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SOME CHARACTERIZATIONS OF REGULAR MODULES

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Abstract

Let M be a left module over a ring R. M is called a Zelmanowitz-regular module if for each $x \in M$ there exists a homomorphism $f : M \to R$ such that $f(x)x = x$. Let Q be a left R-module and $h : Q \to M$ a homomorphism. We call h locally split if for each $x \in M$ there exists a homomorphism $g : M \to Q$ such that $h(g(x)) = x$. M is called locally projective if every epimorphism onto M is locally split. We prove that the following conditions are equivalent:

(1) M is Zelmanowitz-regular.

(2) every homomorphism into M is locally split.

(3) M is locally projective and every cyclic submodule of M is a direct summand of M.

As generalizations of the concept of Von Neumann's regular rings to the module case, there have been considered three types of modules by Fieldhouse [1], Ware [4] and Zelmanowitz [5], each called regular. The Fieldhouse-regular module was defined to be a module whose submodules are pure submodules and the Ware-regular module was defined as a projective module in which every cyclic submodule is a direct summand, while a left module M over a ring R is called a Zelmanowitz-regular module if for each $x \in M$ there exists a homomorphism $f : M \to R$ such that $f(x)x = x$. Now we introduce a notion of locally split homomorphisms to show that a module is Zelmanowitz-regular if and only if every homomorphism into the module is locally split, and by making use of this we prove that Zelmanowitz-regular modules are characterized as locally projective modules whose cyclic submodules are direct summands. For convenience (but at the risk of confusion), we call a module regular if every cyclic submodule of it is a direct summand. Thus, in this terminology, a module is Ware-regular or Zelmanowitz-regular if and only if it is projective regular or locally projective regular respectively. Moreover we shall see that every regular module is Fieldhouse-regular and that Ware-regular and Zelmanowitz-regular modules are also characterized as projective Fieldhouse-regular and locally projective Fieldhouse-regular modules respectively.

Let R be a ring with identity element. By a module we shall throughout mean a unital left R-module, unless otherwise specified. Let Q and M be modules, and let $h : Q \to M$ be a $(R-)$ homomorphism. h is called *locally split* if for any $x_0 \in h(Q)$ there exists a homomorphism $q : M \to Q$ such that $h(q(x_0)) = x_0$.

Proposition 1. *Let $h : Q \to M$ be a locally split homomorphism. Then, for any finite number of $x_1, x_2, \ldots, x_n \in h(Q)$, there exists a homomorphism $q : M \to Q$ such that $h(q(x_i)) = x_i$ for $i = 1, 2, \ldots, n$.*

Proof: In order to prove by induction, suppose that $n > 1$ and our assertion is true for $n - 1$ (instead of n). Then there exists a $q_1 : M \to Q$ such that $h(q_1(x_i)) = x_i$ for $i = 1, 2, \ldots, n-1$. Since $x_n - h(q_1(x_n))$ is in $h(Q)$, there is a $q_2 : M \to Q$ such that $h(q_2(x_n - h(q_1(x_n)))) = x_n - h(q_1(x_n))$. Let $q = q_1 + q_2 - q_2 \circ h \circ q_1 : M \to Q$. Then $h(q(x_n)) = h(q_1(x_n)) + h(q_2(x_n)) - h(q_2(h(q_1(x_n)))) = h(q_1(x_n)) + h(q_2(x_n - h(q_1(x_n)))) = x_n$, and $h(q(x_i)) = h(q_1(x_i)) + h(q_2(x_i)) - h(q_2(h(q_1(x_i)))) = x_i + h(q_2(x_i)) - h(q_2(x_i)) = x_i$ for $i = 1, 2, \ldots, n-1$. Thus q is a desired homomorphism.

Let N be a submodule of a module M. N is called *locally split* in M if the inclusion map $N \to M$ is locally split, i.e., for any $x_0 \in N$ there exists a homomorphism $s : M \to N$ such that $s(x_0) = x_0$. ■

Proposition 2. *Let $h : Q \to M$ be a homomorphism. Denote by h' the epimorphism $Q \to h(Q)$ regarded h as a map onto $h(Q)$. Then h is locally split if and only if h' is locally split and $h(Q)$ is locally split in M.*

Proof: Let x_0 be any element of $h(Q)$. Suppose that h is locally split. Then there exists a homomorphism $q : M \to Q$ such that $h(q(x_0)) = x_0$. This implies that the homomorphism $s = h \circ q : M \to h(Q)$ satisfies $s(x_0) = x_0$, and thus $h(Q)$ is locally split in M. On the other hand, if we denote by $q' : h(Q) \to Q$ the restriction of q to $h(Q)$ then we have $h'(q'(x_0)) = h(q(x_0)) = x_0$, which shows that h' is locally split. Suppose conversely that $h(Q)$ is locally split in M and h' is also locally split. This means that there exist homomorphisms $s : M \to h(Q)$ and $q' : h(Q) \to Q$ such that $s(x_0) = x_0$ and $h'(q'(x_0)) = x_0$. Let $q = q' \circ s : M \to Q$. Then we have $h(q(x_0)) = h'(q'(s(x_0))) = h'(q'(x_0)) = x_0$. Thus h is locally split. ■

Proposition 3. *Let M be a module. Then every locally split submodule of M is pure in M, while every locally split epimorphism from M is pure, i.e., the kernel of the epimorphism is pure in M.*

Proof: Let N be a locally split submodule of M. Let $x_1, x_2, \ldots, x_n \in M$ satisfy the system of linear equations $r_{i1}x_1 + r_{i2}x_2 + \cdots + r_{in}x_n = v_i$ ($i = 1, 2, \ldots, m$), where each $r_{ij} \in R$ and $v_i \in N$. Then, by applying Proposition 1 to $v_1, v_2, \ldots, v_m$ and the inclusion map $N \to M$ (instead of $x_1, x_2, \ldots, x_n$ and $h : Q \to M$), we can find a homomorphism $s : M \to N$ such that $s(v_i) = v_i$ for $i = 1, 2, \ldots, m$. We have then $r_{i1}s(x_1) + r_{i2}s(x_2) + \cdots + r_{in}s(x_n) = s(v_i) = v_i$ ($i = 1, 2, \ldots, m$). Since each $s(x_i)$ is in N, this shows that N is pure in M by Cohn's theorem.

Let next $h : M \to M'$ be an epimorphism and N the kernel of h. Let $x_1, x_2, \ldots, x_n \in M$ satisfy the system of linear equations $r_{i1}x_1 + r_{i2}x_2 + \cdots +$

$r_{in}x_n = v_i$ $(i = 1, 2, \ldots, m)$, where $r_{ij} \in R$ and $v_i \in N$. Then we have $r_{i1}h(x_1)+r_{i2}h(x_2)+\cdots+r_{in}h(x_n) = h(v_i) = 0$ $(i = 1, 2, \ldots, m)$. Suppose that h is locally split. Then since each $h(x_j)$ is in $h(M) = M'$, by applying Proposition 1 to $h(x_1)$, $h(x_2), \ldots, h(x_n)$ and $h : M \to M'$ (instead of $x_1, x_2, \ldots, x_n$ and $h : Q \to M$), we find a homomorphism $q : M' \to M$ such that $h(q(h(x_j))) = h(x_j)$, i.e., $x_j - q(h(x_j)) \in N$ for $j = 1, 2, \ldots, n$. From the above equalities it follows now $r_{i1}q(h(x_1)) + r_{i2}q(h(x_2)) + \cdots + r_{in}q(h(x_n)) = 0$ and therefore $r_{i1}(x_1 - q(h(x_1))) + r_{i2}(x_2 - q(h(x_2)) + \cdots + r_{in}(x_n - q(h(x_n)) = v_i$ $(i = 1, 2, \ldots, m)$. This implies that N is pure in M again by Cohn's theorem. ■

Remark. The notion of locally split submodules was introduced by Ramamurthi and Rangaswamy [2] by the name of strongly pure submodules, and they actually obtained the first half of the preceding proposition.

Theorem 4. *Let M be a left R-module. Then the following conditions are equivalent:*

(1) *M is a Zelmanowitz-regular module.*
(2) *Every homomorphism into M (from any module) is locally split.*
(3) *Every homomorphism $R \to M$ is locally split.*

Proof: (1) ⇒ (2) : Let Q be a module and $h : Q \to M$ a homomorphism. Let x_0 be any element of $h(Q)$. Choose a $z_0 \in Q$ such that $h(z_0) = x_0$. Since M is Zelmanowitz-regular, there exists a homomorphism $f : M \to R$ such that $f(x_0)x_0 = x_0$. Define a homomorphism $q : M \to Q$ by $q(x) = f(x)z_0$ for $x \in M$. Then we have $h(q(x_0)) = f(x_0)h(z_0) = f(x_0)x_0 = x_0$, which shows that h is locally split.

(2) ⇒ (3) is trivial.

(3) ⇒ (1) : Let x_0 be any element of M. Let $g : R \to M$ be the homomorphism defined by $g(r) = rx_0$ for $r \in R$. Then g is locally split, so that there exists a homomorphism $f : M \to R$ such that $x_0 = g(f(x_0)) = f(x_0)x_0$. This shows that M is Zelmanowitz-regular. ■

Now we call a module M a *regular* module if every submodule of M is locally split in M.

Proposition 5. *Let M be a module. Then the following conditions are equivalent:*

(1) *M is a regular module.*
(2) *Every finitely generated submodule of M is a direct summand of M.*
(3) *Every cyclic submodule of M is a direct summand of M.*

Proof: (1) ⇒ (2) : Let $N = Rx_1 + Rx_2 + \cdots + Rx_n$ be a finitely generated submodule of M. Since M is regular, N is locally split and therefore, by applying Proposition 1 to the inclusion map $N \to M$ (instead of $h : Q \to M$), we can find a homomorphism $s : M \to N$ such that $s(x_i) = x_i$ for $i =$

$1, 2, \ldots, n$, or equivalently, $s(x) = x$ for all $x \in N$. This implies that N is a direct summand of M.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1) : Let N be a submodule of M. Let x_0 be any element of N. Then the cyclic submodule Rx_0 is a direct summand of M, which means that there is a homomorphism $s : M \to Rx_0(\subset N)$ such that $s(x_0) = x_0$. Thus N is locally split in M. ∎

It is to be pointed out that every submodule of a regular module is regular too, and every regular module is Fieldhouse-regular, i.e., every submodule is a pure submodule.

A module M is called *locally projective* if every epimorphism onto M (from any module) is locally split. It follows from Proposition 3 that every locally projective module is flat, since a flat module is characterized as a module onto which every epimorphism is pure. The notion of locally projective modules was introduced by Zimmermann-Huisgen [**6**] and also by Raynaud and Gruson [**3**] under the name of flat strict Mittag-Leffler modules. Their definitions are apparently different from the above one. But the following proposition implies that all the definitions coincide (if compared with [**6**], Theorem 2.1 and [**3**], Proposition 2.3.4), and we will give a proof to the proposition for completeness:

Proposition 6. *Let M be a left R-module. Then the following conditions are equivalent:*

(1) M is locally projective.

(2) For any finitely generated submodule M_0 of M, there exist a finitely generated free left R-module F aand homomorphisms $f : M \to F$ and $g : F \to M$ such that $g(f(x)) = x$ for all $x \in M_0$.

(3) For any $x_0 \in M$, there exist a finite number of homomorphisms $f_i : M \to R$ $(i = 1, 2, \ldots, n)$ and elements $y_i \in M$ $(i = 1, 2, \ldots, n)$ such that $f_1(x_0)y_1 + f_2(x_0)y_2 + \cdots + f_n(x_0)y_n = x_0$.

Proof: (1) $\Rightarrow$ (2): Let Q be a free R-module having an epimorphism $h : Q \to M$. Then h is locally split, so that, by applying Proposition 1 to the finite number of generators of M_0, we can find a homomorphism $q : M \to Q$ such that $h(q(x)) = x$ for all $x \in M_0$. Since the image $q(M_0)$ of M_0 is a finitely generated submodule of Q, there exists a finite subset $\{u_1, u_2, \ldots, u_n\}$ of the free basis of Q such that $q(M_0)$ is contained in the finitely generated free submodule $F = Ru_1 + Ru_2 + \cdots + Ru_n$ of Q. Since F is a direct summand of Q, there exists a homomorphism $p : Q \to F$ such that $p(z) = z$ for all $z \in F$. Let $f = p \circ q : M \to F$, and let $g : F \to M$ be the restriction of h to F. Then they clearly satisfy $g(f(x)) = x$ for all $x \in M_0$.

(2) $\Rightarrow$ (3): Let $x_0 \in M$. Since Rx_0 is finitely generated, there exist a finitely generated free R-module F and homomorphisms $f : M \to F$, $g : F \to M$ such that $g(f(x_0)) = x_0$. Let $u_1, u_2, \ldots, u_n$ be a free basis of F. Then we can, for

each i, define a homomorphism $f_i : M \to R$ by $f(x) = f_1(x)u_1 + f_2(x)u_2 + \cdots + f_n(x)u_n$ for $x \in M$. Let $y_i = g(u_i) \in M$ for $i = 1, 2, \ldots, n$. Then we have $x_0 = g(f(x_0)) = f_1(x_0)y_1 + f_2(x_0)y_2 + \cdots + f_n(x_0)y_n$.

(3) $\Rightarrow$ (1): Let Q be any R-module having an epimorphism $h : Q \to M$. Let $x_0 \in M$, and let $f_i : M \to R$ and $y_i \in M$ $(i = 1, 2, \ldots, n)$ be as in (3). Let $z_i \in Q$ be such that $h(z_i) = y_i$ for each i, and define a homomorphism $q : M \to Q$ by $q(x) = f_1(x)z_1 + f_2(x)z_2 + \cdots + f_n(x)z_n$ for $x \in M$. Then we have that $h(q(x_0)) = f_1(x_0)y_1 + f_2(x_0)y_2 + \cdots + f_n(x_0)y_n = x_0$. Thus h is locally split, so that M is locally projective. ■

Proposition 7. *Let M be a locally projective module, and let N be a pure submodule of M. Then N is locally projective and is locally split in M.*

Proof: Let x_0 be any element of N. By the preceding proposition, there exist homomorphisms $f_i : M \to R$ and elements $y_i \in M$ $(i = 1, 2, \ldots, n)$ such that $f_1(x_0)y_1 + f_2(x_0)y_2 + \cdots + f_n(x_0)y_n = x_0$. Since N is pure in M, we can find elements $v_1, v_2, \ldots, v_n$ in N such that $f_1(x_0)v_1 + f_2(x_0)v_2 + \cdots + f_n(x_0)v_n = x_0$ according to Cohn's criterion. Now we define a homomorphism $s : M \to N$ by $s(x) = f_1(x)v_1 + f_2(x)v_2 + \cdots + f_n(x)v_n$ for $x \in M$. Then we have that $s(x_0) = x_0$. Thus N is locally split in M. On the other hand, if we denote by g_i the restriction of f_i to N then clearly we have that $g_1(x_0)v_1 + g_2(x_0)v_2 + \cdots + g_n(x_0)v_n = x_0$, which shows that N is locally projective. ■

Remark. That N is locally projective in Proposition 7 was mentioned in [6, p. 236].

Theorem 8. *Let M be a module. Then the following conditions are equivalent:*

(1) M is a Zelmanowitz-regular module.

(2) M is a locally projective regular module.

(3) M is locally projective and Fieldhouse-regular (i.e., every submodule of M is pure in M).

Proof: (1) $\Rightarrow$ (2): If M is Zelmanowitz-regular, it follows from Theorem 4 that every epimorphism onto M is locally split and every monomorphism into M is locally split, which mean that M is locally projective and regular respectively. (Another proof for the local projectivity of M can be obtained directly from Proposition 6, for that for any $x_0 \in M$ there exists an homomorphism $f : M \to R$ such that $f(x_0)x_0 = x_0$ implies that M satisfies the condition (3) in Proposition 6 with $n = 1$, $f_1 = f$ and $y_1 = x_0$. That a Zelmanowitz-regular module is regular, i.e., every cyclic submodule of the module is a direct summand, is also proved in [5, Theorem 1.6].

(2) $\Rightarrow$ (3) is a consequence of the fact, due to Proposition 3, that every locally split submodule is a pure submodule.

(3) $\Rightarrow$ (1) : Let Q be a module and $h : Q \to M$ a homomorphism. Since $h(Q)$ is a pure submodule of M by assumption, it follows from Proposition 7 that $h(Q)$ is locally projective and is locally split in M. Regarding h as a map onto $h(Q)$ we have an epimorphism $h' : M \to h(Q)$, but the local projectivity of $h(Q)$ implies that h' is locally split. Therefore, by Proposition 2, h is locally split. Thus, M is Zelmanowitz-regular according to Theorem 4. ■

Remark 1. Although we throughout assume that R has an identity element, the paper [5] deals with modules over rings without identity element.

Remark 2. It is pointed out in [6] that over a regular ring a module is locally projective if and only if it is Zelmanowitz-regular. But this can be regarded as a particular case of Theorem 8, because over a regular ring every module is flat and hence is Fieldhouse-regular.

In this connection, we would like to mention of some properties of regular modules and locally projective modules:

1. *If M is a regular R-module then its Jacobson radical $J(M)$ is zero, and if M is a faithful regular R-module then the Jacobson radical $J(R)$ of R is zero.*

The proof is actually given in [4], though regular modules in [4] mean projective regular modules in the present paper. Namely, if x_0 is in $J(M)$ then Rx_0 is a direct summand small submodule of M and therefore $x_0 = 0$, which implies $J(M) = 0$. Since $J(R)M \subset J(M)$, it follows $J(R) = 0$ if M is faithful and regular.

2. *If M is a locally projective R-module then $J(R)M = J(M)$.*

For, let x_0 be in $J(M)$; then by Proposition 6 there exist a finite number of homomorphisms $f_i : M \to R$ and elements y_i in M $(i = 1, 2, \ldots, n)$ such that $f_1(x_0)y_1 + f_2(x_0)y_2 + \cdots + f_n(x_0)y_n = x_0$. Let L be a maximal left ideal of R. Then its inverse image by f_i is either equal to M or a maximal submodule of M and therefore contains $J(M)$, or equivalently, $f_i(J(M)) \subset L$. Since this is true for every maximal left ideal L, it follows $f_i(J(M)) \subset J(R)$ and in particular $f_i(x_0) \in J(R)$. This is true for each $i = 1, 2, \ldots, n$, so that we have $x_0 \in J(R)M$. Thus we know that $J(M) \subset J(R)M$.

3. *A module M is Fieldhouse-regular if (and only if) every finitely generated submodule of M is pure in M.*

This is because Cohn's criterion for purity is concerned only with finite number of elements.

Proposition 9. *Let M be a Zelmanowitz-regular module, and let S be the endomorphism ring of M. Then, as an S-module, M is Zelmanowitz-regular too, and the Jacobson radical $J(S)$ of S is zero.*

Proof: We consider M a right S-module and hence a two-sided R-S-module; thus $st = t \circ s$ for all $s, t \in S$. Let x_0 be an element of M. Then there exists a homomorphism $f : M \to R$ such that $f(x_0)x_0 = x_0$. Let $y \in M$. Then the mapping $x \mapsto f(x)y$ for $x \in M$ is an endomorphism of M, which we denote by $\bar{y} \in S$. If $s \in S$, we have $f(x)(ys) = (f(x)y)s$ for all $x \in M$, i.e., $\overline{ys} = \bar{y}s$.

This implies that the mapping $y \mapsto \bar{y}$ for $y \in M$ is a homomorphism $M \to S$ as S-modules. If we denote this by g then we have $f(x)y = xg(y)$ for all x, $y \in M$. (In the notation in [5], $g(y) = [f, y]$ for all $y \in M$.) It follows in particular that $x_0 = f(x_0)x_0 = x_0g(x_0)$. This shows that the S-module M is Zelmanowitz-regular. Since M is a faithful S-module, we have $J(S) = 0$ according to the above mentioned property 1. ■

Now, clearly a locally projective module is projective if it is finitely generated, but this is true even if it is countably generated:

Proposition 10. *Every countably generated locally projective module is projective.*

Proof: If we observe the fact that every locally projective module is a Mittag-Leffler module, our proposition can be regarded as a particular case of [3], Corollaire 2.2.2. But we shall for completeness give a proof which is valid for our case. Let M be a locally projective R-module with countable generators $x_1, x_2, x_3, \dots$. Let $M_1 = Rx_1$. By Proposition 6 there exist a finitely generated free R-module F_1 and homomorphisms $f_1 : M \to F_1$, $g_1 : F_1 \to M$ such that $g_1(f_1(x)) = x$ for all $x \in M_1$. Let next $M_2 = g_1(F_1) + Rx_2$. Since M_2 is finitely generated, again by Proposition 6, there exist a finitely generated free R-module F_2 and homomorphisms $f_2 : M \to F_2$, $g_2 : F_2 \to M$ such that $g_2(f_2(x)) = x$ for all $x \in M_2$. In this way, for each $n > 1$, we can find a finitely generated free R-module F_n and homomorphisms $f_n : M \to F_n$, $g_n : F_n \to M$ such that $g_n(f_n(x)) = x$ for all $x \in M_n = g_{n-1}(F_{n-1}) + Rx_n$. But this is clearly equivalent to that $g_n(f_n(g_{n-1}(y))) = g_{n-1}(y)$ for all $y \in F_{n-1}$ and $g_n(f_n(x_n)) = x_n$. From this follows then that $g_n \circ f_n \circ g_{n-1} = g_{n-1}$ whence $g_{n-1}(F_{n-1}) \subset g_n(F_n)$ and $x_n \in g_n(F_n)$. Thus we have an ascending chain $g_1(F_1) \subset g_2(F_2) \subset g_3(F_3) \subset \dots$ of submodules of M whose union is equal to M. For simplicity, we put $s_n = g_n \circ f_n : M \to g_n(F_n)$ for each n. Then s is an endomorphism of M satisfying $s_n \circ g_{n-1} = g_{n-1}$ and hence $s_n \circ s_{n-1} = s_{n-1}$ for each $n > 1$. Moreover we point out that $s_n \circ g_r = g_r$ and $s_n \circ s_r = s_r$ whenever $n > r$, because if $r < n$ then $g_r(F_r) \subset g_{n-1}(F_{n-1})$ and so $s_n(g_r(y)) = g_r(y)$ for all $y \in F_r$.

Let F be the direct sum of all F_n's. Then F is a countably generated free R-module. The homomorphisms $g_n : F_n \to M$ for $n = 1, 2, 3, \dots$ together define a homomorphism $g : F \to M$ in the natural manner. The image $g(F)$ is the sum of all $g_n(F_n)$'s and hence is equal to M, because even their union is M. Thus g is an epimorphism. In order to prove that M is projective, it is therefore sufficient to show that g splits, i.e., there exists a homomorphism $f : M \to F$ such that $g \circ f = 1$, the identity map of M. Let now $q_n : F_n \to F$ be the canonical embedding for $n = 1, 2, 3, \dots$ Then we have $g \circ q_n = g_n$ for each n. We shall construct a homomorphism $h_n : F_n \to F$ for each n such that $g \circ h_n = g_n$ and $h_n \circ f_n \circ g_{n-1} = h_{n+1} \circ f_{n+1} \circ g_{n-1}$ if $n > 1$. For this purpose, let first $h_1 = q_1$. Then $g \circ h_1 = g_1$. Suppose $n > 1$ and there is given an $h_n : F_n \to F$ such that $g \circ h_n = g_n$. We define $h_{n+1} = (h_n \circ f_n + q_{n+2} \circ f_{n+2} \circ (1 - s_n)) \circ g_{n+1}$. Then we have $g \circ h_{n+1} = (g \circ h_n \circ f_n + g \circ q_{n+2} \circ f_{n+2} \circ (1 - s_n)) \circ g_{n+1} = (g_n \circ f_n + g_{n+2} \circ f_{n-2} \circ (1 - s_n)) \circ g_{n+1} = (s_n + s_{n+2} \circ (1 - s_n)) \circ g_{n+1} = (s_n + s_{n+2} - s_{n+2} \circ s_n) \circ g_{n+1} = s_{n+2} \circ g_{n+1} = g_{n+1}$. On the other hand, we have $h_{n+1} \circ f_{n+1} \circ g_{n-1} = (h_n \circ f_n + q_{n+2} \circ f_{n+2} \circ (1 - s_n)) \circ s_{n+1} \circ g_{n-1} = (h_n \circ f_n + q_{n+2} \circ f_{n+2} \circ (1 - s_n)) \circ g_n n - 1 = h_n \circ f_n \circ g_{n-1} + q_{n-2} \circ f_{n-2} \circ g_{n-1} - q_{n-2} \circ f_{n-2} \circ s_n \circ g_{n-1} = h_n \circ f_n \circ g_{n-1}$. Thus by induction we get a desired sequence of homomorphisms $h_n (n = 1, 2, 3, \dots)$.

Let $x \in M$. Then there exists an $n > 1$ such that $x \in g_{n-1}(F_{n-1})$ i.e., $x = g_{n-1}(y)$ for some $y \in F_{n-1}$. We have then that $h_n(f_n(x)) = h_n(f_n(g_{n-1}(y))) = h_{n+1}(f_{n+1}(g_{n-1}(y))) = h_{n+1}(f_{n+1}(x))$. Moreover, since $x \in g_n(F_n)$ in this case, by replacing n by $n+1$ we should have that $h_{n+1}(f_{n+1}(x)) = h_{n+2}(f_{n+2}(x))$. Continuing in this way, we confirm that $h_n(f_n(x)) = h_m(f_m(x))$ for every $m > n$. This shows that $h_n(f_n(x))$ is independent of the choice of n so far as x is in $g_{n-1}(F_{n-1})$. Thus by defining $f(x) = h_n(f_n(x))$ for $x \in M$ we have a homomorphism $f : M \to F$, which satisfies $g(f(x)) = g_n(f_n(x)) = x$ (since $x \in g_{n-1}(F_{n-1})$). This completes our proof. ■

It is to be pointed out that the preceding proposition.can be regarded as a generalization of [**5**, Corollary 1.7.].

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