Minimum density of identifying codes of king grids

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Abstract

A set $C \subseteq V(G)$ is an *identifying code* in a graph G if for all $v \in V(G)$, $C[v] \neq \emptyset$, and for all distinct $u, v \in V(G)$, $C[u] \neq C[v]$, where $C[v] = N[v] \cap C$ and $N[v]$ denotes the closed neighbourhood of v in G. The minimum density of an identifying code in G is denoted by $d^*(G)$. In this paper, we study the density of king grids which are strong product of two paths. We show that for every king grid G, $d^*(G) \geq 2/9$. In addition, we show this bound is attained only for king grids which are strong products of two infinite paths. Given $k \geq 3$, we denote by $\mathcal{K}_k$ the (infinite) king strip with k rows. We prove that $d^*(\mathcal{K}_3) = 1/3$, $d^*(\mathcal{K}_4) = 5/16$, $d^*(\mathcal{K}_5) = 4/15$ and $d^*(\mathcal{K}_6) = 5/18$. We also prove that $\frac{2}{9} + \frac{8}{81k} \leq d^*(\mathcal{K}_k) \leq \frac{2}{9} + \frac{4}{9k}$ for every $k \geq 7$.

Keywords: Identifying code, King grid, Discharging Method.

1 Introduction

Let G be a graph. The *neighbourhood* of a vertex v of G, denoted by $N(v)$, is the set of vertices adjacent to v in G, and the *closed neighbourhood* of v is the set $N[v] = N(v) \cup \{v\}$. Given a set $C \subseteq V(G)$, let $C[v] = N[v] \cap C$. We say that C is an *identifying code* of G if $C[v] \neq \emptyset$ for all $v \in V(G)$, and $C[u] \neq C[v]$ for all distinct $u, v \in V(G)$. Clearly, a graph G has an identifying code if and only if it contains no *twins* (vertices $u, v \in V(G)$ with $N[u] = N[v]$).

Let G be a (finite or infinite) graph with bounded maximum degree. For any non-negative integer r and vertex v, we denote by $B_r(v)$ the ball of radius r in G centered at v, that is $B_r(v) = \{x \mid \text{dist}(v, x) \leq r\}$. For any set of vertices $C \subseteq V(G)$, the *density* of C in G, denoted by $d(C, G)$, is defined by

$$d(C, G) = \limsup_{r \to +\infty} \frac{|C \cap B_r(v_0)|}{|B_r(v_0)|} \ ,$$

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where v_0 is an arbitrary vertex in G. The infimum of the density of an identifying code in G is denoted by $d^*(G)$. Observe that if G is finite, then $d^*(G) = |C^*|/|V(G)|$, where C^* is a minimum-size identifying code of G.

The problem of finding low-density identifying codes was introduced in [12] in relation to fault diagnosis in arrays of processors. Particular interest was dedicated to grids as many processor networks have a grid topology. Many results have been obtained on square grids [4,1,9,2,11], triangular grids [12,10], and hexagonal grids [5,7,8]. In this paper, we study *king grids*, which are strong products of two paths. The *strong product* of two graphs G and H, denoted by $G \boxtimes H$, is the graph with vertex set $V(G) \times V(H)$ and edge set :

$$\begin{aligned} E(G \boxtimes H) = \quad & \{(a,b)(a,b') \mid a \in V(G) \text{ and } bb' \in E(H)\} \\ & \cup \{(a,b)(a',b) \mid aa' \in E(G) \text{ and } b \in V(H)\} \\ & \cup \{(a,b)(a',b') \mid aa' \in E(G) \text{ and } bb' \in E(H)\}. \end{aligned}$$

The *two-way infinite path*, denoted by $P_{\mathbb{Z}}$, is the graph with vertex set $\mathbb{Z}$ and edge set $\{\{i, i+1\} \mid \in \mathbb{Z}\}$, and the *one-way infinite path*, denoted by $P_{\mathbb{N}}$, is the graph with vertex set $\mathbb{N}$ and edge set $\{\{i, i+1\} \mid i \in \mathbb{N}\}$. A *path* is a connected subgraph of $P_{\mathbb{Z}}$. For every positive integer k, P_k is the subgraph of $P_{\mathbb{Z}}$ induced by $\{1, 2, \dots, k\}$. A *king grid* is the strong product of two (finite or infinite) paths. The *plane king grid* is $\mathcal{G}_K = P_{\mathbb{Z}} \boxtimes P_{\mathbb{Z}}$, the *half-plane king grid* is $\mathcal{H}_K = P_{\mathbb{Z}} \boxtimes P_{\mathbb{N}}$, the *quater-plane king grid* is $\mathcal{Q}_K = P_{\mathbb{N}} \boxtimes P_{\mathbb{N}}$, and the *king strip of height k* is $\mathcal{K}_k = P_{\mathbb{Z}} \boxtimes P_k$.

In 2002, Charon et al. [3] proved that $d^*(\mathcal{G}_K)$ is 2/9. They provided the tile depicted in Figure 1, which generates a periodic tiling of the plane with periods $(0, 6)$ and $(6, 0)$, yielding an identifying code C_∞ of the bidimensional infinite king grid with density $\frac{2}{9}$.

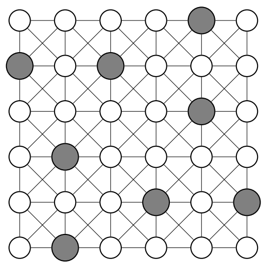

Fig. 1. Tile generating an optimal identifying code of the bidimensional infinite grid. Black vertices are those of the code.

In this paper, using the Discharging Method (see Section 3 of [10] for a detailed presentation of this technique for identifying codes), we provide the following tight general lower bound on the minimum density of identifying codes of king grids.

Theorem 1.1 *If G is a (finite or infinite) king grid, then* $d^*(G) \geq \frac{2}{9}$.

Keeping on, we prove the following.

Theorem 1.2 *If G is a finite king grid, then* $d^*(G) > \frac{2}{9}$.

Finally, we give some bounds for king strips. Pushing further the proof of Theorem 1.1, we prove the following.

Theorem 1.3 *For every $k \geq 6$,* $d^*(\mathcal{K}_k) \geq \frac{2}{9} + \frac{8}{81k}$.

Modifying C_∞, we construct identifying codes of $\mathcal{K}_k$ yielding the following upper bounds.

Theorem 1.4 *For every $k \geq 5$,*

$$d^*(\mathcal{K}_k) \leq \begin{cases} \frac{2}{9} + \frac{6}{18k}, & \text{if } k \equiv 0 \mod 3, \\ \frac{2}{9} + \frac{8}{18k}, & \text{if } k \equiv 1 \mod 3, \\ \frac{2}{9} + \frac{7}{18k}, & \text{if } k \equiv 2 \mod 3. \end{cases}$$

Finally, we show some identifying codes of $\mathcal{K}_3$, $\mathcal{K}_4$, $\mathcal{K}_5$ and $\mathcal{K}_6$ (see Figures 2, 3, 4, and 5.) and prove that they are optimal. This yields the following.

Theorem 1.5 $d^*(\mathcal{K}_3) = 1/3 = 0.333\ldots$ $\quad d^*(\mathcal{K}_4) = 5/16 = 0.3125$
$d^*(\mathcal{K}_5) = 4/15 = 0.2666\ldots$ $\quad d^*(\mathcal{K}_6) = 5/18 = 0.2777\ldots$

Clearly $d^*(\mathcal{K}_1) = 1/2$ (as $\mathcal{K}_1 = \mathcal{S}_1 = P_{\mathbb{Z}}$) and $\mathcal{K}_2$ has no identifying code because it has twins. All these results imply that $\mathcal{G}_K$, $\mathcal{H}_K$ and $\mathcal{Q}_K$ are the unique king grids having an identifying code with density 2/9. (One can easily derive from C_∞ identifying codes with density 2/9 of $\mathcal{H}_K$ and $\mathcal{Q}_K$).

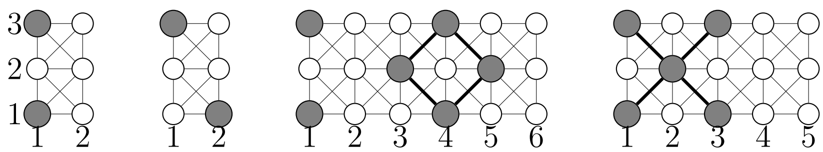

Fig. 2. Four tiles generating optimal identifying codes of $\mathcal{K}_3$ (density 1/3)

2 Sketches of proofs

Sketch of proof of Theorem 1.1. Let G be a king grid and C an identifying code of G. We shall prove that $d(C, G) \geq 2/9$. For this, we use the Discharging

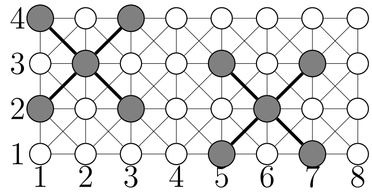

Fig. 3. Tile generating an optimal identifying code of $\mathcal{K}_4$ (density 5/16)

Fig. 4. Two tiles generating optimal identifying codes of $\mathcal{K}_5$ (density 4/15)

Fig. 5. Two tiles generating optimal identifying codes of $\mathcal{K}_6$ (density 5/18)

Method. The initial charge of a vertex v is 1 if $v \in C$ and 0 otherwise. We then apply some local discharging rules. We shall prove that the final charge of every vertex in C is at least 2/9. This would imply the result.

We set $U = V(G) \setminus C$. Given $X \subseteq V(G)$ and $1 \leq i \leq 9$, we denote by X_i (resp. $X_{\geq i}$) the set of vertices in X having exactly i vertices (resp. at least i vertices) in their identifier. An X*-vertex* is a vertex in X. A vertex is *full* if its eight neighbours in $\mathcal{G}_K$ are in G; otherwise it is a *side vertex*.

We first establish the following properties of C.

(i) Two C_2-vertices are not adjacent.

(ii) Every C-vertex has at most one neighbour in U_1.

(iii) Every full C_2-vertex has at least three neighbours in $U_{\geq 3}$.

(iv) Every full C_3-vertex has a neighbour in $U_{\geq 3}$.

(v) Every C_1-vertex (a,b) has no neighbour in U_1 and at most six neighbours in U_2. Furthermore, if it has six neighbours in U_2, then either $\{(a-1,b-2),(a-2,b-1),(a+2,b+1),(a+1,b+2)\} \subseteq C$ or $\{(a+1,b-2),(a+2,b-1),(a-2,b+1),(a-1,b+2)\} \subseteq C$.

A *defective vertex* is a vertex in C_1 with six neighbours in U_2. Let $v=(a,b)$ be a defective vertex. The *team* of v is a set among $\{(a-1,b-2),(a-2,b-1),(a+2,b+1),(a+1,b+2)\}$ and $\{(a+1,b-2),(a+2,b-1),(a-2,b+1),(a-1,b+2)\}$ which is included in C. By Property (v), the team exists. Moreover, by Property (i), at least two vertices of the team are in $C_{\geq 3}$. Those vertices are the *partners* of v.

We apply the following discharging rules.

(R1) Every C-vertex sends $\frac{2}{9i}$ to each of its neighbours in U_i.

(R2) Every defective vertex receives $\frac{1}{54}$ from each of its partners.

Using the above properties, we then prove that the final charge of every vertex v is at least $2/9$. □

Sketch of proof of Theorem 1.2. We only need to prove that, at the end of the proof of Theorem 1.1, one vertex has final charge greater than $2/9$. To do so we shall prove that there is a side C-vertex or a $C_{\geq 3}$-vertex and check that such a vertex has final charge at least $\frac{2}{9}+\frac{1}{27}$. □

Sketch of proof of Theorem 1.3. Using the Discharging Method, we prove that in average, for every column, there is an extra charge of at least $\frac{4}{81}$ on the three top vertices and an an extra charge of at least $\frac{4}{81}$ on the three top vertices. □

Sketch of proof of Theorem 1.5. The bth row of $\mathcal{K}_k$ is $R_b=\{(a,b) \mid a \in \mathbb{Z}\}$. We have $d(C,\mathcal{K}_k)=\frac{1}{k}\sum_{i=1}^{k} d(C,R_k)$. We show that if C is an identifying code of $\mathcal{K}_k$ $(k \geq 3)$, then $d(C,R_1)+d(C,R_2) \geq 1/2$, $d(C,R_k)+d(C,R_{k-1}) \geq 1/2$, $d(C,R_3) \geq 1/3$ and $d(C,R_{k-2}) \geq 1/3$. One easily derives that if C is an identifying code of $\mathcal{K}_5$ (resp. $\mathcal{K}_6$), then $d(C,\mathcal{K}_5) \geq 4/15$. (resp. $d(C,\mathcal{K}_6) \leq 5/18$.)

To prove lower bounds on $d^*(\mathcal{K}_k)$ for $k \in \{3,4\}$, we use the Discharging Method on the columns $Q_a=\{(a,b) \mid 1 \leq b \leq k\}$. Let C be an identifying code of $\mathcal{K}_k$. We set the initial charge of every integer $a \in \mathbb{Z}$ to $\mathrm{chrg}_0(a)=|Q_a \cap C|$. We say that $a \in \mathbb{Z}$ is *satisfied* if its charge is least q_k and *unsatisfied* otherwise, where $q_3=1$ and $q_4=5/4$. We apply five discharging rules, Rule i for $i=1$ to 5 one after another. We denote by $\mathrm{chrg}_i(a)$ the charge of a after applying Rule i.

Rule i : every unsatisfied $a \in \mathbb{Z}$ receives $\min\{\mathrm{chrg}_{i-1}(a-i)-q_k, q_k-\mathrm{chrg}_{i-1}(a)\}$

from $a - i$, if $a - i$ is satisfied (before Rule i).
Finally, we prove that, after these rules, every integer $a \in \mathbb{Z}$ is satisfied. This implies $d(C, \mathcal{K}_k) \geq q_k/k$. □

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