)$. Moreover $2^{-p/p'q}B_1 \le C \le 2^{r'/rq'}B_1(p')^{1/q'}q^{1/q}$.*

Proof: **Sufficiency.** Reducing the interval of integration and using the condition $W_1(T,p,q)$ we deduce

$$\left(\int_z^\infty u\right)^{1/q}\left(\int_0^z v^{-p'/p}\right)^{1/p'}\left(\frac{p'}{r}\right)^{1/r} \le B_1 \tag{3.8}$$

for all $z \in (0,\infty)$. Integrating by parts in (3.5) and using (3.8) we obtain

$$B_1^r = \frac{p'}{r}\left(\int_y^\infty u\right)^{r/q}\left(\int_0^y v^{-p'/p}\right)^{r/p'}\Bigg]_0^\infty +$$

$$\frac{p'}{q}\int_0^\infty\left(\int_y^\infty u\right)^{r/p}\left(\int_0^y v^{-p'/p}\right)^{r/p'} u(y)dy \ge -B_1^r + \frac{p'}{q}B_2^r.$$

Hence $B_2^r \le 2B_1^r\frac{q}{p'}$ and the result now follows from proposition 3.1.

Necessity. Since we are assuming that T is bounded from $L^p(v)$ to $L^q(u)$, we can apply theorem 1 to find u_0, v_0, u_1, v_1 satisfying

$$\|u_0v_1\|_{L^1} \le 1 \tag{3.9}$$

and

$$v = u_0^{-p/p'}v_1,\ u = v_0^{-q/q'}u_1,\ T(u_0) \le Cv_0,\ T^*(u_1) \le 2^{p/p'}Cv_1.$$

Therefore

$$B_1^r = \int_0^\infty\left(\int_y^\infty v_0(t)^{-q/q'}u_1(t)dt\right)^{r/q}\left(\int_0^y u_0(t)v_1(t)^{-p'/p}dt\right)^{r/q'} v(y)^{-p'/p}dy.$$

From $Tu_0 \le Cv_0$ and $T^*(u_1) \le 2^{p/p'}Cv_1$ we deduce that B_1^r is bounded by

$$2^{r/q'}C^{rp'/q'}\left\{\int_0^\infty\left(\int_0^y u_0\right)^{-r/q'}\left(\int_y^\infty u_1\right)^{r/q}\right.$$

$$\left.\left(\int_y^\infty u_1\right)^{-rp'/pq'}\left(\int_0^y u_0\right)^{r/q'} u_0(y)v_1(y)^{-p'/p}dy\right\}$$

$$\le 2^{r/p'}C^{r-1}\left\{\int_0^\infty\left(\int_y^\infty u_1\right)u_0(y)dy\right\}$$

$$\le 2^{(r+p)/p'}C^r\int_0^\infty v_1(y)u_0(y)dy \le 2^{(r+p)/p'}C^r,$$

where the last inequality is due to (3.9). This finishes the proof of proposition 3.2. ■

Remarks. 1. The limiting cases $q = 1$ and $p = \infty$ of proposition 3.2 are also true; they can be deduced directly from theorem 5.

2. There is a similar result for the dual of the Hardy operator, $T^*f(x) = \int_x^\infty f(y)dx$; details are left for the interested reader.

3. Proposition 3.2 can be found in [6] for the case $p = q$ and in [5] for $q < p$. We feel that our proof is easier than that given in [5], page 45.

To end this section we use theorems 2 and 3 to find particular weights for the Laplace transform $\mathcal{L}$ and the Riemann-Liouville operator T_α, $\alpha > 0$.

Corollary 3.3. *Let* $1 < p < \infty$ *and* $a > -1$. *Then*

$$\left\{\int_0^\infty |(\mathcal{L}f)(x)|^p x^a dx\right\}^{1/p} \leq C\left\{\int_0^\infty |f(x)|^p x^{-a+p-2} dx\right\}^{1/p}$$

and $C \leq \Gamma(\beta+1)^{1/p'}\Gamma'(a - \beta\frac{p}{p'} + 2 - p)^{1/p}$ *for all* $-1 < \beta < \frac{p'}{p}a + \frac{p'}{p} - 1$.

Proof: Since $a > -1$ we can choose β such that $-1 < \beta < \frac{p'}{p}a + \frac{p'}{p} - 1$. Let $u_0(x) = x^\beta$ and $v_1(x) = x^{-a+p-2+\beta\frac{p}{p'}}$ so that $u_0^{-p/p'}v_1 = x^{-a+p-2}$. We have $(\mathcal{L}u_0)(x) = \Gamma(\beta+1)/x^{\beta+1}$. Let $v_0(x) = x^{-(\beta+1)}$ and $u_1(x) = x^{a-(\beta+1)\frac{p}{p'}}$ so that $v_0^{-p/p'}u_1 = x^a$. Moreover $(\mathcal{L}u_1)(x) = \Gamma(a-(\beta+1)\frac{p}{p'}+1)/x^{a-(\beta+1)\frac{p}{p'}+1} = \Gamma(a-(\beta+1)\frac{p}{p'}+1)v_1(x)$. Hence the result follows by theorem 3. ■

Corollary 3.4. *Let* $\alpha > 0$, $1 < p < \infty$ *and* $b < p - 1$. *Then*

$$\left\{\int_0^\infty |(T_\alpha f)(x)|^p x^{b-\alpha p} dx\right\}^{1/p} \leq C\left\{\int_0^\infty |f(x)|^p x^b dx\right\}^{1/p}$$

and $C \leq B(\alpha,\beta+1)^{1/p'}B(\alpha,-b-\frac{p}{p'}\beta)^{1/p}\Gamma(\alpha)^{-1}$ *for all* $-1 < \beta < -\frac{p'}{p}b$, *where* B *is the beta function,* $B(s,r) = \int_0^1 (1-x)^{s-1}x^{r-1}dx$.

Proof: Since $b < p-1$ we can choose β so that $-1 < \beta < -\frac{p'}{p}b$. Let $u_0(x) = x^\beta$ and $v_1(x) = x^{b+\beta\frac{p}{p'}}$ so that $u_o^{-p/p'}v_1 = x^b$. A simple calculation shows $(T_\alpha u_0)(x) = x^{\alpha+\beta}B(\alpha,\beta+1)/\Gamma(\alpha)$. Let $v_0(x) = x^{\alpha+\beta}$ and $u_1 = x^{b-\alpha p+\frac{p}{p'}(\alpha+\beta)}$ so that $v^{-p/p'}u_1 = x^{b-\alpha p}$. Again a calculation shows

$$(T_\alpha^* u_1)(x) = x^{b+\frac{p}{p'}\beta}B(\alpha,-\beta-\frac{p}{p'}\beta)/\Gamma(\alpha) = v_1(x)B(\alpha,-\beta-\frac{p}{p'}\beta)/\Gamma(\alpha).$$

The result follows from theorem 3. ■

Corollary 3.5. *Let $1 \le q < p < \infty$ and let a be such that $(q/p) - (q/q') < a < 1 - (q/q')$ (if $q = 1, \frac{1}{p} < a < 1$). Then*

$$\left(\int_0^\infty |\mathcal{L}f(x)|^q x^{-a} dx\right)^{1/q} \le C\left(\int_0^\infty |f(x)|^p (1+x^2)^{p-1} dx\right)^{1/p}.$$

Proof: Suppose $1 < q < p < \infty$. Choose β so that $\beta(\frac{p}{p'} - \frac{q}{q'}) = a + \frac{q}{q'} - 1$. The conditions on a imply $-\frac{1}{p} < \beta < 0$. Let $u_0(x) = x^\beta(1+x^2)^{-1}$ and $v_1(x) = x^{\beta\frac{p}{p'}}$ so that $u_0^{-p/p'} v_1 = (1+x^2)^{p-1}$. Observe that

$$\int_0^\infty u_0 v_1 = \int_0^\infty x^{\beta p}(1+x^2)^{-1} dx < \infty$$

since $\beta p + 1 > 0$ and $\beta p - 1 < 0$. Now

$$(\mathcal{L}u_0)(x) = \int_0^\infty e^{-xt} t^\beta (1+t^2)^{-1} dt \le \int_0^\infty e^{-xt} t^\beta dt = x^{-\beta-1}\Gamma(\beta+1).$$

Let $v_0(x) = x^{-\beta-1}$ and $u_1(x) = x^{-a-(\beta+1)\frac{q}{q'}}$ so that $v_0^{-q/q'} u_1 = x^{-a}$. Moreover $(\mathcal{L}u_1)(x) = v_1(x)\Gamma(-\beta\frac{p}{p'})$. The result follows by applying theorem 2.

The case $q = 1$ follows from part (A) of theorem 5. ■

Corollary 3.6. *Let $1 \le q < p < \infty$ and let a be such that $-1 + \frac{q}{p} - \alpha q < a < -\alpha q$ where $\alpha > 0$. Then*

$$\left(\int_0^\infty |T_\alpha f(x))|^q x^a dx\right)^{1/q} \le C\left(\int_0^\infty |f(x)|^p e^{xb} dx\right)^{1/p}, \ b > 0.$$

Proof: Suppose $1 < q < p < \infty$. Choose β such that $\beta(\frac{p}{p'} - \frac{q}{q'}) = a + q\alpha$. The conditions on a imply $-\frac{1}{p} < \beta < 0$. Let $u_0(x) = x^\beta e^{\frac{p'}{p}xb}$ and $v_1(x) = x^{\beta\frac{p}{p'}}$ so that $u_0^{-p/p'} v_1 = e^{xb}$. Observe that

$$\int_0^\infty u_0 v_1 = \int_0^\infty x^{p\beta} e^{\frac{p'}{p}xb} dx = C\Gamma(p\beta+1) < \infty$$

since $p\beta + 1 > 0$. Now

$$(T_\alpha u_0)(x) = \frac{1}{\Gamma(\alpha)}\int_0^x \frac{t^\beta e^{-\frac{p'}{p}tb}}{(x-t)^{1-\alpha}} dt \le \frac{1}{\Gamma(\alpha)}\int_0^x \frac{t^\beta}{(x-t)^{1-\alpha}} dt$$
$$= x^{\beta+\alpha} B(\alpha, \beta+1)/\Gamma(\alpha).$$

Let $v_0(x) = x^{\beta+\alpha}$ and $u_1(x) = x^{a+\frac{q}{q'}(\beta+\alpha)}$ so that $v_0^{-q/q'} u_1 = x^a$. Moreover $(T_\alpha^* u_1)(x) = v_1(x) B(\alpha, -a - \frac{q}{q'}(\beta+\alpha) - \alpha)/\Gamma(\alpha)$ and the results follow from theorem 2.

The case $q = 1$ follows from part A of theorem 5. ■

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