## CHEBYSHEV COEFFICIENTS FOR L<sup>1</sup>-PREDUALS AND FOR SPACES WITH THE EXTENSION PROPERTY

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Abstract .

We apply the Chebyshev coefficients  $\lambda_f$  and  $\lambda_b$ , recently introduced by the authors, to obtain some results related to certain geometric properties of Banach spaces. We prove that a real normed space E is an  $L^1$ -predual if and only if  $\lambda_f(E) = 1/2$ , and that if a (real or complex) normed space E is a  $\mathcal{P}_1$  space, then  $\lambda_b(E)$  equals  $\lambda_b(\mathbb{K})$ , where  $\mathbb{K}$  is the ground field of E.

In this note, K will be the real or complex field, and E a normed space over K; when we want to state a result only for the real case or the complex case, we will indicate it specificaly. We will use the notations of [2], to which we refer for all concepts of the theory of normed spaces which may appear without defining them here.

If S is a non empty subset of E, the number

$$r(S) = \inf_{y \in E} \sup_{x \in S} \|x - y\|$$

is called the Chebyshev radius of S, and  $\delta(S)$  denotes the diameter of S.

Definition. We will call the finite Chebyshev coefficient of E the real number

$$\lambda_f(E) = \sup\{r(S)/\delta(S) : S \subset E, S \text{ finite, } \delta(S) > 0\},\$$

and the bounded Chebyshev coefficient of E the real number

$$\lambda_{\delta}(E) = \sup\{r(S)/\delta(S) : S \subset E, 0 < \delta(S) < \infty\}.$$

It is easy to prove that, in general,

$$1/2 \leq \lambda_f(E) \leq \lambda_b(E) \leq 1.$$

Moreover, when E is finite dimensional, we have  $\lambda_f(E) = \lambda_b(E)$ . Specifically, the Chebyshev coefficients associated to the scalar fields are

$$\lambda_f(\mathbf{R}) = \lambda_b(\mathbf{R}) = 1/2$$
$$\lambda_f(\mathbf{C}) = \lambda_b(\mathbf{C}) = 1/\sqrt{3}.$$

Let us recall that a  $\mathcal{P}_{\alpha}(\mathbf{K})$  space, where  $\alpha$  is a real number greater than or equal to 1, is a Banach space E for which any of following equivalent condition holds:

- (i) Given two Banach spaces, F and G, a linear isometry into, φ : F → G, and a bounded linear operator, L : F → E, there exists a bounded linear operator L̂ : G → E, which extends L, in the sense of L̂ ∘ φ = L, and such that ||L̂|| ≤ α||L|| (α-extension property).
- (ii) Given a Banach space F, and a linear isometry,  $\phi : E \to F$ , there exists a projection,  $P : F \to \phi(E)$ , such that  $||P|| \leq \alpha$  ( $\alpha$ -projection property).

It is said that a Banach space E is a  $N_{\alpha}$  space, where  $\alpha$  is a real number greater than or equal to 1, when there exists a collection  $(E_{\gamma})_{\gamma \in \Gamma}$  of finite dimensional subspaces of E, which is upwards directed, their union is dense in E and every one of them is a  $\mathcal{P}_{\alpha}(\mathbb{K})$  space. Note that a Banach space is an  $L^{1}$ -predual space if and only if it is a  $N_{\alpha}$  space for every  $\alpha > 1$  ([2, theorem 2, pg. 232]).

**Theorem 1.** If the Banach space E is an  $L^1$ -predual, then

 $\lambda_f(E) = \lambda_f(\mathbb{K}).$ 

Proof: We fix an  $\alpha > 1$ . Given that E is an  $L^1$ -predual, it is a  $N_{\alpha}$  space, and, so, there exists a collection  $(E_{\gamma})_{\gamma \in \Gamma}$  of subspaces of E according to the definition above. We put  $F = \bigcup_{\gamma \in \Gamma} E_{\gamma}$ , which, because it is dense in E, satisfies

 $\lambda_f(E) = \lambda_f(F).$ 

Let S be a finite subset of F with more than one point. There exists  $\gamma \in \Gamma$  such that  $S \subset E_{\gamma}$ , and, if we indicate with subindices the Chebyshev radii in subspaces of E, we have

$$r(S) = \tau_{F}(S) \le \tau_{E_{\gamma}}(S) \le \delta(S) \cdot \lambda_{f}(E_{\gamma}) \le \delta(S) \cdot \alpha \cdot \lambda_{f}(\mathsf{K}),$$

where the last inequality is due to  $E_{\gamma} \in \mathcal{P}_{\alpha}(\mathbb{K})$ .

Therefore,  $\lambda_f(E) = \lambda_f(F) \leq \alpha . \lambda_f(\mathbb{K})$ , for every  $\alpha > 1$ , so  $\lambda_f(E) \leq \lambda_f(\mathbb{K})$ .

On the other hand, by the Hanh-Banach theorem, there exists a projection of norm 1 from E to K, so  $\lambda_f(K) \leq \lambda_f(E)$ .

If \*E is a non-standard enlargement of E, then, over the set  $fin^*E = \{x \in I^*E : \exists y \in E, ||x - y|| \text{ is a finite hyperreal number} \}$  of the finite elements of \*E, consider the equivalence relation "x is infinitely close to y", denoted by  $x \cong y$ , and defined by "||x - y|| is infinitesimal". In the quotient set, denoted  $\hat{E}$ , the norm  $||\hat{x}|| = st||x||, \hat{x} \in \hat{E}$ , is defined, and the resulting normed space is called an infinitesimal hull of E.

Lemma 2.  $\lambda_f(\hat{E}) = \lambda_f(E)$ .

Proof: Let S be a finite subset of E with more than one point. Then, S is a finite subset of fin<sup>\*</sup>E without infinitely close points.

It is obvious that  $\delta(\hat{S}) = \delta(S)$  and  $r(\hat{S}) \leq r(S)$ . We suppose that  $r(\hat{S}) < r(S)$ , and take a real number t such that  $r(\hat{S}) < t < r(S)$ . Then, there exists  $c \in \text{fin}^*E$  such that  $\hat{S} \subset B[\hat{c}, t]$ , and so, ||x - c|| < r(S) for every  $x \in S$ . Since S is finite, \*S = S, and we have a  $c \in *E$  such that ||x - c|| < r(S) for every  $x \in *S$ . Applying the Transfer Principle, there exists a standard element  $c \in E$  such that ||x - c|| < r(S) for every  $x \in S$ , and again because S is finite, this would imply  $S \subset B[c, \rho]$ , with  $\rho = \max_{x \in S} ||x - c|| < r(S)$ . Therefore, it is true  $r(\hat{S}) = r(S)$  and we conclude  $\lambda_f(E) \leq \lambda_f(\hat{E})$ .

Let S be now a finite subset of fin<sup>\*</sup>E with some points not infinitely close. Then, S is a \*-finite subset of \*E with some points not infinitely close and  $\hat{S}$  is a finite subset of  $\hat{E}$  such that  $\delta(\hat{S}) \cong^* \delta(S)$ .

Since the relation  $r(T) \leq \delta(T) \cdot \lambda_f(E)$  is true for every finite subset T of E, by the Transfer Principle, we have  $r(T) \leq \delta(T) \cdot \lambda_f(E)$  for every \*-finite T, and, in particular,  $r(S) \leq \delta(S) \cdot \lambda_f(E) \cong \delta(\hat{S}) \cdot \lambda_f(E)$ .

Let t be a hiperreal number such that t > r(S),  $t \cong \delta(\hat{S})$ ,  $\lambda_f(E)$ . There exists a  $c \in E$  such that  $S \subset B[c, t]$ , and then,

$$\|\hat{x} - \hat{c}\| = st \|x - c\| \cong \|x - c\| \le t \cong \delta(\hat{S}) \lambda_f(E), \forall \hat{x} \in \hat{S}.$$

Since the first and last members are standard, we have  $\|\hat{x} - \hat{c}\| \leq \delta(\hat{S}) \lambda_f(E)$ , for every  $\hat{x} \in \hat{S}$ , so that  $r(\hat{S}) \leq \delta(\hat{S}) \lambda_f(E)$ , and we conclude  $\lambda_f(\hat{E}) \leq \lambda_f(E)$ .

**Theorem 3.** If E is a real Banach space such that  $\lambda_f(E) = 1/2$ , then E is an  $L^1$ -predual.

Proof: We consider an infinitesimal hull  $\hat{E}$  of E. Then,  $\hat{E}$  has the radial intersection property (2,4), that is, given four closed balls in  $\hat{E}$  with the same radius, which intersect in pairs, the total intersection is non empty.

Indeed, let  $\rho$  be a positive real number, and let  $\hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_4 \in \hat{E}$ , such that  $\|\hat{x}_i - \hat{x}_j\| \leq 2\rho$ , for i, j = 1, 2, 3, 4. We take  $S = \{\hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_4\}$ , a finite subset of  $\hat{E}$  with  $\delta(S) \leq 2\rho$ , and, so,  $r(S) \leq \rho$ . For every natural number p, there exists  $\hat{c}_p \in \hat{E}$  such that  $\|\hat{x}_i - \hat{c}_p\| < \rho + (1/2p)$ , for i = 1, 2, 3, 4, and, consequently, there exists  $c_p \in \text{fin}^*E$  such that  $\|x_i - c_p\| < \rho + (1/p)$ , for i = 1, 2, 3, 4. We consider now the sequence  $(c_p)_{p \in \mathbb{N}}$  in  $^*E$ , which can be enlarged to an internal sequence  $(c_p)_{p \in \mathbb{N}}$  in  $^*E$ . The set of index  $p \in ^*\mathbb{N}$  such that  $\|x_i - c_p\| < \rho + (1/p)$ , for i = 1, 2, 3, 4 is an internal subset of  $^*\mathbb{N}$  containing all standard natural numbers, and, so, if we work in a suitably saturated model (cf. [3]), it contains an infinite index,  $\omega \in ^*\mathbb{N}$ . Then, the element  $c_{\omega} \in ^*E$  is

finite, because  $||x_i - c_{\omega}|| \le \rho$ , so that we can take  $\hat{c}_{\omega} \in \hat{E}$  thus verifying that  $||\hat{x}_i - \hat{c}_{\omega}|| \le \rho$ , for i = 1, 2, 3, 4.

Therefore,  $\hat{E}$  is an  $L^1$ -predual ([2, theorem 6, pg. 212]), that is,  $\hat{E}'$  is an  $L^1$  space. Projecting  $\hat{E}'$  over E' by means of the function  $T \in \hat{E}' \to T|_E \in E'$ , we have that E' is also an  $L^1$  space ([2, theorem 3, pg. 162]), and then E is an  $L^1$ -predual.

Lemma 4.  $\lambda_b(E)$  is the infimum of the positive real numbers r such that for every  $\gamma > 0$ , whenever  $(x_{\alpha})_{\alpha \in I} \subset E$  is a  $\gamma$ -Cauchy net (that is, given  $\varepsilon > 0$ , there exists  $\alpha_0 \in I$  such that  $||x_{\alpha} - x_{\beta}|| \leq \gamma + \varepsilon$ , for every pair of subindices  $\alpha, \beta \in I$  greater than or equal to  $\alpha_0$ ),  $(x_{\alpha})_{\alpha \in I}$  has some  $r\gamma$ -limit x in E (that is, given  $\varepsilon > 0$ , there exists  $\alpha_0 \in I$  such that  $||x_{\alpha} - x|| \leq r\gamma + \varepsilon$ , for every subindex  $\alpha \in I$  greater than or equal to  $\alpha_0$ ).

**Proof:** Let S be a bounded subset of E, with more than one point. For every natural number n, we consider  $S^{(n)} = S \times \{n\}$  and the bijection

$$S \longrightarrow S^{(n)}$$
$$x \longmapsto x^{(n)} = (x, n).$$

We take now  $I = \bigcup_{n \in \mathbb{N}} S^{(n)}$  with the order relation

$$\alpha \leq \beta \Longleftrightarrow \alpha = \beta \lor (\alpha = x^{(n)}, \beta = y^{(m)}, n < m),$$

which makes I a directed set. Over it, we build the net  $(x_{\alpha})_{\alpha \in I}$  defined by  $x_{\alpha} = x$  if  $\alpha \in I$  is such that  $\alpha = x^{(n)}$ , for some  $n \in \mathbb{N}$ . Thus,  $(x_{\alpha})_{\alpha \in I}$  is a  $\delta(S)$ -Cauchy net in E, with range S, and such that for every  $y \in S$  and every  $\alpha \in I$ , there exists a  $\beta \in I$ ,  $\beta \geq \alpha$ , which satisfies  $x_{\beta} = y$ , that is, for every  $y \in S$  there exists and infinite index  $\beta$  which satisfies  $x_{\beta} = y$ .

We call  $\lambda'(E)$  the infimum of the positive real numbers r such that for every  $\gamma > 0$ , every  $\gamma$ -Cauchy net has an  $r\gamma$ -limit in E, and let t be greater than  $\lambda'(E)$ . The previously builded net  $(x_{\alpha})_{\alpha \in I}$  has a  $t\delta(S)$ -limit  $x \in E$ , and, so,  $||x_{\alpha} - x|| \leq t\delta(S)$  for every infinite index  $\alpha$ . Hence, we have  $||y - x|| \leq t\delta(S)$  for every  $y \in S$ , and, because both members are standard,  $||x - y|| \leq t\delta(S)$  for every  $y \in S$ , that is,  $S \subset B[s, t\delta(S)]$ . Thus,  $r(S) \leq t\delta(S)$  for every  $t > \lambda'(E)$ , and  $\lambda_{\delta}(E) \leq \lambda'(E)$ .

Conversely, if  $(x_{\alpha})_{\alpha \in I}$  is a  $\gamma$ -Cauchy net in E for some  $\gamma > 0$ , we put  $S_{\alpha} = \{x_{\beta} : \beta \in I, \beta \geq \alpha\}$ , for every  $\alpha \in I$ . Thus, every set  $S_{\alpha}$  is bounded and we can suppose that it has more than one point (otherwise the proof is trivial); therefore, given  $\varepsilon > 0$ , there exists  $\alpha \in I$  such that  $\delta(S_{\alpha}) < \gamma + \varepsilon/\lambda_b(E)$ . Then,

$$r(S_{\alpha}) \leq \delta(S_{\alpha}) \cdot \lambda_{b}(E) < (\gamma + \varepsilon/\lambda_{b}(E)) \cdot \lambda_{b}(E) = \gamma \cdot \lambda_{b}(E) + \varepsilon_{c}$$

and, so, there exists,  $c_{\epsilon} \in E$  such that  $||x_{\beta} - c_{\epsilon}|| \leq \gamma \cdot \lambda_{b}(E) + \varepsilon$  for every  $\beta \geq \alpha$ . Hence,  $c_{\epsilon}$  is a  $(\gamma \cdot \lambda_{b}(E) + \varepsilon)$ -limit of  $(x_{\alpha})_{\alpha \in I'}$  and, since this is true for every  $\varepsilon > 0$  and for every  $\gamma$ -Cauchy net in E, it follows that  $\lambda'(E) \leq \lambda_{b}(E)$ . **Theorem 5.** If E is a  $\mathcal{P}_1(\mathbb{K})$  space, then  $\lambda_b(E) = \lambda_b(\mathbb{K})$ .

Proof: If we embed K into E by means of a linear isometry, identifying it with a onedimensional subspace of E, the Hahn-Banach theorem assures the existence of a projection of norm 1,  $P: E \to K$ , whence we deduce  $\lambda_{\delta}(K) \leq \lambda_{\delta}(E)$ .

We will prove the reciprocal inequality in several stages:

(I) In the first place, we observe that if  $E \in \mathcal{P}_1(\mathbb{K})$  and  $\hat{E}$  is an infinitesimal hull of E, then  $\lambda_b(E) \leq \lambda_b(\hat{E})$ , because, considering the canonical linear isometry  $E \to \hat{E}$ , there exists a contractive projection  $\hat{E} \to E$ .

(II) Let  $\Gamma$  be a non empty set. We denote by  $l^{\infty}(\Gamma, \mathsf{K})$  the set of all bounded functions from  $\Gamma$  to  $\mathsf{K}$ , with the uniform norm. Giving to  $\Gamma$  the discrete topology, we know that  $l^{\infty}(\Gamma, \mathsf{K})$  is linearly isometric to the space  $C(\beta\Gamma, \mathsf{K})$  of continuous functions with values in  $\mathsf{K}$  defined over the Stone-Čech compactification of  $\Gamma$ ; so,  $l^{\infty}(\Gamma, \mathsf{K}) \in \mathcal{P}_1(\mathsf{K})$  ([2]), and  $\lambda_b(l^{\infty}(\Gamma, \mathsf{K})) \leq \lambda_b(\hat{l}^{\infty}(\Gamma, \mathsf{K}))$ , by (I).

(III) We suppose that  $\Gamma$  is a finite set. We will prove that, in this case,  $\lambda_b(l^{\infty}(\Gamma, \mathbb{K})) \leq \lambda_b(\mathbb{K})$ .

Because  $l^{\infty}(\Gamma, K)$  is a finite dimensional, we know that its bounded and finite Chebyshev coefficients are equal, as are those of K.

Let  $\rho$  be a real number,  $\rho > \lambda_f(\mathbf{K})$ , and let  $S = \{x_1, \ldots, x_n\}$  be a finite subset of  $l^{\infty}(\Gamma, \mathbf{K})$ . Fixed  $\gamma \in \Gamma$ , we consider the finite subset of  $\mathbf{K} S_{\gamma} = \{x_1(\gamma), \ldots, x_n(\gamma)\}$ ; then,  $r(S_{\gamma})/\delta(S_{\gamma}) \leq \lambda_f(\mathbf{K}) < \rho$ , and

$$r(S_{\gamma}) < \rho.\delta(S_{\gamma}) = \rho. \max_{1 \le i, j \le n} |x_i(\gamma) - x_j(\gamma)| \le \rho. \max_{1 \le i, j \le n} ||x_i - x_j|| = \rho.\delta(S).$$

Then, there exists a centre  $c_{\gamma} \in \mathbb{K}$  such that  $S_{\gamma} \subset B[c_{\gamma}, \rho.\delta(S)] \subset \mathbb{K}$ . We define thus a function from  $\Gamma$  in  $\mathbb{K}$  which associates  $c_{\gamma}$  to every  $\gamma$ , and which satisfies

$$\|x_i - c\| = \sup_{\gamma \in \Gamma} |x_i(\gamma) - c| \le \rho.\delta(S), i = 1, \dots, n,$$

so that  $S \subset B[c, \rho.\delta(S)] \subset l^{\infty}(\Gamma, \mathbb{K})$ . Therefore,  $\tau(S) \leq \rho.\delta(S)$ , and we conclude  $\lambda_f(l^{\infty}(\Gamma, \mathbb{K})) \leq \lambda_f(\mathbb{K})$ .

(IV) The inequality  $\lambda_b(l^{\infty}(\Gamma, \mathbb{K})) \leq \lambda_b(\mathbb{K})$  is valid also when  $\Gamma$  is an infinite set.

Indeed, let  $(x_{\alpha})_{\alpha \in I}$  be a  $\gamma$ -Cauchy net in  $l^{\infty}(\Gamma, \mathbb{K})$ , for  $\gamma > 0$ . We take  $X_0 = \mathbb{K} \cup \Gamma \cup I$  in order to build a superstructure  $\mathcal{X}$  with base  $X_0$  and over it a polysaturated nonstandard model satisfying the  $\aleph_0$ -isomorphism property (cf. [3, sec. 0.4.]). In this case, we can identify  $\hat{l}^{\infty}(\Gamma, \mathbb{K})$  to  $\hat{l}^{\infty}(\omega, \mathbb{K})$ , for every infinite natural number  $\omega$ , since  $\hat{l}^{\infty}(\omega, \mathbb{K})$  is isometrically isomorphic to  $\hat{l}^{\infty}(\mathbb{N}, \mathbb{K})$ , and this is so to  $\hat{l}^{\infty}(\Gamma, \mathbb{K})$  ([4, theorem 2.11]).

Let p be a natural number. There exists an index  $\alpha_p \in I$  such that  $||x_{\alpha} - x_{\beta}|| < \gamma + 1/(2p)$ , when  $\alpha, \beta \in {}^*I, \alpha, \beta \geq \alpha_p$ . We consider the set

 $S = \{x_{\alpha} : \alpha \in {}^*I, \alpha \geq \alpha_p\}$ , an internal \*-bounded subset of  $l^{\infty}(\omega, \mathbf{K})$ , where we can apply (III), and so

where  $t = \lambda_b(\mathbb{K})(\gamma + 1/p)$ . Since r(S) < t, there exists  $c_p \in l^{\infty}(\omega, \mathbb{K})$  such that  $S \subset B[c_p, t]$  and  $\hat{c}_p \in \hat{l}^{\infty}(\omega, \mathbb{K}) = \hat{l}^{\infty}(\Gamma, \mathbb{K})$  satisfies  $||\hat{x}_{\alpha} - \hat{c}_p|| < t$ , for  $\alpha \in I, \alpha \geq \alpha_p$ .

Bearing in mind that  $l^{\infty}(\Gamma, \mathbb{K}) \in \mathcal{P}_1(\mathbb{K})$ , consider the natural embedding  $l^{\infty}(\Gamma, \mathbb{K}) \to \hat{l}^{\infty}(\Gamma, \mathbb{K})$ ; then, there exists a projection of norm 1,  $P : \hat{l}^{\infty}(\Gamma, \mathbb{K}) \to l^{\infty}(\Gamma, \mathbb{K})$ , and  $x = P(\hat{c}_p)$  is an element of  $l^{\infty}(\Gamma, \mathbb{K})$  which satisfies

$$||x_{\alpha} - x|| = ||P(\hat{x}_{\alpha}) - P(\hat{c}_p)|| \le ||\hat{x}_{\alpha} - \hat{c}_p|| \le t, \alpha \in I, \alpha \ge \alpha_p.$$

Thus, for every  $\rho > \lambda_b(\mathbb{K})$ , taking a  $p \in \mathbb{N}$  greater than the real number  $\lambda_b(\mathbb{K})/\gamma(\rho - \lambda_b(\mathbb{K}))$ , we have

$$\|x_{\alpha} - x\| \leq \lambda_{b}(\mathbf{K})(\gamma + \frac{1}{p}) < \rho\gamma, \ \alpha \in I, \ \alpha \geq \alpha_{p},$$

that is, x is a  $\rho\gamma$ -limit of  $(x_n)_n$  in  $l^{\infty}(\Gamma, \mathbb{K})$ . Since this is valid for every  $\gamma$ -Cauchy net in  $l^{\infty}(\Gamma, \mathbb{K})$  and for any  $\gamma > 0$  we conclude by Lemma 4 that  $\lambda_b(l^{\infty}(\Gamma, \mathbb{K})) \leq \lambda_b(\mathbb{K})$ .

(V) Now, we can embed E linearly and isometrically into  $l^{\infty}(E, \mathbb{K})$  by means of the application

$$\phi : E \longrightarrow l^{\infty}(E, \mathbb{K})$$
  

$$x \longrightarrow \phi(x) : E \longrightarrow \mathbb{K}$$
  

$$y \longrightarrow \phi(x)(y) = f_{y}(x)$$

where  $f_y$  is a continuous linear functional from E to K which satisfies  $||f_y|| = 1$ and  $f_y(y) = ||y||$ , the existence of which is guaranteed by the Hahn-Banach theorem. So, there exists a projection of norm 1,  $l^{\infty}(E, K) \to E$ , which permits us to deduce the inequality  $\lambda_b(E) \leq \lambda_b(l^{\infty}(E, K))$ , and, from the result in the preceding paragraph,  $\lambda_b(E) \leq \lambda_b(K)$ .

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