Publicacions Matemàtiques, Vol 34 (1990), 389–396.

RELATION BETWEEN THE BOUNDARY POINT SPECTRUM OF A GENERATOR AND OF ITS ADJOINT

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Abstract

In this paper we get some relations between the boundary point spectrum of the generator A of a C_0-semigroup and the generator A^* of the dual semigroup. These relations combined with the results due to Lyubich-Phóng and Arendt-Batty, yield stability results on positive C_0-semigroups.

After the classical stability theorem due to Liapunov, a lot of contributions have been made in order to get generalizations to infinite dimension of this theorem. One of the more recent results in this context is the following stability theorem due to Arendt-Batty [1] and Lyubich-Phóng [7].

(I) " *Let $(T(t))_{t\geq 0}$ be a bounded C_0-semigroup on a Banach space E with generator A. Denote by $\sigma(A)$ the spectrum of A. If $\sigma(A)\cap i\mathbb{R}$ is countable and no eigenvalues of A^* lies on the imaginary axis, then $(T(t))_{t\geq 0}$ is uniformly stable (i.e., $\lim_{t\to\infty} T(t)f = 0$ for all $f \in E$).* "

If in (I) we assume that the Banach space E is reflexive, then we can reformulate it in a more suitable form, requiring the absence of imaginary eigenvalues of A instead of its adjoint A^*. More precisely, we have:

(II) " *Let $(T(t))_{t\geq 0}$ be a bounded C_0-semigroup on a reflexive Banach space E with generator A. If $\sigma(A)\cap i\mathbb{R}$ is countable and no eigenvalues of A lies on the imaginary axis, then $(T(t))_{t\geq 0}$ is uniformly stable.* "

The above result points out the importance of knowing the relationship between the boundary point spectrum of the generator A of a C_0-semigroup and the boundary point spectrum of its adjoint A^*. This note is devoted to study this problem.

Let E be a Banach lattice with positive cone E_+. The principal ideal in E generated by some $f \in E_+$ is denoted by E_f. Recall that $f \in E_+$ is called a quasi-interior point of E_+ if E_f is dense in E (see [11]). For each $f \in E_+$ the principal ideal E_f endowed with the gauge function p_f of $[-f, f]$ is an AM-space with unit f, and the canonical embedding $E_f \to E$ is continuous (see, [**11**, II.7.2]). If A is the generator of a C_0-semigroup $(T(t))_{t\geq 0}$, we denote the spectrum (point, residual spectrum) of A by $\sigma(A)(p\sigma(A), r\sigma(A))$, the resolvent set by $\rho(A) := \mathbb{C} \sim \sigma(A)$ and the resolvent by $R(\lambda, A) := (\lambda - A)^{-1}$ for $\lambda \in p(A)$. The spectral bound of A is defined by

$$s(A) := \sup\{Re\lambda : \lambda \in \sigma(A)\},$$

and the boundary point spectrum of A by

$$p\sigma_b(A) = \{\lambda \in p\sigma(A) : Re\lambda = s(A)\}.$$

If $(T(t))_{t\geq 0}$ is C_0-semigroup on E with generator A, it is known that the adjoint operator $(T(t)^*)_{t\geq 0}$ need not be a C_0-semigroup on E^*. We define $E^{\odot}$ to be the subspace of E^* on which $(T(t)^*)_{t\geq 0}$ is strongly continuous, $T(t)^{\odot} := T(t)_{|E^{\odot}}$ and $A^{\odot}$ the generator of the C_0-semigroup $(T(t)^{\odot})_{t\geq 0}$ (see [**3**] and [**8**]).

By adapting the proof of Satz 3.2 in [**12**] to the semigroup context one can obtain the following result. We include the proof for sake of completeness.

Proposition 1. *Let $(T(t))_{t\geq 0}$ be a positive C_0-semigroup on the Banach lattice E with generator A. If the resolvent of A grows slowly (i.e., the set $\{(\lambda - A)R(\lambda, A) : \lambda > s(A)\}$ is bounded in $\mathcal{L}(E)$), then*

$$p\sigma_b(A) \subset p\sigma_b(A^*).$$

Proof: In view of the rescaling procedure we assume that $s(A) = 0$. Let $\alpha \in \mathbb{R}$ such that $Af = i\alpha f$ with $0 \neq f \in E$. Let $\{a_n\}$ be a sequence of strictly positive real numbers converging to 0. Since $f \neq 0$, by the Hahn-Banach theorem we can find an element $\phi \in E^*$, $\|\phi\| = 1$ and $\langle \phi, f\rangle = 1$. Then,

$$\langle R(a_n + i\alpha, A^*)\phi, f\rangle = \langle \phi, \int_0^\infty e^{-(a_n+i\alpha)t} T(t)f\, dt\rangle$$
$$= \int_0^\infty e^{-(a_n+i\alpha)t}\langle \phi, T(t)f\rangle\, dt = \int_0^\infty e^{-(a_n+i\alpha)t}\langle \phi, e^{i\alpha t} f\rangle\, dt$$
$$= \int_0^\infty e^{-a_n t}\, dt = \frac{1}{a_n}.$$

Thus, if we set

$$\phi_n := \frac{R(a_n + i\alpha, A^*)\phi}{\|R(a_n + i\alpha, A^*)\phi\|}$$

we have

$$\langle \phi_n, f\rangle = \frac{1}{a_n\|R(a_n + i\alpha, A^*)\phi\|}. \tag{1}$$

Since $R(., A^*)$ grows slowly, there is an $M > 0$ such that

$$\|R(a_n, A^*)\| \leq \frac{M}{a_n} \text{ for every } n \in \mathbb{N}. \tag{2}$$

Moreover

$$\|R(\lambda, A^*)\| = \|R(\lambda, A)\| \leq \|R(Re\lambda, A)\| = \|R(Re\lambda, A^*)\|.$$

Hence, by (2) we have

$$\|R(a_n + i\alpha, A^*)\| \leq \frac{M}{a_n} \text{ for every } n \in \mathbb{N}. \tag{3}$$

As a consequence of (1) and (2) we obtain

$$\langle \phi_n, f \rangle \geq \frac{1}{M} > 0, \text{ for every } n \in \mathbb{N}. \tag{4}$$

On the other hand, by Alaoglu's theorem there exists a subnet $\{\phi_{n_k}\}$ of $\{\phi_n\}$ such that $\Psi := \sigma(E^*, E) - \lim_k \phi_{n_k} \in E^*$. Then, by (4), $\langle \Psi, f \rangle \geq \frac{1}{M} > 0$. Hence, $\Psi \neq 0$.

By the definition of the element ϕ_n we have that

$$A^* \phi_{n_k} = (a_{n_k} + i\alpha)\phi_{n_k} - \frac{\phi}{\|R(a_{n_k} + i\alpha, A^*)\phi\|}.$$

Now, since

$$\left\| \frac{\phi}{\|R(a_{n_k} + i\alpha, A^*)\phi\|} \right\| \leq \|f\| a_{n_k}$$

and

$$\sigma(E^*, E) - \lim_k (a_{n_k} + i\alpha)\phi_{n_k} = i\alpha\Psi,$$

it follows that $\sigma(E^*, E) - \lim_k A^* \phi_{n_k} = i\alpha\Psi$. Finally, since A^* has $\sigma(E^*, E)$-closed graph, $\Psi \in D(A^*)$ and $A^*\Psi = i\alpha\Psi$. Therefore, $i\alpha \in p\sigma_b(A^*)$. ∎

A C_0-semigroup $(T(t))_{t\geq 0}$ in a Banach space E is called *weakly almost periodic* if for any $f \in E$ the orbit $\{T(t)f : t \geq 0\}$ is weakly relatively compact (see [6]). Evidently, if E is reflexive, every bounded C_0-semigroup in E is weakly almost periodic. In order to find another class of weakly almost periodic C_0-semigroups we need the following result, which is well known in ergodic theory (see for instance [10]).

Lemma 1. *Let E be a Banach space and E_0 a subset of E such that* span(E_0) *is dense in E. If $(T(t))_{t\geq 0}$ is a C_0-semigroup in E, the following conditions are equivalent:*

(i) *$(T(t))_{t\geq 0}$ is weakly almost periodic.*
(ii) *$(T(t))_{t\geq 0}$ is bounded and for any $f \in E_0$ the orbit $\{T(t)f : t \geq 0\}$ is weakly relatively compact.*

Definition 1. Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on the Banach lattice E. $(T(t))_{t\geq 0}$ is called a *submarkovian semigroup* if is positive and there exists a quasi-interior point $f \in E_+$ such that $T(t)f \leq f$ for every $t \geq 0$.

Recall that a Banach lattice E is said to have order continuous norm if every order convergent net in E norm converges. It is known that every reflexive or weakly sequentially complete Banach lattice has order continuous norm (see [11]). So, every Lebesgue space $L^p(\mu)(1 \leq p < \infty)$ has order continuous norm.

Proposition 2. *If E is a Banach lattice with order continuous norm, then every bounded submarkovian semigroup on E is weakly almost periodic.*

Proof: By hypothesis, there exists a quasi-interior point $f \in E_+$ such that the orbit $\{T(t)f : t \geq 0\}$ is contained in $[0, f]$. So, this orbit is weakly relatively compact, since $[0, f]$ is $\sigma(E, E^*)$-compact (see, [11, II, Thm 5.10]). Then, since E_f is dense in E, the proposition follows from the above lemma. ■

In the following result, whose short proof is due to G. Greiner (private communication), we obtain the converse inclusion to the one obtained in Proposition 1.

Theorem 1. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on a Banach space E with generator A. Then, if $(T(t))_{t\geq 0}$ is weakly almost periodic,*

$$p\sigma(A^*) \cap i\mathbb{R} \subset p\sigma(A) \cap i\mathbb{R}.$$

Proof: Let $\alpha \in \mathbb{R}$ be such that $A^*\phi = i\alpha\phi$ with $\phi \in D(A^*)$, $\phi \neq 0$. Let $f \in E$ with $\langle \phi, f\rangle \neq 0$. As $(T(t))_{t\geq 0}$ is weakly almost periodic, the Krein-Smulian theorem shows that $M := \overline{co}\{e^{-i\alpha t}T(t)f : t \geq 0\}$ is weakly compact. Now, approximating the integral by Riemann sums we notice that

$$\lambda R(\lambda + i\alpha, A)f = \lambda \int_0^\infty e^{-\lambda t} e^{-i\alpha t} T(t) f \, dt \in M.$$

Hence, there exists a net $\lambda_j \downarrow 0$ such that

$$g := \sigma(E, E^*) - \lim_j \lambda_j R(\lambda_j + i\alpha, A) f \in M.$$

Since ϕ is weakly continuous we have that

$$\langle \phi, g\rangle = \lim_j \langle \phi, \lambda_j R(\lambda_j + i\alpha, A)f\rangle = \lim_j \langle \lambda_j R(\lambda_j + i\alpha, A^*)\phi, f\rangle = \langle \phi, f\rangle \neq 0,$$

from where it follows that $g \neq 0$. Finally,

$$R(1 + i\alpha, A)g = \sigma(E, E^*) - \lim_j R(1 + i\alpha, A)\lambda_j R(\lambda_j + i\alpha, A)f =$$

$$\sigma(E, E^*) - \lim_j \frac{\lambda_j}{1 - \lambda_j}[R(\lambda_j + i\alpha, A)f - R(1 + i\alpha, A)f] = g.$$

Therefore $Ag = i\alpha g$, and so $i\alpha \in p\sigma(A)$. ■

Corollary 1. *Let $(T(t))_{t\geq 0}$ be a bounded C_0-semigroup on a Banach space E with generator A. Assume that*

(i) *E is reflexive*

or

(ii) *E is a Banach lattice having order continuous norm and $(T(t))_{t\geq 0}$ is submarkovian.*

Then, $p\sigma(A^*)\cap i\mathbb{R} = p\sigma(A)\cap i\mathbb{R}$.

In the following result we will see that it is possible to avoid the boundedness condition in (ii) of the above corollary.

Theorem 2. *Let E be a Banach lattice having order continuous norm and let $(T(t))_{t\geq 0}$ be a submarkovian semigroup on E. Then*

$$p\sigma(A^*)\cap i\mathbb{R} \subset p\sigma(A)\cap i\mathbb{R}.$$

Proof: Let $\alpha \in \mathbb{R}$ be such that $A^*\phi = i\alpha\phi$ with $\phi \in D(A^*)$, $\phi \neq 0$. If $f \in E_+$ is the quasi-interior point such that $T(t)f \leq f$ for every $t \geq 0$, let $F := (E^*, f)$ be the AL-space associated to f considering f as an element of E^{**} (see [**11**, p. 113]). Then, F is the completion of E^* with respect to the norm $q_f(\Psi) := \langle|\Psi|, f\rangle$. Then, since $T(t)f \leq f$ for every $t \geq 0$, each $T(t)^*$ induces a positive operator $S(t) \in \mathcal{L}(F)$. Moreover,

$$\begin{aligned}\|S(t)\|_{\mathcal{L}(F)} = &\sup\{\langle|S(t)\Psi|, f\rangle : \Psi \in F, \langle|\Psi|, f\rangle \leq 1\} \leq \\ &\sup\{\langle|\Psi|, T(t)f\rangle : \Psi \in F, \langle|\Psi|, f\rangle \leq 1\} \leq 1.\end{aligned}$$

By ([**11**, IV. Exer. 9(a)]), $F^* = (E^{**})_f$. Now, since E has order continuous norm, E is an order ideal of E^{**} (see [**11**, II. 5.10]), hence $F^* = E_f$. Thus, for every $\Psi \in E^*$, $\sigma(F, F^*) - \lim_{t\to 0^+}(S(t)\Psi - \Psi) = 0$.

Then, having in mind that $\|S(t)\| \leq 1$ for all $t \geq 0$ and E^* is dense in F, we have that $\sigma(F, F^*) - \lim_{t\to 0^+}(S(t)\Psi - \Psi) = 0$ for every $\Psi \in F$. From here it follows that $(S(t))_{t\geq 0}$ is a C_0-semigroup on F (see [**4**, I Thm. 1.23]). Let B be the generator of $(S(t))_{t\geq 0}$. By ([**9**, Chap. 2, Thm. 1.3]) we have that $D(A^*) \subset D(B)$ and $A^*\Psi = B\Psi$ for every $\Psi \in D(A^*)$. Hence, $B\Psi = i\alpha\Psi$, so $s(B) \geq 0$. Now, since $\|S(t)\| \leq 1$ for all $t \geq 0$, it follows that $s(B) = 0$. Thus, the resolvent of B grows slowly. Applying Proposition 1, there exists $0 \neq z \in E_f$ such that $B^*z = i\alpha z$. Consequently, $z \in D(B^{\odot})$ (see [**8**, A-I, 3.4]) and since the dual norm of F is p_f, we have that there exists

$$p_f - \lim_{t\to 0^+} \frac{S(t)^*z - z}{t} = B^{\odot}z = B^*z.$$

Then, as p_f is finer than the norm of E restricted to E_f, we conclude that $z \in D(A)$ and $Az = B^*z = i\alpha z$. ■

If every submarkovian semigroup on a Banach lattice with order continuous norm were bounded, then, the above theorem would be included in Corollary 1. This does not hold, as the following example shows.

Example 1. Consider $\mathbb{R}_+$ with Lebesgue measure m and define $E_p := L^p(\mathbb{R}_+, dm) \cap L^1(\mathbb{R}_+, e^x dm)$, $1 < p < \infty$, which is a Banach lattice having order continuous norm with respect to the norm

$$N_p(f) := \left(\int_0^\infty |f(x)|^p \, dm(x) \right)^{1/p} + \int_0^\infty |f(x)| e^x \, dm(x),$$

but which is not reflexive. For $t \geq 0$ define the opertators $T(t)$ in E_p by $T(t)f(x) := f(x+t)$. Then, it is not difficult to show that $\|T(t)\| = 1$ for all $t \geq 0$ and $(T(t))_{t\geq 0}$ is a positive C_0-semigroup on E_p. Consider $(S(t))_{t\geq 0}$ the rescaled semigroup $S(t) := e^t T(t)$ for every $t \geq 0$. Then $(S(t))_{t\geq 0}$ is not bounded but is submarkovian since $f(x) := e^{-2x}$, is a quasi-interior point of E_p and

$$S(t)f(x) = e^t e^{-2(x+t)} = e^{-t} e^{-2x} \leq f(x).$$

The semigroup $(T(t))_{t\geq 0}$ in this example is a modification, given in [**3**], of an example due to Greiner-Voigt-Wolff [**5**].

Example 2. This example shows that the hypothesis that E has order continuous norm in Theorem 2 is essential. Consider the Banach lattice $E := C(\mathbb{R}_+)$ of all real-valued continuous functions on $\mathbb{R}_+$ which vanish at infinite. Let q be the function $q(x) := -x (x \in \mathbb{R}_+)$. Then the multiplication operator $M_q : f \to qf$ with maximal domain generates the multiplication semigroup $T(t)f := e^{tq} f$ and it is known (see [**8**, A-III, 2.3]) that

$$0 \in r\sigma_b(M_q) = p\sigma_b((M_q)^*) \not\subset p\sigma_b(M_q) = \phi.$$

As a consequence of (I) and Theorem 1 we obtain the following generalization of (II).

Theorem 3. *Let $(T(t))_{t\geq 0}$ be a weakly almost periodic C_0-semigroup with generator A. If $\sigma(A) \cap i\mathbb{R}$ is countable and $p\sigma(A) \cap i\mathbb{R} = \phi$, then $(T(t))_{t\geq 0}$ is uniformly stable.*

Remark 2. The above theorem can be also obtained as consequence of the characterization of almost periodic semigroups given in [**2**].

In the same way we get Theorem 3. if we use the stability result given in ([**8**, C-IV, 1.5]) instead of the result (I), we get the following result.

Theorem 4. *Let $(T(t))_{t\geq 0}$ be a positive C_0-semigroup on the Banach lattice E with generator A. Assume that $(T(t))_{t\geq 0}$ is eventually norm-continuous. If $(T(t))_{t\geq 0}$ is weakly almost periodic, then the following two assertions are equivalent:*

(i) $(T(t))_{t\geq 0}$ *is uniformly stable,*
(ii) $0 \notin p\sigma(A)$.

Remark 3. In the two above theorems the condition "$(T(t))_{t\geq 0}$ is weakly almost periodic" is necessary, since if $(T(t))_{t\geq 0}$ is the diffusion semigroup on $L^1(\mathbb{R}^n)$ generated by the Laplacian Δ, $(T(t))_{t\geq 0}$ is not uniformly stable because $0 \in r\sigma(\Delta)$, but, as all the solutions of $\Delta f = 0$ are either constant or unbounded, $o \notin p\sigma(\Delta)$. Observe that this example also shows that the weak almost periodicity condition is necessary in Theorem 1 and the hypothesis "$(T(t))_{t\geq 0}$ submarkovian" is necessary in Theorem 2.

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Rebut el 20 de Febrer de 1990