ASYMPTOTIC EQUIVALENCE
OF IMPULSIVE DIFFERENTIAL EQUATIONS
IN A BANACH SPACE

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Abstract

By means of Schauder's fixed point theorem sufficient conditions for asymptotic equivalence of impulsive equations in a Banach space are found.

1. Introduction

Impulsive differential equations are a powerful apparatus for mathematical simulation in physics, chemical technology, ecology, biotechnologies, impulse technology and industrial robotics. That is why the theory of impulsive differential equations develops actively in the last 20 years [4], [8-13], [15]. The presence of phenomena specific for the impulsive differential equations such as *beating* of the solutions, merging of the solutions, bifurcation, dying of the solutions, loss of the property of autonomy, etc. makes this theory richer than the theory of ordinary differential equations and, on the other hand, in the investigation of impulsive differential equations a preliminary modification of the methods used is necessary.

In relation to the mathematical simulation in theoretical physics recently the investigation of impulsive differential equations in abstract spaces began [1], [2]. In the present paper the asymptotic equivalence of impulsive differential equations in a Banach space is investigated. We shall note that the asymptotic equivalence of systems of ordinary differential equations has been investigated by many authors [3], [5-7], [14], [16].

2. Preliminary notes

Consider the impulsive equations

(1) $$\frac{dx}{dt} = A(t)x\,(t \notin \{t_j\}_{j=1}^{\infty})$$

(2) $$x(t_j^+) = L_j x(t_j^-)\,(j = 1, 2, 3, \ldots)$$

and

(3) $$\frac{dy}{dt} = A(t)y + f(t, y)(t \notin \{t_j\}_{j=1}^{\infty})$$

(4) $$y(t_j^+) = (L_j + H_j)y(t_j^-)\,(j = 1, 2, 3, \ldots)$$

in a real or complex Banach space X. Here t varies in the interval $[\bar{t}, \infty)$; $A(t)$: $X \to X$ (for each $t \in [\bar{t}, \infty)$) is a linear bounded operator and the operator-valued function $A(\cdot)$ is continuously continuable from (t_{j-1}, t_j) to $[t_{j-1}, t_j]$ $(j = 1, 2, 3, \ldots, t_0 = \bar{t})$; $L_j : X \to X$ $(j = 1, 2, 3, \ldots)$ are linear bounded operators having bounded inverse ones and $L_j X = X$; the function $f : [\bar{t}, \infty) \times X \to X$ is continuosly continuable from $(t_{j-1}, t_j) \times X$ to $[t_{j-1}, t_j] \times X$ $(j = 1, 2, 3, \ldots)$; H_j : $X \to X$ $(j = 1, 2, 3, \ldots)$; the sequence $T = \{t_j\}_{j=1}^{\infty}$ satisfies the condition

$$\bar{t} < t_1 < t_2 < \cdots \to \infty$$

We assume that the solutions of equations (1), (2), (3) and (4) are continuous from the left, thus in the conditions of a jump (2) and (4) instead of $\bar{t}_j$ we shall write t_j. If x is a solution of equation (1), (2) on the interval $[a, \infty)$, we shall write $x \in ((1), (2))_a$ and analogously of equation (3), (4). An arbitrary solution of impulsive equation (1), (2) defined on some subinterval of $[\bar{t}, a)$ is continuable to the whole interval which for impulsive equation (3), (4) in general is not true.

Denote by $W(t, \tau)(\bar{t} \leq t, \tau < \infty)$ the evolutionary operator of impulsive equation (1), (2) associating with each element $x_0 \in X$ the solution $x(t) = W(t, \tau)x_0$ of (1), (2) with condition $x(\tau) = x_0$.

The following equalities are valid

$$W_t'(t, \tau) = A(t)W(t, \tau)(t \notin \{t_j\}_{j=1}^{\infty}), W(t_j^+, \tau) = L_j W(t_j, \tau)$$

Assume that the space X is split up in a direct sum of the subspaces X_1 and X_2, i.e. $X = X_1 \dot{+} X_2$ and let P_1, P_2 be the corresponding complementary projectors $(P_1 + P_2 = I$, where I is the identity operator). Set $W_i(t_1, t_2) = W(t_1)P_i W^{-1}(t_2)(i = 1, 2)$, where $W(t) = W(t, \bar{t})$.

Let $K \subseteq [\bar{t}, \infty)$. Denote by $S(K, X)$ the Banach space of all bounded functions $x : K \to X$ which are continuous for $t \neq t_j \in K$ continuous from the left, have a limit from the right, provided with the norm $|||x||| = \sup_{t \in K} \|x(t)\|$.

Lemma 1. *Let $F \subset S([\alpha, \infty), X)(\alpha \geq \bar{t})$ and let the following conditions hold:*

1. *There exists a function $\varphi_0 \in S([\alpha, \infty), X)$ such that $\|\varphi(t) - \varphi_0(t)\| \xrightarrow[t \to \infty]{} 0$ uniformly with respect to $\varphi \in F$.*
2. *The set F is equicontinuous on each interval $[\alpha, \infty) \cap (t_{j-1}, t_j] (j = 1, 2, 3, \ldots)$.*
3. *For each $t \in [\alpha, \infty)$ the set $\{\varphi(t) : \varphi \in F\}$ is relatively compact in X.*

Then F is relatively compact in $S([\alpha, \infty), X)$.

Lemma 1 is proved in a standard way by the construction for any $\varepsilon > 0$ of an ε-lattice for F in the space $S([\alpha, \infty), X)$.

3. Main results

Set $h_j = L_j^{-1} H_j$ and $x_j = x(t_j)$ where $x(t)$ is an arbitrary function of t.

Definition 1. The impulsive equations (1), (2) and (3), (4) are called asymptotically equivalent if for any bounded solution $x \in ((1), (2))_{\bar{t}}$ there exists $\alpha \in [\bar{t}, \infty)$ and a solution $y \in ((3), (4))_\alpha$ such that $y(t) - x(t) \xrightarrow[t \to \infty]{} 0$ and, conversely, for any bounded solution $y \in ((3), (4))_{\bar{t}}$ there exists a solution $x \in ((1), (2))_{\bar{t}}$ for which $y(t) - x(t) \xrightarrow[t \to \infty]{} 0$.

Theorem 1. *Let the following conditions hold:*

1. *There exists a bounded solution $x \in ((1), (2))_{\bar{t}}$.*
2. *There exists a constant $R > 0$ for which*

$$(5) \quad \int_{\bar{t}}^{t} \sup_z{}' \|W_1(t, s)f(s, z)\| \, ds + \sum_{\bar{t} < t_j < t} \sup_z{}'' \|W_1(t, t_j)h_j(z)\| +$$

$$+ \sup_{\bar{t} \leq r \leq t} \left[\int_t^\infty \sup_z{}' \|W_2(\tau, s)f(s, z)\| \, ds + \sum_{t \leq t_j < \infty} \sup_z{}'' \|W_2(\tau, t_j)h_j(z)\| \right] \xrightarrow[t \to \infty]{} 0$$

where $\sup_z'$ is taken over the set $\{z \in X : \|z - x(s)\| \leq R\}$ and $\sup_z''$ over the set $\{z \in X : \|z - x_j\| \leq R\}$.

3. *The set $f\{(t, z) : \bar{t} \leq t < \infty, \|z - x(t)\| \leq R\}$ is relatively compact in X.*
4. *For $j = 1, 2, 3, \ldots$ the mappings H_j are continuous on the balls $B_j = \{z \in X : \|z - x_j\| \leq R\}$ and the set $\bigcup_{j=1}^{\infty} h_j(B_j)$ is relatively compact in X.*

Then for some $\alpha \in [\bar{t}, \infty)$ there exists a solution $y \in ((3, (4))_\alpha$ for which $y(t) - x(t) \xrightarrow[t \to \infty]{} 0$. If, moreover, the left-hand side of the relation in (5) (as a

function of t) belongs to the space $L_p(\bar{t}, \infty)$ for some $p \in [1, \infty]$, then $\|y(t) - x(t)\| \in L_p(\alpha, \infty)$.

Proof: Denote the left-hand side of relation (5) by $F(t)$ and choose $\alpha \geq \bar{t}$ so large that $F(t) \leq R$ for $t \geq \alpha$. The set $D = \{z \in S([\alpha, \infty), X) : \||z - x|_{[\alpha, \infty)}\|| \leq R\}$ is nonempty, convex, closed and bounded.

On D define the operator Q by means of the formula

$$(6) \quad (Qz)(t) = x(t) + \int_\alpha^t W_1(t, s) f(s, z(s)) \, ds + \sum_{\alpha \leq t_j < t} W_1(t, t_j) h_j(z_j) -$$
$$- \int_t^\infty W_2(t, s) f(s, z(s)) \, ds - \sum_{t \leq t_j < \infty} W_2(t, t_j) h_j(z_j)$$

where $z \in D, t \in [\alpha, \infty)$. The convergence of the improper integral and the series in (6) follws from the fact that $F(t) < \infty$.

We shall show that the operator Q satisfies the conditions of Schauder's fixed point theorem.

1. $Q : D \to D$. In fact, from the uniform convergence of the improper integral and the series in the right-hand side of equality (6) and from the choice of α it follows that $Qz \in S([\alpha, \infty), X)$ and that $\|(Qz)(t) - x(t)\| \leq R, \alpha \leq t < \infty$.

2. The operator Q is continuous. In fact, let $\epsilon > 0$ be an arbitrarily chosen number and let $\xi \geq \alpha$ be choosen so that $F(s) < \frac{\epsilon}{8}$ for $s \geq \xi$. Let $z, z_1 \in D$ and $\alpha \leq t \leq \xi$. Then

$$\|(Qz)(t) - (Qz_1)(t)\| \leq \int_\alpha^t \|W_1(t, s)[f(s, z(s)) - f(s, z_1(s))]\| \, ds +$$
$$+ \int_t^\xi \|W_2(t, s)[f(s, z(s)) - f(s, z_1(s))]\| \, ds +$$
$$+ \sum_{\alpha \leq t_j < t} \|W_1(t, t_j)[h_j(z_j) - h_j(z_{1j})]\| +$$
$$+ \sum_{t \leq t_j < \xi} \|W_2(t, t_j)[h_j(z_j) - h_j(z_{1j})]\| +$$
$$+ \int_\xi^\infty \|W_2(t, s)[f(s, z(s)) - f(s, z_1(s))]\| \, ds +$$
$$+ \sum_{\xi \leq t_j < \infty} \|W_2(t, t_j)[h_j(z_j) - h_j(z_{1j})]\|$$

Due to the continuity of the functions f and h_j for a fixed function z_1 a number $\eta > 0$ can be found such that for $\||z - z_1\|| < \eta$ each of the first four addends in the right-hand side of the last inequality should be smaller than

$\frac{\varepsilon}{8}$. But then the right-hand side of the last inequality becomes smaller than $4\frac{\varepsilon}{8} + 2\frac{\varepsilon}{8} + 2\frac{\varepsilon}{8} = \varepsilon$. For $t > \xi$ the following inequalities are valid

$$\|(Qz)(t) - (Qz_1)(t)\| \le \|(Qz)(t) - x(t)\| + \|(Qz_1)(t) - x(t)\| < 2\frac{\varepsilon}{8} < \varepsilon$$

Hence the operator Q is continuous.

3. We shall show that the set QD is relatively compact in D. For this purpose we shall show that the conditions of Lemma 1 are fulfilled for $F = QD$, $\varphi_0 = x$. Condition 1 of Lemma 1 follows immediately from relation (5).

We shall show that condition 2 of Lemma 1 holds. Let $z \in D$ and $t', t'' \in (t_{k-1}, t_k]$ for some K where $t' \le t''$. Then, after elementary transformations we obtain the equality

$$
\begin{aligned}
(Qz)(t') - (Qz)(t'') = & x(t') - x(t'') - \int_{t'}^{t''} W_1(t'', s) f(s, z(s))\, ds - \\
& - \int_{\alpha}^{t'} [W_1(t'', s) - W_1(t', s)] f(s, z(s))\, ds + \\
& + \sum_{\alpha \le t_j < t_k} [W_1(t', t_j) - W_1(t'', t_j)] h_j(z_j) - \\
& - \int_{t'}^{t''} W_2(t', s) f(s, z(s))\, ds - \\
& - \int_{t''}^{\infty} [W_2(t', s) - W_2(t'', s)] f(s, z(s))\, ds - \\
& - \sum_{j=k}^{\infty} [W_2(t', t_j) - W_2(t'', t_j)] h_j(z_j),
\end{aligned}
$$

from which we deduce the estimate

$$
\begin{aligned}
\|(Qz)(t') - (Qz)(t'')\| \le & \|x(t') - x(t'')\| + \int_{t'}^{t''} \|W_1(t'', s) f(s, z(s))\|\, ds + \\
& + \int_{\alpha}^{t'} \|W(t') - W(t'')\| \cdot \|P_1 W^{-1}(s) f(s, z(s))\|\, ds + \\
& + \sum_{\alpha \le t_j < t_k} \|W(t') - W(t'')\| \|P_1 W^{-1}(t_j) h_j(z_j)\| + \\
& + \int_{t'}^{t''} \|W_2(t', s) f(s, z(s))\|\, ds + \\
& + \int_{t''}^{\infty} \|[W(t') - W(t'')] P_2 W^{-1}(s) f(s, z(s))\|\, ds + \\
& + \sum_{j=k}^{\infty} \|[W(t') - W(t'')] P_2 W^{-1}(t_j) h_j(z_j)\|
\end{aligned}
$$

Denote by $I_1, \ldots, I_7$ the successive addends in the right-hand side of the last inequality and assume that $t'' - t' \to 0$. Then $I_1 \to 0$ since $x \in S([\bar{t}, \infty), X)$. $I_2, I_5 \to 0$ because of the uniform (with respect to z, t', t'') boundedness of the integrand. $I_3 \to 0$ and $I_4 \to 0$ since under the signs of the integral and of the sum the first factor tends uniformly to zero and the second one is uniformly bounded. Represent I_6 for $t \geq t_k$ in the form

$$ I_6 = \int_{t''}^{t} + \int_{t}^{\infty} \leq \int_{t''}^{t} \|W(t') - W(t'')\| \cdot \|P_2 W^{-1}(s)f(s, z(s))\| \, ds + 2F(t) $$

The second addend in the last inequality can be made arbitrarily small for a sufficiently large value of t. For t fixed and $t'' - t'$ small enough the first addend can also be made uniformly arbitrarily small (analogously to I_3). Hence $I_6 \to 0$. Analogously it is proved that $I_7 \to 0$.

Thus the equicontinuity of the set QD on the interval $(t_{k-1}, t_k]$ is proved.

In order to verify condition 3 of Lemma 1, first we shall show that for $t \in [\alpha, \infty)$ the set $\{\int_t^{\infty} W_2(t,s)f(s, z(s)) \, ds : z \in D\}$ is relatively compact in X. Let $\varepsilon > 0$ be arbitrarily chosen. Choose $\xi \geq t$ so that $\| \int_{\xi}^{\infty} W_2(t,s)f(s, z(s)) \, ds\| \leq \frac{\varepsilon}{3}$ for all $z \in D$. The set $\{W^{-1}(s)f(s, z(s)) : t \leq s \leq \xi, z \in D\}$ is relatively compact in X. Hence the sets $\{\int_t^{\xi} W^{-1}(s)f(s, z(s)) \, ds : z \in D\}$ and $\{\int_t^{\xi} W_2(t,s)f(s, z(s)) \, ds : z \in D\}$ are also relatively compact subsets of X. For the last set we choose a finite $\frac{\varepsilon}{3}$ -lattice- $\{\int_t^{\xi} W_2(t,s)f(s, z_i(s)) \, ds : i = 1, \ldots, N; z_i \in D\}$ and obtain that the set $\{\int_t^{\infty} W_2(t,s)f(s, z_i(s)) \, ds : i = 1, \ldots, N\}$ forms an ε-lattice for the set $\{\int_t^{\infty} W_2(t,s)f(s, z(s)) \, ds : z \in D\}$ which is therefore relatively compact in X.

Analogously the relative compactness in X of the remaining addends is verified, hence of the sum in the right-hand side of equality (6) for arbitrary fixed $t \in [\alpha, \infty)$ as well when z runs over the set D. This enables us to apply Lemma 1.

From Schauder's theorem it follows that the operator Q has a fixed point $y \in D$. We shall show that $y \in ((3), (4))_\alpha$.

Let $t \in [\alpha, \infty)\backslash\{t_j\}_{j=1}^{\infty}$. Then, in view of the fact that x is a solution of impulsive equation (1), (2) and of the equality $W_1(t,t) + W_2(t,t) = I$, we obtain

$$ (7) \quad y'(t) = (Qy)'(t) = $$

$$ = x'(t) + W_1(t,t)f(t, y(t)) + A(t) \int_{\alpha}^{t} W_1(t,s)f(s, y(s)) \, ds + $$

$$ + A(t) \sum_{\alpha \leq t_j < t} W_1(t,t_j)h_j(y_j) + W_2(t,t)f(t, y(t)) - $$

$$ - A(t) \int_{t}^{\infty} W_2(t,s)f(s, y(s)) \, ds - A(t) \sum_{t < t_j < \infty} W_2(t,t_j)h_j(y_j) = $$

$$ = A(t)y(t) + f(t, y(t)) $$

Let $t = t_k \in [\alpha, \infty)$. Then

$$(8) \quad y(t_k^+) = (Qy)(t_k^+) =$$

$$= L_k x_k + \int_\alpha^{t_k} L_k W_1(t_k, s) f(s, y(s)) \, ds + \sum_{\alpha \le t_j \le t_k} L_k W_1(t_k, t_j) \cdot h_j(y_j) -$$

$$- \int_{t_k}^\infty L_k W_2(t_k, s) f(s, y(s)) \, ds - \sum_{j=k+1}^\infty L_k W_2(t_k, t_j) h_j(y_j) =$$

$$= L_k[y_k + W_1(t_k, t_k) h_k(y_k) + W_2(t_k, t_k) h_k(y_k)] =$$

$$= L_k y_k + H_k y_k$$

Finally, as a consequence of the definition of the operator Q we obtain the estimate

$$\|y(t) - x(t)\| = \|(Qy)(t) - x(t)\| \le F(t) \quad (\alpha \le t < \infty)$$

which implies all assertions of Theorem 1. ∎

Remark 1. For $\dim X < \infty$ Theorem 1 still holds without condition 3 and the second part of condition 4.

Theorem 2. *Let a solution* $y \in ((3), (4))_{\bar{t}}$ *exist satisfying the condition*

$$(9) \quad \int_{\bar{t}}^t \|W_1(t, s) f(s, y(s))\| \, ds + \sum_{\bar{t} < t_j < t} \|W_1(t, t_j) h_j(y_j)\| +$$

$$+ \int_t^\infty \|W_2(t, s) f(s, y(s))\| \, ds + \sum_{t \le t_j < \infty} \|W_2(t, t_j) h_j(y_j)\| \underset{t \to \infty}{\to} 0$$

Then there exists a solution $x \in ((1), (2))_{\bar{t}}$ *for which* $x(t) - y(t) \underset{t \to \infty}{\to} 0$. *If for some* $p \in [1, \infty]$ *the left-hand side of relation (9) belongs to the space* $L_p(\bar{t}, \infty)$, *then* $\|x(t) - y(t)\| \in L_p(\bar{t}, \infty)$ *too.*

Proof: For $t \ge \bar{t}$ define the function x by means of the formula

$$x(t) = y(t) - \int_{\bar{t}}^t W_1(t, s) f(s, y(s)) \, ds - \sum_{\bar{t} < t_j < t} W_1(t, t_j) h_j(y_j) +$$

$$+ \int_t^\infty W_2(t, s) f(s, y(s)) \, ds + \sum_{t \le t_j < \infty} W_2(t, t_j) h_j(y_j)$$

By a straightforward verification, analogously to (7) and (8), it is established that $x \in ((1), (2))_{\bar{t}}$ and has the necessary properties. ∎

Theorems 1 and 2 immediately imply the following theorem of asymptotic equivalence of impulsive equations (1), (2) and (3), (4):

Theorem 3. *Let for any $R > 0$ the following conditions hold:*

1.

$$(10) \quad \int_{\bar{t}}^{t} \sup_{\|z\| \leq R} \|W_1(t,s)f(s,z)\| \, ds + \sum_{\bar{t} < t_j < t} \sup_{\|z\| \leq R} \|W_1(t,t_j)h_j(z)\| +$$

$$+ \sup_{\bar{t} \leq \tau \leq t} \left(\int_{t}^{\infty} \sup_{\|z\| \leq R} \|W_2(\tau,s)f(s,z)\| \, ds + \sum_{t \leq t_j < \infty} \sup_{\|z\| \leq R} \|W_2(\tau,t_j)h_j(z)\| \right) \xrightarrow[t \to \infty]{} 0$$

2. *The set $f\{(t,z) : \bar{t} \leq t < \infty, \|z\| \leq R\}$ is relatively compact in X.*

3. *The mappings $H_j (j = 1,2,3,\dots)$ are continuous and the set $\bigcup_{j=1}^{\infty} h_j\{z \in X : \|z\| \leq R\}$ is relatively compact in X.*

Then the impulsive equations (1), (2) and (3), (4) are asymptotically equivalent. Moreover, if the left-hand side of relation (10) belongs for some $p \in [1,\infty]$ to the space $L_p(\bar{t},\infty)$, then $\|y(t) - x(t)\| \in L_p(\alpha,\infty)$ as well.

Remark 2. For $\dim X < \infty$ Theorem 3 still holds without condition 2 and the second part of condition 3.

References

1. BAINOV D.D., KOSTADINOV S.I. AND MYSHKIS A.D., Bounded and periodic solutions of differential equations with impulse effect in a Banach space, *Differential and integral equations* **1**, no. 2 (1988), 223–230.

2. BAINOV D.D., ZABREIKO P.P. AND KOSTADINOV S.I., Stability of the general exponent of nonlinear impulsive differential equations in a Banach space, *International J. of theoretical physics* **27**, no. 3 (1988), 373–380.

3. BROUWER, F., On the asymptotic relationships between solutions of two systems of ordinary differential equations, *J. of differential equations* **6** (1969), 527–543.

4. DISHLIEV A.B. AND BAINOV D.D., Sufficient conditions for absence of *beating* in systems of differential equations with impulses, *Applicable Anal.* **18** (1985), 67–73.

5. FUTAK J., Asymptotic and integral equivalence of functional and ordinary differential equations, *Fasciculi Mathematici* **15** (1984), 97–108.

6. HAŠČAK AND ŠVEC M., Integral equivalence of two systems of differential equations, *Czechoslovak Mathematical Journal* **32** (107) (1982), 423–436.

7. HAŠČAK A., Integral equivalence between a nonlinear system and its nonlinear perturbation, *Math. Slovaca* **34**, no. 4 (1984), 393–404.

8. HU, SCHOUCHUAN AND LAKSHMIKANTHAM V., Periodic boundary value problems for second order impulsive differential systems, To appear in *J. of Nonlinear Analysis.*

9. HU, SCHOUCHUAN, LAKSHMIKANTHAM V., Existence and continuation results for impulsive differential equations, To appear.

10. KULEV G.K. AND BAINOV D.D., Asymptotic stability of systems with impulses by the direct method of Lyapunov, *Bull. Austral. Math. Soc.* **38** (1988), 113–123.

11. KULEV G.K. AND BAINOV D.D., Second method of Lyapunov and comparison principle for systems with impulse effect, *J. of Computational and Applied Mathematics* **23** (1988), 305–321.

12. LAKSHMIKANTHAM V.AND LIU XINZHI, Stability for impulsive differential systems in terms of two measures, *Appl. Math. Comp.* (to appear).

13. LAKSHMIKANTHAM V. AND LIU XINZHI, On quasi-stability for impulsive differential systems, To appear in *J. Nonlinear Analysis*.

14. MARLIN J.A. AND STRUBLE R.A., Asymptotic equivalence of nonlinear systems, *J. of differential equations* **6** (1969), 578–596.

15. MIL'MAN V.D. AND MYSHKIS A.D., On the stability of motion in the presence of impulses, *Sibirsk. Mat. J.* **1** (1960), 223–237 (in Russian).

16. SIMEONOV P.S. AND BAINOV D.D., Asymptotic equivalence of two systems of differential equations with impulse effect, *Systems control letters* **3** (1983), 297–301.

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