Direct kinematics of CDPR with extra cable orientation sensors: the 2 and 3 cables case with perfect measurement and ideal or elastic cables

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Abstract Direct kinematics (DK) of cable-driven parallel robots (CDPR) based only on cable lengths measurements is a complex issue even with ideal cables and consequently even harder for more realistic cable models. A natural way to simplify the DK solving is to add sensors. We consider here sensors that give a partial or complete measurement of the cable direction at the anchor points and spatial CDPR with 2/3 cables and we assume that these measurements are exact. We provide a solving procedure and maximal number of DK solutions for an extensive combination of sensors while considering two different cables models: ideal and linearly elastic without deformation.

1 Introduction

We consider cable-driven parallel robot (CDPR) with 3 cables whose output point on the base is A_i and anchor point B_i on the platform. The known distance between B_i, B_j will be denoted d_{ij} and length of cable i will be denoted ρ_i. Solving the direct kinematics (DK) problem with only as input the ρ's is clearly an issue in parallel robotics. Although relatively well mastered for parallel robots with rigid legs, it is still an open issue for CDPR. Even if we assume ideal cable (with no elasticity and no deformation of the cable due to its own mass) the DK problem leads to a larger number of equations than in the rigid leg case [8] and consequently to solving problems [1, 2, 6, 11, 9, 10, 19], although finding all solutions is possible at the expense of a rather large computation time [5]. If we assume linearly elastic cables similar solving problem arise[15]. All the proposed DK algorithms exhibit a large computation time that prohibits their use in a real-time context. In this case fast and safe algorithms have been proposed [15, 19]: still several DK solutions may

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exist even in a small neighborhood around the previous pose so that the proposed algorithms will fail.

An intuitive approach to avoid the non-unicity problem and to speed up the solving time of the DK is to add sensors that provide additional information on the cable beside the cable lengths, as already proposed for classical parallel robots [7, 12, 14, 18]. A natural candidate will be to measure the cable tensions as they play an important role in the solving. Unfortunately force measurements are usually noisy and measuring these tensions on a moving platform submitted to various mechanical noises appears to be difficult [13, 16]. Although several attempts have been made of integrating force sensing in CDPR, none of them have presented clear result about the reliability of the measurement.

In this paper we are considering another measurement possibility which consists in getting complete or partial information on the cable direction at the anchor points A. These measurement are, figure 1:

- the angle θ_V between the x axis and the vertical plane that includes the cable
- the angle θ_H between the horizontal direction of the cable plane and the cable

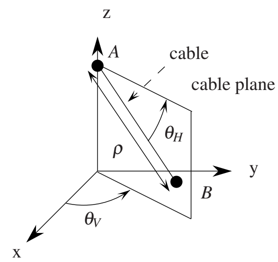

Fig. 1 Orientation sensors may provide the value of θ_V and/or θ_H

Realizing such measurement has already been considered: for example our CDPR MARIONET-Assist uses a simple rotating guide at A whose rotation is measured by a potentiometer in order to obtain the measurement of θ_V while our CDPR MARIONET-VR is instrumented with a more sophisticated cable guiding system which allows for the measurement of both θ_V and θ_H (figure 2). For measuring theses angles we may also consider a vision system as proposed in [4]. If ρ, θ_V, θ_H are known, then the location of B is fixed. If only ρ, θ_V are known, then B lies on a circle $\mathscr{C}_V$ centered at A which belong to the vertical cable plane. If only ρ, θ_H are known, then B lies on a horizontal circle $\mathscr{C}_H$ whose center U and radius can easily be calculated as function of ρ, θ_H. To characterize the sensor arrangement we will use the following notation:

- $\theta_V^j \theta_H^j$ indicates that the cable j has both θ_V, θ_H sensors
- $\theta_V(_H)^j$ indicates that the cable j has only $\theta_V(_H)$ sensor

We will also use $n\theta_V\theta_H$ to indicate that n cables have all both θ_V, θ_H sensors. Whenever needed xb_i, yb_i, zb_i will denote the coordinates of B_i while xa_i, ya_i, za_i

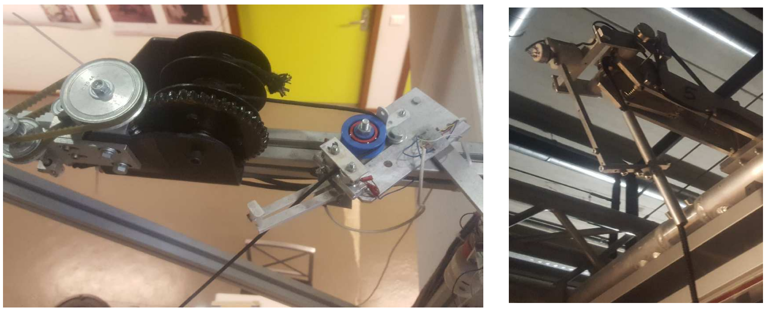

Fig. 2 On the left the rotation guide of MARIONET-Assist which allows for the measurement of θ_V. On the right the system used on MARIONET-VR for the measurement of both θ_V and θ_H

are the coordinates of A_i. In some cases and angle α_i will appear and we define T_i as $tan(\alpha_i/2)$. We may have also to use the mechanical equilibrium equations:

$$\mathscr{F} = \mathbf{J^{-T}}\tau \tag{1}$$

where $\mathscr{F}$ is the external wrench applied on the platform, assumed here to be only the force applied by the gravity, $\mathbf{J^{-T}}$ is the transpose of the inverse kinematic Jacobian and τ the vector of the 3 tensions in the cable. Equations (1) are a set of 6 constraint equations. Furthermore if G denotes the center of mass of the platform there are constants l_i, k_i such that

$$\mathbf{OG} = l_1\mathbf{B_1B_2} + l_2\mathbf{B_1B_3} + l_3\mathbf{B_1B_2} \times \mathbf{B_1B_3} \tag{2}$$

$$\mathbf{OB_3} = k_1\mathbf{B_1B_2} + k_2\mathbf{B_1G} + k_3\mathbf{B_1B_2} \times \mathbf{B_1G} \tag{3}$$

Our objective is to consider an exhaustive set of sensor arrangements and number of sensors and for each of them to determine the computational effort that is required to solve the DK, together with an upper bound on the maximal number m of solutions that may be obtained. As we have redundant information we will consider in each case only one square system leading to a closed-form solution (thereby faster than the DK algorithm based only only on cable lengths) and whenever possible one leading to a minimal number of solutions. The closed-form solution will be obtained through an elimination process leading to a univariate polynomial. Elimination may lead to a polynomial whose degree is higher than the minimal one: in this work we have tried to provide solution with the lowest degree but we cannot claim for minimality. For this preliminary, but exhaustive, work we will consider a spatial CDPR with only 2 and 3 cables. Furthermore we will assume that all measurements are exact, including the cable lengths. Clearly this assumption is not realistic but our purpose is to pave the way to a more complete analysis.

2 Ideal cable

For an ideal cable the shape of the cable is the straight line between A and B and cable tension does not affect the length of the cable.

2.1 The 3 ideal cables case

- **case** $3\theta_V\theta_H$**, 6 extra sensors**: this case is trivial as the measurements provide directly the coordinates of all three B and therefore a single solution of the DK
- **case** $\theta_V^1\theta_H^1-\theta_V^2\theta_H^2-\theta_V(_H)^3$**, 5 extra sensors**: in that case the locations of $B_1, B2$ are known. Consequently B_3 must lie on a circle C_3 lying in a plane that is perpendicular to B_1B_2 and whose center is located on the line B_1B_2, while B_3 is also located on the circle $\mathscr{C}_V^3$ Hence B_3 is located at the intersection of two circles, this leading to one of two solutions whose calculation involves solving a univariate quadratic polynomial. Note that the case $\theta_V^1\theta_H^1-\theta_V^2\theta_H^2-\theta_H^3$ is similar if we substitute $\mathscr{C}_V^3$ by $\mathscr{C}_H^3$.
- **case** $\theta_V^1\theta_H^1-\theta_V^2\theta_H^2$**, 4 extra sensors**: as in the previous case B_3 lies on the circle C_3 and is also located on a sphere centered at A_3 with radius ρ_3. The intersection of this sphere with C_3 leads usually to 2 intersection points and involves solving a univariate quadratic polynomial. Hence the DK may have at most 2 solutions
- **case** $\theta_V^1\theta_H^1-\theta_V^2-\theta_V^3$**, 4 extra sensors**: B_2 lies on a circle C_2 that is in a plane perpendicular to A_2B_1, whose center lies on the line A_2B_1 and whose radius may easily be calculated being given B_1, ρ_2, d_{12}. It lies also on the circle $\mathscr{C}_V^2$. Consequently there are two possible locations for B_2 that are obtained by solving a univariate quadratic polynomial. In the same manner B_3 lies on a circle that is perpendicular to B_1A_3 and whose center is located on this line while B_3 also belongs to $\mathscr{C}_V^3$, thereby leading to two possible locations for this point that are obtained by solving a univariate quadratic polynomial. Hence there may be at most 4 possible poses for the platform. Note that changing θ_V to θ_H for any of the cables 2 and 3 will lead to the same result
- **case** $\theta_V^1\theta_H^1-\theta_V^2$**, 3 extra sensors**: in that case B_1 is fixed and B_2 lies on the circle $\mathscr{C}_V^2$. At the same time B_2 lies on the sphere centered at B_1 with radius d_{12}. Consequently there are two possible locations for B_2 whose calculation amounts to solving a univariate quadratic polynomial.. For each of these locations as seen in the previous sections there are up to 2 possible location for B_3. In summary there are up to four DK solutions that are obtained by solving two univariate quadratic polynomial. Note that the case $\theta_V^1\theta_H^1-\theta_H^2$ is similar.
- **case** $3-\theta_V$**, 3 extra sensors**: in that case each of the three B_i is constrained to lie on a known circle $\mathscr{C}_V^i$. The CDPR is therefore equivalent to a $3-RS$ whose DK may lead to 16 solutions that are obtained by solving a 16th order univariate polynomial. The case $3-\theta_H$ will be similar.
- **case** $\theta_V^1\theta_H^1$**, 2 extra sensors**: in that case B_1 has a fixed position while for $j=2,3$ B_j lies on a circle perpendicular to the the line B_1A_j whose center M_j lies on this

line with a radius r_j than can easily be calculated. Hence $\mathbf{OB_j}$ may be written as $\mathbf{OM_j} + r_j \cos\alpha_j \mathbf{u_j} + r_j \sin\alpha_j \mathbf{v_j}$ where $\mathbf{u_j}, \mathbf{v_j}$ are two arbitrary unit vectors perpendicular to $\mathbf{B_1A_j}$ and perpendicular to each other while α_j is an unknown angle that parametrizes the location of B_j on its circle. A constraint is that $||B_2B_3|| = d_{23}$ but this provides only one constraint for the 2 unknowns α_2, α_3. We have therefore to look at the mechanical equilibrium equations that involve the 3 unknown tensions in the cable τ_j. Using the 3 first equations of the equilibrium (1) allows one to determine τ_1, τ_2, τ_3 as functions of α_2, α_3. Reporting this result in the last equation of the equilibrium enables us to obtain a second constraint on α_2, α_3. The 2 constraint equations are transformed into algebraic equations by using the Weierstrass substitution and calculating the resultant of these two equations leads to a univariate polynomial of degree 8, leading to up to 8 solutions for the DK

- **case $\theta_H^1 - \theta_H^2$, 2 extra sensors**: in that case B_1, B_2 are moving on the horizontal circles $\mathscr{C}_H^1, \mathscr{C}_H^2$. Hence we have $\mathbf{OB_j} = \mathbf{OU_j} + r_j \cos\alpha_j \mathbf{x} + r \sin\alpha_j \mathbf{y}$ for $j = 1, 2$. Then we have the constraint equations $||\mathbf{B_1B_2}||^2 = d_{12}^2$, $||\mathbf{B_1B_3}||^2 = d_{13}^2$, $||\mathbf{B_2B_3}||^2 = d_{23}^2$, $||\mathbf{A_3B_3}||^2 = \rho_3^2$ which is a set of 4 equations in the 5 unknowns $\alpha_1, \alpha_2, xb_3, yb_3, zb_3$. Hence the geometrical condition are not sufficient to determine the DK solution(s). The mechanical equilibrium equations (1) introduces three new unknowns τ_1, τ_2, τ_3 and 6 constraints. The 3 first equations of the mechanical equilibrium are linear in τ_1, τ_2, τ_3: solving this system leads to the 6th equation of the mechanical equilibrium, $||\mathbf{B_2B_3}||^2 - d_{23}^2 - ||\mathbf{A_3B_3}||^2 + \rho_3^2$ and $||\mathbf{B_1B_3}||^2 - d_{13}^2 - ||\mathbf{A_3B_3}||^2 + \rho_3^2$ being linear in xb_3, yb_3, zb_3. Consequently we have 3 linear equations in xb_3, yb_3, zb_3 that may be solved in these unknowns. It remain the equations $||\mathbf{B_1B_2}||^2 = d_{12}^2 (A)$ and the 4th and 5th equations of the mechanical equilibrium. These two later equations may be factored and have a common factor (B) whose cancellation will ensure that these 2 equations are satisfied. Then equations (A) and (B) are functions of the sine and cosine of α_1, α_2: using the Weierstrass substitution allows one to obtain 2 algebraic equations in T_1, T_2 whose resultant in T_2 is a univariate polynomial in T_1 of degree 12.
- **case θ_H^1, 1 extra sensors**: this case is somewhat similar to the previous one: we have now as unknown α_1, xb_3, yb_3, zb_3 and xb_2, yb_2, zb_2. with the additional constraint $||\mathbf{A_2B_2}||^2 = \rho_2^2$. As previously we solve the mechanical equilibrium equation to get τ_1, τ_2, τ_3 and the other constraints to obtain xb_3, yb_3, zb_3. We end up with a system of 4 equations $||\mathbf{B_1B_2}||^2 = d_{12}^2$, $||\mathbf{A_2B_2}||^2 = \rho_2^2$, $||\mathbf{A_3B_3}||^2 = \rho_3^2$ and the 4th equation of the mechanical equilibrium in the 4 unknowns $\alpha_1, xb_2, yb_2, zb_2$. The difference of the two first equations is linear in xb_2 and the last equation is linear in zb_2. Therefore 2 equations remain in the unknowns α_1, yb_2: the resultant in yb_2 leads to a polynomial in $T_1 = \tan(\alpha_1/2)$ which factors out in polynomials of degree 6, 8, 16 and 24.

Table 1 summarizes the previous results for the 3-cables case (the complexity indicates the degree of the polynomials that have to be solved). It must be noted that even a single sensor allows one to drastically reduce the computational effort to get all the DK solutions.

case	number of sensors	complexity	number of solution
$3\theta_V\theta_H$	6	1	1
$\theta_V^1\theta_H^1 - \theta_V^2\theta_H^2 - \theta_V(_H)^3$	5	2	2
$\theta_V^1\theta_H^1 - \theta_V^2\theta_H^2$	4	2	2
$\theta_V^1\theta_H^1 - \theta_V(_H)^2 - \theta_V(_H)^3$	4	2,2	4
$\theta_V^1\theta_H^1 - \theta_V(_H)^2$	3	2,2	4
$3 - \theta_V(_H)$	3	16	16
$\theta_V^1\theta_H^1$	2	8	8
$\theta_H(V)^1 - \theta_H(V)^1$	2	12	12
$\theta_H(V)^1$	1	6,8,16,24	54

Table 1 For ideal cable: sensors arrangement, total number of sensors, complexity of the solving and maximal number of DK solution(s)

2.2 The 2 ideal cables case

We should not forget that although the CDPR has 3 cables it may end up in a pose where only 2 cables are under tension, the remaining one being slack. Without losing generality we may assume that cable 1 and 2 are under tension and cable 3 is slack. A direct consequence is that the platform fully lies in the vertical plane that includes A_1, A_2, B_1, B_2 and G.

- **case** $\theta_V^1\theta_H^1 - \theta_V^2\theta_H^2$, 4 sensors: a necessary condition to have the platform in the vertical plane including A_1, A_2 is

$$(ya_2 - ya_1)/(xa_2 - xa_1) = \tan(\theta_V^1) = -\tan(\theta_V^2) \tag{4}$$

  If this condition is fulfilled then the locations of B_1, B_2 are fixed. There are then 2 possible locations for G: one below B_1B_2 (which is stable) and one above B_1B_2 (unstable). By choosing an appropriate frame both locations may be determined by solving a linear equation. Using equation (3) we may determine the location of B_3 and check if $\rho_3 > ||\mathbf{A_3B_3}||$ for confirming the slackness of cable 3.
- **case** $\theta_V^1\theta_H^1 - \theta_V^2$, 3 sensors: we use equation (4) to check if A_1, A_2, B_1, B_2 may be in the same vertical plane (and this is the only use of θ_V^2). If this is so, then B_1 is in a fixed location, while B_2 belongs to a circle centered in B_1 with radius d_{12} and to a circle centered in A_2 with radius ρ_2. Hence there are two possible locations for B_2 that are obtained by solving a quadratic polynomial. The two possible location of B_3 for each location of B_2 are obtained using the same method as in the previous item for checking the slackness of cable 3
- **case** $\theta_V^1\theta_H^1$, 2 sensors: here we cannot check if A_1, A_2, B_1, B_2 are in the same vertical plane but we still may use the same method than in the previous item and we may obtain up to 4 solutions for the DK, two of them being unstable.
- **case** $\theta_V^1 - \theta_V^2$, 2 sensors: if condition (4) holds, then the CDPR becomes a planar CDPR with 2 cables and it is known that to obtain the DK solutions we will have to solve two univariate polynomials of degree 12 [10]

- **case** $\theta_H^1 - \theta_H^2$, 2 sensors: here we will assume that A_1, A_2, B_1, B_2 are in the same vertical plane. Being given the sensor measurements we are able to get the location of B_1, B_2 in this plane, which will to check if the condition $||\mathbf{B_1B_2}|| = d_{12}$ holds. If this is the case we may solve the DK by using the procedure described for the $\theta_V^1\theta_H^1 - \theta_V^2\theta_H^2$ case.
- **case** θ_H^1, 1 sensor: here again we will proceed under the assumption that A_1, A_2, B_1, B_2 are in the same vertical plane and use the procedure for the $\theta_V^1\theta_H^1 - \theta_V^2$ case to obtain up to 4 DK solutions
- **case** θ_V^1, 1 sensor: the sensor measurement allows to check if A_1, A_2, B_1 lie in the same vertical plane. If this is so we resort to the procedure for solving the planar 2-cable DK problem, i.e. solving two univariate polynomials of degree 12 [10]

3 Elastic cable

The shape of the cable is still the straight line between A and B but the cable length and its length at rest ρ_r (which is the variable that is controlled and estimated from the winch motion) are related to the cable tension τ by:

$$\tau = k(\rho - \rho_r) \quad \text{if } \rho \geq \rho_r, \;\; 0 \text{ otherwise} \tag{5}$$

where k is the known stiffness of the cable. There is no deformation of the cable whose shape is the straight line between A and B. The same measurement system as for the ideal cable may be implemented and we use the same notation for describing the sensor arrangement. The difference with the ideal case is that the measurement of both θ_V, θ_H is no more sufficient to determine the location of the B as the cable length is no more known (and so is the radius of the circles $\mathscr{C}_V, \mathscr{C}_H$)..

- **case** $3\theta_V\theta_H$**, 6 extra sensors**:
  the 2 sensors on a given cable j provide the cable direction unit vector $\mathbf{u^j}$ and the three first equations of the mechanical equilibrium may be written as

  $$\sum_{j=1}^{j=3}\mathbf{u_x^j}k(\rho_j - \rho_r^j) = 0 \quad \sum_{j=1}^{j=3}\mathbf{u_y^j}k(\rho_j - \rho_r^j) = 0 \quad \sum_{j=1}^{j=3}\mathbf{u_z^j}k(\rho_j - \rho_r^j) = mg$$

  These 3 equations constitute a linear system in the ρ_j that can be solved to obtain these variables. We have then $\mathbf{OB_j} = \mathbf{OA_j} + \rho_j\mathbf{u_j}$ that allow to determine the unique pose of the platform.
- **case** $2 - \theta_V\theta_H - \theta_V^3$**, 5 extra sensors**:
  the ρ may be determined using the same method than in the previous case but they are now function of α_3, the angle used to define B_3 on its vertical circle $\mathscr{C}_V^3$. The constraint $||\mathbf{B_1B_2}||^2 = d_{12}^2$ factors out in a polynomial of degree 2 and a polynomial of degree 4, leading to 6 possible DK solutions.
- **case** $\theta_V^1\theta_H^1 - \theta_V^2\theta_H^2$**, 4 extra sensors**:
  the unknowns are the 3 ρ and the three coordinates of B_3. The ρ can be de-

termined by solving the first three equations of (1). We consider the 6th equation of the mechanical equilibrium (1) and the two constraints $||\mathbf{B_1B_2}||^2 = d_{12}^2$, $||\mathbf{A_3B_3}||^2 = \rho_3^2$. We compute in sequence the resultant with respect to xb_3, yb_3 of these 3 constraints to get a univariate polynomial in zb_3. This polynomial factors out in 3 polynomials of degree 4. Hence there are at most 12 DK solutions.

- **case $\theta_V^1\theta_H^1 - \theta_V(H)^2 - \theta_V(H)^3$, 4 extra sensors**:
  The ρ can be determined by solving the first three equations of (1) which are functions of α_2, α_3, the two angles that allow to determine the location of B_2, B_3 on the $\mathscr{C}_V^2, \mathscr{C}_V^3$ circles. The 6th equation of the mechanical equilibrium (1) factors out in 4 polynomials of degree (2,2) in T_2, T_3 and one polynomial of degree (4,4) in T_2, T_3, while $||\mathbf{B_1B_2}||^2 = d_{12}^2$ is a polynomial P of degree (4,8) in T_2, T_3. Taking all resultants in T_3 of all factors of the 6th equation of the mechanical equilibrium with P leads to 5 polynomials of degree 6, 6, 6, 12 and 12 in T_2.
- **case $\theta_V(H)^1 - \theta_V(H)^2 - \theta_V(H)^3$, 3 extra sensors**:
  the unknowns here are the 3 angles α_i that define the location of the B_i on the vertical circle $\mathscr{C}_V$ and the ρ's which may be obtained by solving the first three equations of (1). The constraint $||\mathbf{B_1B_2}||^2 = d_{12}^2$, $||\mathbf{B_1B_3}||^2 = d_{13}^2$ and the 6th equation of the mechanical equilibrium are functions of T_1, T_2, T_3. Successive resultants in T_1, T_2 leads to a univariate polynomial in T_3 which factors out in 6 polynomials of degree 72, one polynomial of degree 12, one of degree 24, 2 of degree 8 and two of degree 4. So an upper bound on the number of solutions is 492, a number which is most probably overestimated.
- **case $\theta_V^1\theta_H^1 - \theta_V(H)^2$, 3 extra sensors**:
  the unknowns are the 3 ρ, the 3 coordinates of B_3 and the angle α_2 that allow to define the position of B_2 on its vertical circle $\mathscr{C}_V^2$. We use the 3 first equations of the mechanical equilibrium (1) to determine xb_3, yb_3, zb_3. The 6th equation of the mechanical equilibrium, which is linear in ρ_1, will be used to calculate this unknown. The resultant R_1 of the constraints $||\mathbf{A_3B_3}||^2 - \rho_3^2$ and $||\mathbf{B_2B_3}||^2 = d_{23}^2$ in ρ_3 is a function of ρ_2, α_2. The constraint $||\mathbf{B_1B_2}||^2 = d_{12}^2$ is only function of α_2, ρ_2. The resultant of this equation and of R_1 in ρ_2 is only a function of α_2. Using the Weierstrass substitution this resultant factors out in 2 polynomials in T_2 of degree 12 and 40
- **case $\theta_V^1\theta_H^1$, 2 extra sensors**:
  the unknowns are the 3 ρ and the 3 coordinates of B_2, B_3. The constraint equations are the 6 equations of the mechanical equilibrium (1), the two equations $||\mathbf{A_jB_j}||^2 - \rho_j^2$ for $j \in [2,3]$ and the 3 equations $||\mathbf{B_iB_j}||^2 = d_{ij}^2$ with $i, j > i \in [1,3], i \neq j$. We first use the 3 first equation of the mechanical equilibrium to determine xb_2, yb_2, zb_2. If we consider the difference between $||\mathbf{B_1B_3}||^2 = d_{13}^2$ and $||\mathbf{A_3B_3}||^2 - \rho_3^2$ and the 6th equation of the mechanical equilibrium we have a linear system in xb_3, yb_3. If we report the solution of this system into the remaining equations, then the 5th equation of the mechanical equilibrium is linear in zb_3. The remaining equations are now functions of ρ_1, ρ_2, ρ_3. Successive resultants in $\rho_1, \rho 2$ leads to a univariate polynomial in ρ_3 which factors out in polynomial of degree 162, 104, 68, 48, 22, 20, 8, 7 and 3, leading to a maximum of 442 solutions, a number which is most probably overestimated

- **case $\theta_V(_H)^1 - \theta_V(_H)^2$, 2 extra sensors**:
  the unknowns are the 3 ρ, the 2 angles α_1, α_2 that are used to determine the location of B_1, B_2 on their vertical circle $\mathscr{C}_V$ and the 3 coordinates of B_3. The constraint equationss are the 6 equations of the mechanical equilibrium (1), the equation $||\mathbf{A_3B_3}||^2 - \rho_3^2$ and the 3 equations $||\mathbf{B_iB_j}||^2 = d_{ij}^2$ with $i, j > i \in [1,3], i \neq j$. We first use the 3 first equation of the mechanical equilibrium to determine xb_3, yb_3, zb_3. The 6th equation of the mechanical equilibrium is linear in ρ_1. The resultant of $||\mathbf{B_2B_3}||^2 = d_{23}^2$ and of the 4th equation of the mechanical equilibrium with the constraint $||\mathbf{B_1B_2}||^2 = d_{12}^2$ allows one to obtain 2 equations free of ρ_2. The Weierstrass substitution is then used to obtain 2 polynomials P_1, P_2 in T_1, T_2 which factor out in several polynomials with $P_1 = \prod R_i$ and $P_2 = \prod S_j$. When considering the resultant of all possible combinations of P_i, Q_j we get polynomials in T_1 only of degree 936, 240, 112, 72, 8, 4, 4 and hence the maximum number of solutions is 1376. . Trials have shown that the polynomials of degree 936, 240 may have real roots.

Table 2 summarizes the result for the 3 elastic cables case.

case	number of sensors	complexity	max number of solutions
$3\theta_V\theta_H$	6	1	1
$\theta_V^1\theta_H^1 - \theta_V^2\theta_H^2 - \theta_V(_H)^3$	5	2,4	6
$\theta_V^1\theta_H^1 - \theta_V^2\theta_H^2$	4	4,4,4	12
$\theta_V^1\theta_H^1 - \theta_V(_H)^2 - \theta_V(_H)^3$	4	6,6,6,12,12	44
$\theta_V(_H)^1 - \theta_V(_H)^2 - \theta_V(_H)^3$	3	6 × 72,12,24,2×8,2×4	492
$\theta_V^1\theta_H^1 - \theta_V(_H)^2$	3	40, 12	52
$\theta_V^1\theta_H^1$	2	162,104, 68, 48, 22, 20, 8, 7,3	442
$\theta_V(_H)^1 - \theta_V(_H)^2$	2	936, 240, 112, 72, 8, 4, 4	1376

Table 2 For elastic cable: sensors arrangement, total number of sensors, complexity of the solving and maximal number of DK solution(s)

3.1 The 2 elastic cables case

- **case $\theta_H^1 - \theta_H^2$, 2 extra sensors**: the unknowns are ρ_1, ρ_2 and the 2 first equations of the mechanical equilibrium are linear in these variables. The solution is unique for the planar CDPR but has 2 DK solutions for the spatial CDPR, see section 2.2
- **case θ_H^1, 1 extra sensors**: the unknowns are ρ_1, ρ_2 and the 2 coordinates of B_2 in the CDPR plane. The 2 first equations of the mechanical equilibrium are used to determine these later unknowns. The third equation of the mechanical equilibrium becomes linear in ρ_1. After solving the constraint $||\mathbf{A_2B_2}||^2 = \rho_2^2$ becomes a polynomial of degree 4 in ρ_2.

4 Analysis and uncertainty

As seen in tables 1 and 2 the complexity of the calculation of the DK solution(s) and their maximal number increases very quickly as soon as the number of sensors is getting lower than 6 (in which case we get a single solution both for the ideal and elastic cables). Taking into account measurement uncertainty is not the purpose of this paper but our first trial with our measurement system (see figure 2) has shown that we cannot expect a high accuracy, especially when the cable tension is low. Furthermore the accuracy ΔB of the location of B based on the orientation sensors and assuming an exact measurement of ρ is $\Delta B = \rho\Delta\theta$ where $\Delta\theta$ is the sensor error. This implies that for large CDPR where ρ is much larger than 1 we may expect large error on the coordinates of the B. However it may be thought that the parallel structure may overall decrease this influence. To examine this point we have considered a simple planar CDPR with 2 cables connected at the same point. We assume that the ρ, θ are measured respectively with an accuracy $\pm\Delta\rho, \pm\Delta\theta$. For a given pose x_0, y_0 of the CDPR these uncertainties induce an error on the location of the CDPR and its real pose lies in a closed region around the nominal pose. To determine the border of this region we consider the poses $x_0 + r\cos(\alpha), y_0 + r\sin(\alpha)$ along a specific direction defined by the angle α, poses that are at a distance r from the pose x_0, y_0 For a given α a simple otimization procedure allows one to determine the maximum of r, i.e. the maximal positioning error that is compatible with $\Delta\rho, \Delta\theta$ along the direction defined by α. Starting from $\alpha = 0$ we increment α by a step of 5 degrees until we reach 360 degrees, giving us a reasonable approximation of the border of the region in which the CDPR will lie. The calculation of r at each α allows us to calculate a good approximation of the minimal, maximal and mean value for the maximal positioning error. We are thus able to calculate these variables as a function of $\Delta\theta$ for a fixed value of $\Delta\rho$. When $\Delta\theta$ is large the positioning error is just influenced by $\Delta\rho$ but when $\Delta\theta$ decreases their will be a switching point at which the maximal positioning error will start to decrease due to the influence of $\Delta\theta$. Hence this switching point indicates how accurate should be the measurement in $\Delta\theta$ in order to obtain a better accuracy than the one based on $\Delta\rho$ only. Figure 3 shows this function for the CDPR with $A_1 = (0,0)$, $A_2 = (10,0)$, the pose $x_0 = 5\sqrt{2}/2, y_0 = -5\sqrt{2}/2$ and $\Delta\rho = 0.01$. It may be seen that the switching point occurs around 0.1 degrees. Therefore the orientation measurement must be highly accurate to provide a better accuracy than the one obtained by using only the cable lengths.

5 Conclusion

As solving the DK of CDPR based only on the cable lengths is a complex task it is worth investigating how additional sensors may help this solving. Note that these additional sensor(s) may also be used for other tasks such as auto-calibration [3], identification [17] or workspace limit detection and consequently may be worth the limited additional cost. In this paper we have investigated sensors that provide par-

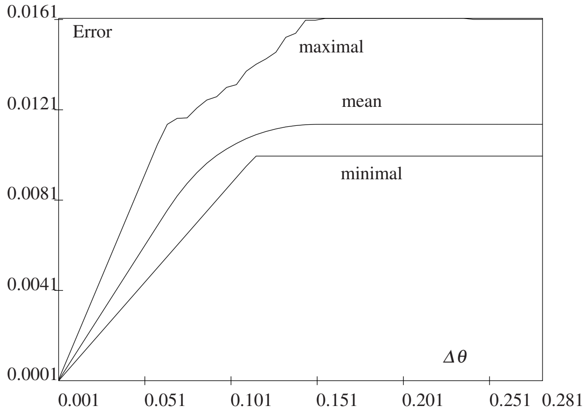

Fig. 3 Minimal, maximal and mean positioning error for $\Delta\rho = 0.01$ as a function of $\Delta\theta$ in degree

tial or complete information on the cable orientation and have examined the effect of sensor number and arrangement on the DK solving for CDPR with 2 or 3 cables, ideal or elastic. This is a necessary work but also preliminary: it should be extended to CDPR with more than 3 cables. Furthermore we have assumed perfect sensor measurements which is an unrealistic hypothesis, and consequently the influence of the uncertainties on the DK solving has to be studied. We have shown on a 2dof planar CDPR that the uncertainty on the orientation sensor measurement must be very low to have an influence on the accuracy of the estimation of the DK solutions but this influence has to be studied in detail in more general cases. However cable orientation measurement, even with an uncertainty interval, may provide useful information for a numerical method solving the DK with the cable lengths only, allowing to safely eliminate possible DK solutions. Indeed some of these solutions may lead to angles that lie outside their measurement intervals and thus can be eliminated. Finally the use of extra orientation sensors has also to be investigated to manage redundantly actuated CDPR, singularity and sagging cables.

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