View metadata, citation and similar papers at core.ac.uk
brought to you by CORE
provided by INRIA a CCSD electronic archive server

Finite-time estimation for linear time-delay systems via homogeneous method

Gang Zheng, Haoping Wang

To cite this version:

Gang Zheng, Haoping Wang. Finite-time estimation for linear time-delay systems via homogeneous method. International Journal of Control, Taylor & Francis, 2019, 92 (6), pp.1252-1263. 10.1080/00207179.2017.1390255 . hal-01649434

HAL Id: hal-01649434

https://hal.inria.fr/hal-01649434

Submitted on 27 Nov 2017

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Finite-time estimation for linear time-delay systems via homogeneous method

Gang Zheng and Haoping Wang *

Abstract

This paper presents a finite-time observer for linear time-delay systems with commensurate delay. Unlike the existing observers in the literature which converge asymptotically, the proposed observer provides a finite-time estimation. This is realized by using the well-known homogeneous technique, and the results are also extended to investigate the estimation problem for linear time-delay systems with unknown inputs. Simulation results are presented in order to illustrate the feasibility of the proposed method.

1 Introduction

Time-delay systems describe a wide range of dynamic processes arising often in chemical, biological and economic applications. The observer design problem of linear system without delays has already been solved for the case without unknown inputs in Luenberger (1966) and for the case with unknown inputs in Hostetter and Meditch (1973); Wang et al. (1975). However, for time-delay systems which are used sometimes to model practical applications (see Richard (2003); Sename (2001)), the observer design problem is not trivial, especially when the system states contain multiple commensurate delays.

For this issue, different techniques have been proposed in the literature. Roughly speaking, there mainly exist two categories: treating the delay as variable or as operator. For the methods treating the delay as variable, such as the work of Salamon (1980); Darouach (2001); Germani et al. (2001); Darouach (2006), they normally studied linear or nonlinear systems with only one delay, and focused on seeking Lyapunov-Krasovskii functions to prove the convergence of the proposed observer. It is worth noting that most of the existing observers in this category are asymptotic. The second category introduces the delay operator which enables us to treat more general time-delay systems with multiple commensurate delays Sename (1997); Emre and Khargonekar (1982); Bejarano and Zheng (2014). The study of such a general system with multiple delays is motivated by different concrete applications, such as cold rolling mills system Malek-Zavarei and Jamshidi (1987) and chemical reactor train Nguang (2000); Gao and Ding (2007) where the studied plants can be modeled as linear time-delay systems with multiple delays. To design observers for such systems, Fattouh et al. (1999) proposed an unknown input asymptotic observer with dynamic gain for linear systems with commensurate delays in state, input and output variables, while the output was not affected by the unknown inputs. Recently, in Zheng et al. (2015), an asymptotic observer is studied for linear time-delay system with unknown inputs which affects as well the outputs.

All those mentioned observers provide only asymptotic estimation. However, in some applications it is desired to have a fast estimation in a finite time Shi et al. (2015); Zhang et al. (2015). There exist different techniques, such as algebraic differentiator Mboup et al. (2007), high order sliding mode observer/differentiator Levant (2003), impulsive observer Engel and Kreisselmeier (2002), or homogeneous observer/differentiator Perruquetti et al. (2008). In Zhai and Xia (2016b), the finite-time tracking control for nonlinear teleoperation systems is addressed, where the model uncertainties, actuator saturation, asymmetric time-varying delays, and passive/non-passive external forces are considered. In Zhai and Xia (2016a), an adaptive finite-time control law has been proposed for nonlinear teleoperation system in the presence of asymmetric time-varying delays, by introducing a switching-technique-based error filtering.

*Gang Zheng is with INRIA Lille-Nord Europe, 40, avenue Halley, 59650 Villeneuve d'Ascq, France. Haoping Wang is with School of Automation Nanjing University of Science and Technology, 210094 Nanjing, China

Up to now, for a general class of time-delay systems with multiple delays, there exist few results on finite-time observer (even for linear system with one or multiple delays). In Langueh et al. (2016), an impulsive finite-time observer is proposed, where two simple Luenberger-like observes are combined to achieve the finite-time estimation of the trajectory. However, such a finite-time estimation is not continuous, and this will lead to discontinuous controller if the estimations are used in the closed-loop control. Another approach to achieve the finite-time estimation is based on the technique of homogeneity. In Zheng and Wang (2016), the authors used such a technique to realize the continuous finite-time estimation for linear systems with multiple delays and known inputs. This paper is an extension of Zheng and Wang (2016) to investigate the simultaneous continuous finite-time estimation of the trajectory and the unknown input for linear systems with multiple delays and unknown inputs. Roughly, **the contributions of this paper** are twofold. Firstly, we present an observer with a finite-time convergence for linear time-delay system with known/unknown inputs by using homogeneous technique. And secondly we extend this result to design a finite-time observer to estimate the unknown inputs as well.

This paper adopts the method based on ring theory since it enables us to reuse some useful techniques developed for linear systems without delays. The following notations will be used in this paper. $\mathbb{R}$ is the field of real numbers. The set of positive integers is denoted by $\mathbb{N}$. I_p means $p \times p$ identity matrix. $\mathbb{R}[\delta]$ is the polynomial ring over the field $\mathbb{R}$ and $\mathbb{R}^n[\delta]$ is the $\mathbb{R}[\delta]$-module whose elements are vectors of dimension n and whose entries are polynomials. By $\mathbb{R}^{q\times s}[\delta]$ we denote the set of matrices of dimension $q \times s$, whose entries are in $\mathbb{R}[\delta]$. For a matrix $M(\delta)$, $\text{rank}_{\mathbb{R}[\delta]}M(\delta)$ means the rank of the matrix $M(\delta)$ over $\mathbb{R}[\delta]$. We denote $\text{Inv}_S[M(\delta)] = \{\Psi_i(\delta)\}_{1\leq i\leq r}$ as the set of invariant factors of the Smith form of $M(\delta)$.

The outline of this paper is as follows: The problem statement is explained in Section 2. Definitions and assumptions are given in Section 3. Section 4 presents a finite-time observer for a class of linear time-delay systems with known inputs by using homogeneous technique. This method has been extended to treat linear time-delay systems with unknown inputs in Section 5, where the simultaneous finite-time estimations of the trajectory and the unknown inputs are both investigated. Two examples and the associated simulations are given in Section 6 to show the effectiveness of the proposed method.

2 Problem statement

In this paper, we firstly consider the following class of linear systems with commensurate delays and known inputs:

$$\begin{aligned} \dot{x}(t) &= \sum_{i=0}^{k_a} \bar{A}_i x(t-ih) + \sum_{i=0}^{k_b} \bar{B}_i u(t-ih) \\ y(t) &= \sum_{i=0}^{k_c} \bar{C}_i x(t-ih) \end{aligned} \tag{1}$$

where the vector $x(t) \in \mathbb{R}^n$, the known input vector $u(t) \in \mathbb{R}^m$, and the output vector $y(t) \in \mathbb{R}^p$, $\bar{A}_i$, $\bar{B}_i$ and $\bar{C}_i$ are matrices of appropriate dimension. The initial condition $\varphi(t)$ is a function $\varphi(t) : [-kh, 0] \to \mathbb{R}^n$ ($k = \max\{k_a, k_b, k_c\}$) where h represents the basic commensurate delay, and it is assumed that system (1) admits a unique smooth solution.

In order to simplify the analysis, let us introduce the delay operator δ (see Williams and Zakian (1977)) such that $x(t-kh) = \delta^k x(t)$, $k \geq 0$. Let $\mathbb{R}[\delta]$ be the polynomial ring of δ over the field $\mathbb{R}$. After having introduced the delay operator δ, system (1) can be written as follows:

$$\begin{aligned} \dot{x}(t) &= \tilde{A}(\delta)x(t) + \tilde{B}(\delta)u(t) \\ y(t) &= \tilde{C}(\delta)x(t) \end{aligned} \tag{2}$$

where $\tilde{A}(\delta) = \sum_{i=0}^{k_a} \bar{A}_i\delta^i$, $\tilde{B}(\delta) = \sum_{i=0}^{k_b} \bar{B}_i\delta^i$ and $\tilde{C}(\delta) = \sum_{i=0}^{k_c} \bar{C}_i\delta^i$.

As for $x(t, \varphi, u)$, we mean the solution of system (2) with the initial condition equal to $\varphi(t)$ and the input equal to u. In the same way, we define $y(t, \varphi, u) = \tilde{C}(\delta)x(t, \varphi, u)$, which is the output of system (1) when $x(t) = x(t, \varphi, u)$. When treating with the observer design for time-delay systems, it is desired to use actual and past information of measurement. For this, let us firstly recall the definition of backward observability introduced in Bejarano and Zheng (2014).

Definition 1 *Bejarano and Zheng (2014) System (1) is said to be backward observable on* $[t_1, t_2]$ *if and only if for each* $\tau \in [t_1, t_2]$*, there exist* $t_1' < t_2' \le \tau$ *such that for every initial condition* φ*:*

$$y(t, \varphi, 0) = 0, \forall t \in [t_1\prime, t_2\prime] \text{ implies } x(\tau, \varphi, 0) = 0$$

For the general linear system (2) with commensurate delays which can appear both in the state and in the output, we are interested in designing a finite-time observer by using only actual and past values of the measurement.

For system (2), Hou et al. (2002) has proposed a simple Luenberger-like observer, and sufficient conditions were deduced to calculate the gains of the proposed observer. This method was extended to treat unknown input case by Zheng et al. (2015). This paper will show how to adapt the results of Hou et al. (2002) and Zheng et al. (2015) to design a finite-time observe for (2) by using homogeneous technique.

3 Definitions and assumptions

Since system (2) is described by the polynomial matrices over $\mathbb{R}[\delta]$, therefore let us firstly give some useful definitions of unimodularity and the change of coordinates over the polynomial ring $\mathbb{R}[\delta]$.

Definition 2 *For a given polynomial matrix* $A(\delta) \in \mathbb{R}^{n\times q}[\delta]$*, it is said to be left (or right) unimodular over* $\mathbb{R}[\delta]$ *if there exists* $A_L^{-1}(\delta) \in \mathbb{R}^{q\times n}[\delta]$ *with* $n \ge q$ *(or* $A_R^{-1}(\delta) \in \mathbb{R}^{n\times q}[\delta]$ *with* $n \le q$*), such that* $A_L^{-1}(\delta)A(\delta) = I_q$ *(or* $A_R^{-1}(\delta)A(\delta) = I_n$*). A square matrix* $A(\delta) \in \mathbb{R}^{n\times n}[\delta]$ *is said to be unimodular over* $\mathbb{R}[\delta]$ *if* $A_L^{-1}(\delta) = A_R^{-1}(\delta)$*.*

Definition 3 *For* $x(t) \in \mathbb{R}^n$ *defined in (1),* $z(t) = T(\delta)x(t)$ *with* $T(\delta) \in \mathbb{R}^{n_z\times n}[\delta]$ *and* $n_z \ge n$ *is said to be a causal generalized change of coordinates over* $\mathbb{R}[\delta]$ *if* $\text{rank}_{\mathbb{R}[\delta]}T(\delta) = n$*. Moreover, it is said to be a bicausal generalized change of coordinates over* $\mathbb{R}[\delta]$ *if* $T(\delta)$ *is left unimodular over* $\mathbb{R}[\delta]$*.*

Let us recall that for any polynomial matrix $D(\delta) \in \mathbb{R}^{p\times m}[\delta]$ with $\text{rank}_{\mathbb{R}[\delta]}D(\delta) = k \le \min\{p, m\}$, it was shown in Hou et al. (2002) that $D(\delta) \in \mathbb{R}^{p\times m}[\delta]$ is left unimodular over $\mathbb{R}[\delta]$ if and only if $\text{rank}_{\mathbb{R}[\delta]}D(\delta) = m \le p$ and $\text{Inv}_S[D(\delta)] \subset \mathbb{R}$.

Then, we can define the following polynomial matrix over $\mathbb{R}[\delta]$:

$$\mathcal{O}_l(\delta) = \begin{bmatrix} \tilde{C}(\delta) \\ \tilde{C}(\delta)\tilde{A}(\delta) \\ \vdots \\ \tilde{C}(\delta)\tilde{A}(\delta)^{l-1} \end{bmatrix} \in \mathbb{R}^{pl\times n}[\delta] \tag{3}$$

where $l \in \mathbb{N}$, and make the following assumption.

Assumption 1 *It is assumed that there exists a least integer* $l^* \in \mathbb{N}$ *such that* $\mathcal{O}_{l^*}(\delta)$ *defined in (3) is left unimodular over* $\mathbb{R}[\delta]$*, i.e.* $\text{rank}_{\mathbb{R}[\delta]}\mathcal{O}_{l^*}(\delta) = n$ *and* $\text{Inv}_S\mathcal{O}_{l^*}(\delta) \subset \mathbb{R}$*.*

Remark 1 *If no delays are involved in the matrices* $\tilde{A}(\delta)$ *and* $\tilde{C}(\delta)$*, i.e.* $\tilde{A}(\delta) = \tilde{A}$ *and* $\tilde{C}(\delta) = \tilde{C}$*, then Assumption 1 is equivalent to require that there exists an integer* l^* *such that*

$$\text{rank} \begin{bmatrix} \tilde{C} \\ \tilde{C}\tilde{A} \\ \vdots \\ \tilde{C}\tilde{A}^{l^*-1} \end{bmatrix} = n$$

Due to Cayley-Hamilton Theorem, system (2) without delays is observable if and only if this rank condition is satisfied with $l^* = n$*. In this sense, Assumption 1 can be seen as an extension of the classical Kalman rank condition for linear time-invariant system to treat linear system with commensurate delays.*

It is clear to see that if Assumption 1 is satisfied, then system (2) is backward observable (see Hou et al. (2002)). In other words, the above assumption guarantees that the studied system is backward observable.

Definition 4 *The dynamics $\dot{\hat{x}} = f(\hat{x}, \delta, y, u)$ with $\hat{x} \in \mathbb{R}^n$ and the user-chosen function f, is an asymptotic observer of (2) if*

$$\lim_{t\to\infty} ||\hat{x}(t) - x(t)|| = 0.$$

It is said to be a finite-time observer if there exists a finite-time $T_s > 0$, such that

$$||\hat{x}(t) - x(t)|| = 0, \forall t \geq T_s.$$

The following will present how to design a finite-time observer to estimate the vector $x(t)$ of (2) in Section 4. An extension to treat unknown input case will be discussed in Section 5 where both finite-time estimations for $x(t)$ and the unknown inputs will be investigated.

4 Finite-time observer

4.1 Transformation into blocks

Before presenting the finite-time observer, let us recall one useful result stated in Hou et al. (2002).

Lemma 1 *Hou et al. (2002) There exists a bicausal generalized change of coordinates $z = \tilde{T}(\delta)x$ which transforms $\tilde{A}(\delta)$ and $\tilde{C}(\delta)$ of (2) into the following form:*

$$\begin{array}{l}\tilde{C}(\delta)\tilde{T}_L^{-1}(\delta) = \tilde{C}_0 \\ \tilde{T}(\delta)\tilde{A}(\delta)\tilde{T}_L^{-1}(\delta) = \tilde{A}_0 + \tilde{F}(\delta)\tilde{C}_0\end{array} \tag{4}$$

where the matrices $\tilde{F}(\delta) = [\tilde{F}_1^T(\delta), \ldots, \tilde{F}_{l^*}^T(\delta)]$, and

$$\begin{array}{l}\tilde{A}_0 = \begin{bmatrix} 0 & I_p & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I_p \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{pl^* \times pl^*} \\ \tilde{C}_0 = \begin{bmatrix} I_p & 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{p \times pl^*}\end{array} \tag{5}$$

if and only if there exists a least integer $l^* \in \mathbb{N}$ such that $\mathcal{O}_{l^*}(\delta)$ is left unimodular over $\mathbb{R}[\delta]$. Moreover, the bicausal generalized change of coordinates $z = \tilde{T}(\delta)x$ with $\tilde{T}(\delta) = \mathrm{col}\{\tilde{T}_1(\delta), \ldots, \tilde{T}_{l^*}(\delta)\}$ is defined as follows:

$$\left\{\begin{array}{l}\tilde{T}_1(\delta) = \tilde{C}(\delta) \\ \tilde{T}_{i+1}(\delta) = \tilde{T}_i(\delta)\tilde{A}(\delta) - \tilde{F}_i(\delta)\tilde{C}(\delta), \ \textit{for } 1 \leq i \leq l^* - 1\end{array}\right. \tag{6}$$

with $\tilde{F}_i(\delta)$ being determined through the following equation:

$$[\tilde{F}_{l^*}(\delta), \ldots, \tilde{F}_1(\delta)] = \tilde{C}(\delta)\tilde{A}^{l^*}(\delta)[\mathcal{O}_{l^*}(\delta)]_L^{-1}. \tag{7}$$

Let us remark that if Assumption 1 is satisfied, then there always exists a least integer $l^* \leq n$ such that $\mathcal{O}_{l^*}(\delta)$ is left unimodular over $\mathbb{R}[\delta]$. In other words, if Assumption 1 is satisfied, then $\tilde{A}(\delta)$ and $\tilde{C}(\delta)$ of system (2) can be transformed into (4) via $\tilde{T}(\delta)$ defined in (7). This result enables us to state the following theorem.

Theorem 1 *If Assumption 1 is satisfied, then there exists a change of coordinates $z = T(\delta)x$ such that system (2) can be transformed into the following system with blocks:*

$$\dot{z}_i(t) = A_i z(t) + F_i(\delta)y + B_i(\delta)u(t), \ \textit{for } 1 \leq i \leq p \tag{8}$$

$$y_i(t) = C_i z_i = z_{i,1} \tag{9}$$

where $z_i = \text{col}\{z_{i,1}, \ldots, z_{i,l^*}\} \in \mathbb{R}^{l^*}$,

$$\begin{aligned} C_i &= [1, 0 \cdots, 0] \in \mathbb{R}^{l^*} \\ A_i &= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{l^* \times l^*} \end{aligned} \tag{10}$$

$F_i(\delta)$ *and* $B_i(\delta)$ *are defined in* (12).

Proof. According to Lemma 1, if Assumption 1 is satisfied, then by introducing $\xi = \tilde{T}(\delta)x$ where $\tilde{T}(\delta)$ is defined in (6), we get

$$\begin{aligned} \dot{\xi}(t) &= \tilde{T}(\delta)\tilde{A}(\delta)\tilde{T}_L^{-1}(\delta)\xi(t) + \tilde{T}(\delta)\tilde{B}(\delta)u(t) \\ y(t) &= \tilde{C}(\delta)\tilde{T}_L^{-1}(\delta)\xi \end{aligned}$$

Based on the equality given in (4), the above equations become

$$\begin{aligned} \dot{\xi}(t) &= \tilde{A}_0\xi(t) + \tilde{F}(\delta)y + \tilde{T}(\delta)\tilde{B}(\delta)u(t) \\ y(t) &= \tilde{C}_0\xi \end{aligned}$$

where $\tilde{A}_0$, $\tilde{C}_0$ and $\tilde{F}(\delta)$ are given in (5) and (7), respectively.

Denote Q as the elementary matrix $Q = \text{col}\{Q_1, \ldots, Q_p\}$ with

$$Q_i = \begin{bmatrix} 0 & \cdots & 0 & \underset{i\text{th}}{1} & 0\cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0\cdots & 0 & \underset{(i+p)\text{th}}{1} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & \cdots & 0 & 0 & 0\cdots & 0 & 0 & 0 & \cdots & 0 & \underset{(i+(l^*-1)p)\text{th}}{1} \end{bmatrix} \in \mathbb{R}^{l^* \times pl^*} \tag{11}$$

for $1 \leq i \leq p$. Introducing the second transformation $z = \text{col}\{z_1, \ldots, z_p\} = Q\xi$ with

$$z_i = \begin{bmatrix} z_{i,1} \\ z_{i,2} \\ \vdots \\ z_{i,l^*} \end{bmatrix} = Q_i\xi = \begin{bmatrix} \xi_i \\ \xi_{i+p} \\ \vdots \\ \xi_{i+(l^*-1)p} \end{bmatrix}$$

then a straightforward calculation yields

$$\begin{aligned} Q\tilde{A}_0Q^{-1} &= \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_p \end{bmatrix} \text{ with } A_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \\ \tilde{C}_0Q^{-1} &= \begin{bmatrix} C_1 & 0 & \cdots & 0 \\ 0 & C_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C_p \end{bmatrix} \text{ with } C_i = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \end{aligned}$$

Finally, with the transformation $z = T(\delta)x$ where $T(\delta) = Q\tilde{T}(\delta)$, system (2) can be transformed into (9) with

$$\begin{aligned} F(\delta) &= Q_i\tilde{F}(\delta) \\ B(\delta) &= Q_i\tilde{B}(\delta) \end{aligned} \tag{12}$$

■

4.2 Observer design

Based on the result stated in Theorem 1, with the change of coordinates $z = T(\delta)x$, system (2) can be transformed into (9), which can be written into the following compact form:

$$\begin{aligned} \dot{z} &= Az + F(\delta)y + B(\delta)u \\ y &= Cz \end{aligned} \tag{13}$$

with $A = \text{diag}\{A_1, \cdots, A_p\}$, $C = \text{diag}\{C_1, \cdots, C_p\}$, $F(\delta) = \text{col}\{F_1(\delta), \cdots, F_p(\delta)\}$ and

$$B(\delta) = \text{col}\{B_1(\delta), \cdots, B_p(\delta)\}$$

For the transformed system with compact form, let us consider the following dynamics:

$$\begin{aligned} \dot{\hat{z}} &= A\hat{z} + F(\delta)y + B(\delta)u + K\lceil y - C(\delta)\hat{z}\rfloor^{\alpha} \\ \hat{x} &= T_L^{-1}(\delta)\hat{z} \end{aligned} \tag{14}$$

where $K \in \mathbb{R}^{pl^* \times pl^*}$ is a constant matrix defined as:

$$K = \text{diag}\{K_1, \cdots, K_j, \cdots, K_p\} \tag{15}$$

with

$$K_j = \text{diag}\{k_{j,1}, \cdots, k_{j,l^*}\}$$

where $k_{i,j}$ for $1 \leq i \leq p$ and $1 \leq j \leq l^*$ are chosen such that the matrix $(A_i - K_i C_i)$ is Hurwitz.

$$\lceil y - C(\delta)\hat{z}\rfloor^{\alpha} = \begin{bmatrix} |y_1 - C_1(\delta)\hat{z}|^{\alpha}\text{sgn}(y_1 - C_1(\delta)\hat{z}) \\ \vdots \\ |y_p - C_p(\delta)\hat{z}|^{\alpha}\text{sgn}(y_p - C_p(\delta)\hat{z}) \end{bmatrix} \tag{16}$$

and for $1 \leq i \leq p$

$$|y_i - C_i(\delta)\hat{z}|^{\alpha}\text{sgn}(y_i - C_i(\delta)\hat{z}) = \begin{bmatrix} |y_i - C_i(\delta)\hat{z}|^{\alpha_1}\text{sgn}(y_i - C_i(\delta)\hat{z}) \\ \vdots \\ |y_i - C_i(\delta)\hat{z}|^{\alpha_{l^*}}\text{sgn}(y_i - C_i(\delta)\hat{z}) \end{bmatrix}$$

where α_j for $1 \leq j \leq l^*$ are defined as those in Perruquetti et al. (2008):

$$\alpha_j = j\beta - (j-1) \tag{17}$$

with $\beta \in (1 - \frac{1}{l^*}, 1)$. Before proving the finite-time convergence, let us recall the following result stated in Perruquetti et al. (2008) for homogeneous observer:

Lemma 2 *Perruquetti et al. (2008) The following dynamics*

$$\begin{bmatrix} \dot{\xi}_1 \\ \vdots \\ \dot{\xi}_{l^*-1} \\ \dot{\xi}_{l^*} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_{l^*-1} \\ \xi_{l^*} \end{bmatrix} - \begin{bmatrix} l_1|\xi_1|^{\alpha_1}\text{sgn}(\xi_1) \\ \vdots \\ l_{n-1}|\xi_1|^{\alpha_{l^*-1}}\text{sgn}(\xi_1) \\ l_n|\xi_1|^{\alpha_{l^*}}\text{sgn}(\xi_1) \end{bmatrix}$$

with α_j for $1 \leq j \leq l^*$ being defined in (17), is finite-time stable, if l_j for $1 \leq j \leq l^*$ are chosen such that $\begin{bmatrix} -l_1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -l_{l^*-1} & 0 & 0 & \cdots & 1 \\ -l_{l^*} & 0 & 0 & \cdots & 0 \end{bmatrix}$ *is Hurwitz.*

Remark 2 *It is worth noting that the convergence of the proposed finite-time observer (14) depends on the result stated in Lemma 2. As stated in Perruquetti et al. (2008), unlike classical finite-time sliding mode observer which relies on discontinuous output injections and on a step-by-step procedure (might be harmful for high order systems), the homogeneous observer is based on continuous output injections, and no chattering phenomenon will happen.*

Theorem 2 *If Assumption 1 is satisfied, then the dynamics (14) is a finite-time observer of system (2), i.e., there exists a finite time $T_{s_1} \in (0, +\infty)$ such that $||\hat{x}(t) - x(t)|| = 0$ for all $t \geq T_{s_1}$.*

Proof. Denote $e = x - \hat{x}$ and $\varepsilon = z - \hat{z}$, then according to system (2) and (14), their observation error dynamics can be written as:

$$\dot{\varepsilon} \;= A\varepsilon - K\lceil C\varepsilon \rfloor^{\alpha}$$

whose ith block ε_i can be written as:

$$\begin{bmatrix} \dot{\varepsilon}_{i,1} \\ \vdots \\ \dot{\varepsilon}_{i,l^*-1} \\ \dot{\varepsilon}_{i,l^*} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_{i,1} \\ \vdots \\ \varepsilon_{i,l^*-1} \\ \varepsilon_{i,l^*} \end{bmatrix} - \begin{bmatrix} k_{i,1}|\varepsilon_{i,1}|^{\alpha_1}\mathrm{sgn}(\varepsilon_{i,1}) \\ \vdots \\ k_{i,l^*-1}|\varepsilon_{i,1}|^{\alpha_{l^*-1}}\mathrm{sgn}(\varepsilon_{i,1}) \\ k_{i,l^*}|\varepsilon_{i,1}|^{\alpha_{l^*}}\mathrm{sgn}(\varepsilon_{i,1}) \end{bmatrix}$$

By choosing a positive constant $\beta \in (1 - \frac{1}{l^*}, 1)$, according to Lemma 2, if

$$\alpha_i = i\beta - (i-1), \text{ for } 1 \leq i \leq l^*$$

and $k_{i,j}$ are chosen such that the following matrix: $\begin{bmatrix} -k_{i,1} & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -k_{i,l^*-1} & 0 & 0 & \cdots & 1 \\ -k_{i,l^*} & 0 & 0 & \cdots & 0 \end{bmatrix}$ is Hurwitz, then ε_i is finite-time stable for $1 \leq i \leq p$. Therefore, there exists a finite time $T_{s_1} \in (0, +\infty)$ such that

$$||\varepsilon(t)|| = 0, \forall t \geq T_{s_1}$$

Note $d_\delta = \deg\left[T_L^{-1}(\delta)\right]$ as the highest polynomial order of δ in $T_L^{-1}(\delta)$, since $e = T_L^{-1}(\delta)\varepsilon = \tilde{T}_L^{-1}(\delta)Q^{-1}\varepsilon$, thus we have

$$||e(t)|| = 0, \forall t \geq T_{s_1} + d_\delta \times h$$

where h represents the basic commensurate delay given in (1). Therefore, we proved that system (14) is a finite-time observer of system (2). ■

5 Extension to systems with unknown inputs

In this section, we extend the previous result to deal with linear time-delay systems with unknown inputs. Firstly, a change of coordinates will be applied to transform the studied system into a new form which does not depend on the unknown inputs, but depend on the derivative of the output. Then the same technique is used to design a finite-time observer to estimate both $x(t)$ and the unknown input.

5.1 Assumptions and transformation

Consider now the following class of linear time-delay systems with unknown inputs:

$$\begin{aligned} \dot{x}(t) &= \textstyle\sum_{i=0}^{k_a} \bar{A}_i x(t-ih) + \sum_{i=0}^{k_b} \bar{B}_i u(t-ih) + \sum_{i=0}^{k_e} \bar{E}_i w(t-ih) \\ y(t) &= \textstyle\sum_{i=0}^{k_c} \bar{C}_i x(t-ih) \end{aligned} \tag{18}$$

where $x(t) \in \mathbb{R}^n$, the known input $u(t) \in \mathbb{R}^m$, and the output vector $y(t) \in \mathbb{R}^p$ are defined as those in (1), $w(t) \in \mathbb{R}^q$ represents the unknown inputs (such as disturbances for example). By introducing the delay operator δ as before, system (18) can be then written as follows:

$$\begin{aligned} \dot{x}(t) &= \tilde{A}(\delta)x(t) + \tilde{B}(\delta)u(t) + \tilde{E}(\delta)w(t) \\ y(t) &= \tilde{C}(\delta)x(t) \end{aligned} \tag{19}$$

where $\tilde{A}(\delta) = \sum_{i=0}^{k_a} \bar{A}_i\delta^i$, $\tilde{B}(\delta) = \sum_{i=0}^{k_b} \bar{B}_i\delta^i$, $\tilde{C}(\delta) = \sum_{i=0}^{k_c} \bar{C}_i\delta^i$ and $\tilde{E}(\delta) = \sum_{i=0}^{k_e} \bar{E}_i\delta^i$.

When considering unknown input $w(t)$, the backward observability requires that Definition 1 is satisfied for all unknown input $w(t)$. Interested reader can refer to Definition 2 in Bejarano and Zheng (2014) for the details. Compared to system (1) without unknown input, the backward observation of $x(t)$ in (19) cannot be judged simply by Assumption 1. When $w(t)$ is taken into account, as it was described in Bejarano and Zheng (2014), Silverman and Monlinari algorithms (see Silverman (1969); Molinari (1976)) can be used to test the backward observability of $x(t)$. For this, let us define $N_k(\delta)$ as the matrices generated by the following algorithm:

- Initialization: $\Delta_1 \triangleq 0$, $H_1 \triangleq 0$, $G_1(\delta) \triangleq \tilde{C}(\delta)$, $N_1(\delta) \triangleq \Delta_1$;
- Iteration: For $k \geq 1$, calculate the unimodular matrix $P_k(\delta)$ over $\mathbb{R}[\delta]$ such that it transforms $\Delta_k(\delta)\tilde{E}(\delta)$ into its Hermite form: $P_k(\delta)\Delta_k(\delta)\tilde{E}(\delta) = \begin{bmatrix} H_{k+1}(\delta) \\ 0 \end{bmatrix}$, then we obtain

$$\begin{bmatrix} H_{k+1}(\delta) & G_{k+1}(\delta) \\ 0 & \Delta_{k+1}(\delta) \end{bmatrix} = P_k(\delta) \begin{bmatrix} \Delta_k(\delta)\tilde{E}(\delta) & \Delta_k(\delta)\tilde{A}(\delta) \\ H_k(\delta) & G_k(\delta) \end{bmatrix}$$

and note $N_{k+1}(\delta) \triangleq \begin{bmatrix} N_k(\delta) \\ \Delta_{k+1}(\delta) \end{bmatrix}$.

When applying the above algorithm to system (19), it has been proven in Bejarano and Zheng (2014) that there always exists a least integer $k^* \in \mathbb{N}$, independent of the choice of $P_k(\delta)$, such that $\text{Inv}_S N_{k^*+1}(\delta) = \text{Inv}_S N_{k^*}(\delta)$. Moreover, it was proven as well $x(t)$ of system (19) is backward observable if $\text{rank}_{\mathbb{R}[\delta]} N_{k^*}(\delta) = n$ and $\text{Inv}_S N_{k^*}(\delta) \subset \mathbb{R}$. Based on this result, instead of Assumption 1 for the case without unknown input, the following assumption was imposed for the case with unknown input $w(t)$.

Assumption 2 *It is assumed that there exists a least integer $k^* \in \mathbb{N}$ for system (19) such that $\text{rank}_{\mathbb{R}[\delta]} N_{k^*}(\delta) = n$ and $\text{Inv}_S N_{k^*}(\delta) \subset \mathbb{R}$.*

Remark 3 *This assumption guarantees that $x(t)$ of system (19) is backward observable, thus we can design an observer to estimate it. Moreover, it can be easily shown that Assumption 2 is equivalent to Assumption 1 if no unknown input exists.*

Before proposing the finite-time observer, we need as well the following assumption.

Assumption 3 *For the polynomial matrices $\tilde{E}(\delta)$ and $\tilde{C}(\delta)$ defined in system (19), it is assumed that*

$$\text{Inv}_S \begin{bmatrix} \tilde{C}(\delta)\tilde{E}(\delta) \\ \tilde{E}(\delta) \end{bmatrix} = \text{Inv}_S \begin{bmatrix} \tilde{C}(\delta)\tilde{E}(\delta) \end{bmatrix} \tag{20}$$

Remark 4 *Assumption 3 is not restrictive since it coincides with the well-known matching condition*

$$\text{rank}\left[\tilde{C}\tilde{E}\right] = \text{rank}\tilde{E}$$

if no delays are involved in $\tilde{C}(\delta)$ and $\tilde{E}(\delta)$.

It has been shown in Langueh et al. (2016) that, if Assumptions 2 and 3 are satisfied, then there exists a polynomial matrix $\tilde{L}(\delta)$ such that $\tilde{E}(\delta) = \tilde{L}(\delta)\tilde{C}(\delta)\tilde{E}(\delta)$. Assumption 3 is crucial since it enables us to transform system (19) into a simple form. For this, let us calculate the derivative of y of (19):

$$\dot{y} = \tilde{C}(\delta)\tilde{A}(\delta)x(t) + \tilde{C}(\delta)\tilde{B}(\delta)u(t) + \tilde{C}(\delta)\tilde{E}(\delta)w(t)$$

Thus, if Assumption 3 is satisfied, which implies the existence of $\tilde{L}(\delta)$ such that $\tilde{E}(\delta) = \tilde{L}(\delta)\tilde{C}(\delta)\tilde{E}(\delta)$, then multiplying the above equation by $\tilde{L}(\delta)$ gives

$$\tilde{L}(\delta)\dot{y} = \tilde{L}(\delta)\tilde{C}(\delta)\tilde{A}(\delta)x(t) + \tilde{L}(\delta)\tilde{C}(\delta)\tilde{B}(\delta)u(t) + \tilde{E}(\delta)w(t)$$

Substituting the above equation back into (19) yields:

$$\begin{aligned} \dot{x}(t) &= \left[I - \tilde{L}(\delta)\tilde{C}(\delta)\right]\tilde{A}(\delta)x(t) + \left[I - \tilde{L}(\delta)\tilde{C}(\delta)\right]\tilde{B}(\delta)u(t) + \tilde{L}(\delta)\dot{y} \\ y(t) &= \tilde{C}(\delta)x(t) \end{aligned} \tag{21}$$

or equivalently, by noting $S(\delta) = I - \tilde{L}(\delta)\tilde{C}(\delta)$, we have

$$\begin{aligned} \dot{x}(t) &= S(\delta)\tilde{A}(\delta)x(t) + S(\delta)\tilde{B}(\delta)u(t) + \tilde{L}(\delta)\dot{y} \\ y(t) &= \tilde{C}(\delta)x(t) \end{aligned} \tag{22}$$

Note that the deduced system (22) now does not depend on the unknown input $w(t)$, and it has been proven in Langueh et al. (2016) that if Assumptions 2 and 3 are both satisfied, then there exists a least integer $l^* \in \mathbb{N}$ such that

$$\mathcal{O}_{w,l^*}(\delta) = \begin{bmatrix} \tilde{C}(\delta) \\ \tilde{C}(\delta)S(\delta)\tilde{A}(\delta) \\ \vdots \\ \tilde{C}(\delta)\left[S(\delta)\tilde{A}(\delta)\right]^{l^*-1} \end{bmatrix} \tag{23}$$

is left unimodular over $\mathbb{R}[\delta]$.

Consequently, for the deduced system (22) which is independent of unknown input $w(t)$, we can reuse the result presented in Section 4 to design a finite-time observer for the purpose of estimating $x(t)$ and $w(t)$. More precisely, replacing $\tilde{A}(\delta)$ and $\tilde{C}(\delta)$ by $S(\delta)\tilde{A}(\delta)$ and $S(\delta)\tilde{C}(\delta)$, then following the procedure presented in Section 4.1 yields the associated matrices A, C, $F(\delta)$ and a change of coordinates $z = T(\delta)x$ such that system (22) can be transformed into

$$\begin{aligned} \dot{z} &= Az + F(\delta)y + B(\delta)u + L(\delta)\dot{y} \\ y &= Cz \end{aligned} \tag{24}$$

with $A = \text{diag}\{A_1, \cdots, A_p\}$, $C = \text{diag}\{C_1, \cdots, C_p\}$, with A_i and C_i being defined in (10), $B(\delta) = T(\delta)S(\delta)\tilde{B}(\delta)$ and $L(\delta) = T(\delta)\tilde{L}(\delta)$.

5.2 Finite-time estimation of $x(t)$

Compared to (13), the transformed system (22) contains the derivative of y, therefore the structure of the finite-time observer needs to be adapted as follows:

$$\begin{aligned} \dot{\zeta} &= M(\delta)\zeta + J(\delta)y + \Omega(\delta)u + K\lceil y - C(\delta)\hat{z}\rfloor^{\alpha} \\ \hat{z} &= \zeta + L(\delta)y \\ \hat{x} &= T_L^{-1}(\delta)\hat{z} \end{aligned} \tag{25}$$

where $K \in \mathbb{R}^{pl^* \times pl^*}$ is a constant matrix defined in (15), and the symbol $\lceil y - C(\delta)\hat{x}\rfloor^{\alpha}$ is defined in (16). The matrices $M(\delta)$, $J(\delta)$ and $\Omega(\delta)$ need to be determined to guarantee that $\hat{x}(t)$ will converge to $x(t)$ in the finite-time.

Theorem 3 *Suppose that Assumptions 2 and 3 are satisfied for system (19). Then system (25) is a finite-time observer of system (19) if*

$$\begin{aligned} M(\delta) &= A \\ \Omega(\delta) &= B(\delta) \\ J(\delta) &= F(\delta) + M(\delta)L(\delta) \end{aligned} \tag{26}$$

where A, $B(\delta)$, $F(\delta)$ and $L(\delta)$ are given in (24), i.e., there exists a finite time $T_{s_1} \in (0, +\infty)$ such that $||\hat{x}(t) - x(t)|| = 0$ for all $t \geq T_{s_1}$.

Proof. The proof of this theorem is quite similar to that of Theorem 2. Denote $e = x - \hat{x}$ and $\varepsilon = z - \hat{z}$, then according to (19) and (25), the observation error dynamics can be written as:

$$\begin{aligned}\dot{\varepsilon} = &\ Az + F(\delta)y + B(\delta)u + L(\delta)\dot{y} - \dot{\zeta} - L(\delta)\dot{y} \\ = &\ Az + F(\delta)y + B(\delta)u - M(\delta)(\hat{z} - L(\delta)y) - J(\delta)y - \Omega(\delta)u - K\lceil y - C(\delta)\hat{z}\rfloor^{\alpha} \\ = &\ Az - M(\delta)\hat{z} + [F(\delta) + M(\delta)L(\delta) - J(\delta)]\,y + [B(\delta) - \Omega(\delta)]\,u - K\lceil y - C(\delta)\hat{z}\rfloor^{\alpha}\end{aligned}$$

So, if we choose the matrices $M(\delta)$, $J(\delta)$ and $\Omega(\delta)$ as those defined in (26), then the above equation becomes

$$\dot{\varepsilon} = \ A\varepsilon - K\lceil C\varepsilon\rfloor^{\alpha}$$

Following the same arguments used in the proof of Theorem 2, we can then conclude that system (25) is a finite-time observer of system (19). ■

5.3 Finite-time estimation of $w(t)$

With the finite-time estimation of $x(t)$ by the dynamics (25), the possibility to estimate the unknown input $w(t)$ can be guaranteed by the following result.

Theorem 4 *Suppose that Assumptions 2 and 3 are both satisfied for system (19). Then the unknown input $w(t)$ is finite-time estimable if $\tilde{E}(\delta)$ is left unimodular over $\mathbb{R}[\delta]$.*

Proof. From (19), it is obvious that if $\tilde{E}(\delta)$ is left unimodular over $\mathbb{R}[\delta]$, then there exists $\tilde{E}_L^{-1}(\delta)$ with $\tilde{E}_L^{-1}(\delta)\tilde{E}(\delta) = I$ such that

$$w(t) = \tilde{E}_L^{-1}(\delta)\dot{x}\,(t) - \tilde{E}_L^{-1}(\delta)\tilde{A}\,(\delta)\,x\,(t) - \tilde{E}_L^{-1}(\delta)\tilde{B}\,(\delta)\,u\,(t)$$

According to Theorem 3, $\hat{x}(t)$ defined in (25) converges in finite-time to $x(t)$ if Assumptions 2 and 3 are both satisfied. Thus, the derivative of $\hat{x}$, noted as $\dot{\hat{x}}$, will converge as well in finite-time to $\dot{x}$. In such a situation, $w(t)$ can be estimated in finite-time by the following equation:

$$\hat{w}(t) = \tilde{E}_L^{-1}(\delta)\dot{\hat{x}}\,(t) - \tilde{E}_L^{-1}(\delta)\tilde{A}\,(\delta)\,\hat{x}\,(t) - \tilde{E}_L^{-1}(\delta)\tilde{B}\,(\delta)\,u\,(t) \tag{27}$$

■

From Theorem 4, it is clear that the estimation of $w(t)$ depends on the computation of the derivative of $\hat{x}$, which in fact can be realized by different types of differentiators, such as algebraic differentiator Mboup et al. (2007), high order sliding mode differentiator/observer Levant (2003), or homogeneous differentiator/observer Perruquetti et al. (2008). In the following, we adopt the homogeneous observer to realize the computation of $\dot{\hat{x}}_i$:

$$\begin{cases} \dot{\eta}_{i,1} = \eta_{i,2} - \lambda_2\lceil \eta_{i,1} - \hat{x}_i\rfloor^{\bar{\alpha}_1} \\ \dot{\eta}_{i,2} = -\lambda_1\lceil \eta_{i,1} - \hat{x}_i\rfloor^{\bar{\alpha}_2} \end{cases} \tag{28}$$

for $1 \le i \le n$, where $\bar{\alpha}_1 = \bar{\beta}$ and $\bar{\alpha}_2 = 2\bar{\beta} - 1$ with $\bar{\beta} \in (\frac{1}{2}, 1)$, λ_1 and λ_2 are chosen such that $\begin{bmatrix} -\lambda_2 & 1 \\ -\lambda_1 & 0 \end{bmatrix}$ is Hurwitz.

Based on the definition of $\hat{x}(t)$ in (25) and on the assumption that $x(t)$ is smooth, there exists a finite time T_{s_2} such that for all $t \ge T_{s_2}$ we have $\eta_{i,1}\,(t) = \hat{x}_i\,(t)$ and $\eta_{i,2}(t) = \dot{\hat{x}}_i\,(t)$ for $1 \le i \le n$.

By noting

$$\eta_{:,1} = [\eta_{1,1}, \cdots, \eta_{n,1}]^T$$

and

$$\eta_{:,2} = [\eta_{1,2}, \cdots, \eta_{n,2}]^T$$

finally the proposed observer is the combination of (25) and (28) as follows:

$$\begin{cases} \dot{\zeta} = M(\delta)\zeta + J(\delta)y + \Omega(\delta)u + K\lceil y - C(\delta)\hat{z}\rfloor^{\alpha} \\ \hat{z} = \zeta + L(\delta)y \\ \hat{x} = T_L^{-1}(\delta)\hat{z} \\ \dot{\eta}_{i,1} = \eta_{i,2} - \lambda_2\lceil \eta_{i,1} - \hat{x}_i\rfloor^{\bar{\alpha}_1} \\ \dot{\eta}_{i,2} = -\lambda_1\lceil \eta_{i,1} - \hat{x}_i\rfloor^{\bar{\alpha}_2} \\ \hat{w} = \tilde{E}_L^{-1}(\delta)\eta_{:,2} - \tilde{E}_L^{-1}(\delta)\tilde{A}\,(\delta)\,\eta_{:,1} - \tilde{E}_L^{-1}(\delta)\tilde{B}\,(\delta)\,u \end{cases} \tag{29}$$

Lemma 3 *Suppose that Assumptions 2 and 3 are satisfied for system (19) and $\tilde{E}(\delta)$ is left unimodular over $\mathbb{R}[\delta]$. Then there exists a finite time $T_{s_2} \in (T_{s_1}, +\infty)$ with T_{s_1} being the setting time of finite-time convergence of $\hat{x}$ to $x(t)$, such that*

$$||\hat{w}(t) - w(t)|| = 0, \forall t > T_{s_2}$$

where $\hat{w}(t)$ is defined in (29).

Proof. According to Theorem 3, if Assumptions 2 and 3 are both satisfied for system (19), then we have

$$||\hat{x}(t) - x(t)|| = 0, \forall t \geq T_{s_1}$$

Due to the property of homogeneous observer, there exists T_{s_2} with $T_{s_2} > T_{s_1}$ such that, for all $t \geq T_{s_2}$, we have $\eta_{i,1}(t) = \hat{x}_i(t)$ and $\eta_{i,2}(t) = \dot{\hat{x}}_i(t)$ for $1 \leq i \leq n$. Therefore, if $\tilde{E}(\delta)$ is left unimodular over $\mathbb{R}[\delta]$, according to (27), we can conclude that $||\hat{w}(t) - w(t)|| = 0$ for all $t > T_{s_2}$. ■

6 Illustrative example

In order to highlight the effectiveness of the proposed finite-time observer, let us consider the following two examples.

6.1 Academic example

Let us consider a linear time-delay systems of the form (19) with the following matrices:

$$\tilde{A}(\delta) = \begin{bmatrix} \delta^2 & 1 & \delta \\ \delta & \delta & 1+\delta \\ 1 & \delta & \delta^2 \end{bmatrix}, \tilde{B}(\delta) = \begin{bmatrix} 1 & \delta \\ 0 & 1 \\ 1+\delta^2 & 1 \end{bmatrix}, \tilde{E}(\delta) = \begin{bmatrix} 1 \\ 1 \\ \delta \end{bmatrix}$$

and

$$\tilde{C}(\delta) = \begin{bmatrix} 1 & 0 & 0 \\ \delta & 1 & 0 \end{bmatrix}$$

It can be checked that there exists $l^* = 2$ such that $\text{rank}_{\mathbb{R}[\delta]} N_2(\delta) = 3$ and $\text{Inv}_S N_2(\delta) = \{1, 1, 1\} \subset \mathbb{R}$. Therefore, Assumption 2 is satisfied, and the studied system is backward observable. In order to design the proposed observer, we need to check whether the condition (20) is satisfied. Since $\tilde{C}(\delta)\tilde{E}(\delta) = \begin{bmatrix} 1 \\ 1+\delta \end{bmatrix}$ and it is clear that

$$\text{Inv}_S \begin{bmatrix} \tilde{C}(\delta)\tilde{E}(\delta) \\ \tilde{E}(\delta) \end{bmatrix} = \text{Inv}_S \begin{bmatrix} \tilde{C}(\delta)\tilde{E}(\delta) \end{bmatrix} = \{1\}$$

thus all conditions of Theorem 3 are satisfied, and we can follow the proposed procedure to design a finite-time observer to estimate $x(t)$.

Also, it is easy to check that $\tilde{E}(\delta)$ is left unimodular over $\mathbb{R}[\delta]$, and we can choose

$$\tilde{E}_L^{-1}(\delta) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \tag{30}$$

Therefore all conditions of Theorem 4 are satisfied and we can estimate the unknown input $w(t)$ in a finite-time.

Following the procedure presented in Sections 5.2 and 5.3, we obtain

$$\tilde{L}(\delta) = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ \delta & 0 \end{bmatrix}$$

such that $\tilde{E}(\delta) = \tilde{L}(\delta)\tilde{C}(\delta)\tilde{E}(\delta)$, thus

$$S(\delta) = \text{I} - L(\delta)C(\delta) = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ -\delta & 0 & 1 \end{bmatrix}$$

With this $S(\delta)$, we get

$$\mathcal{O}_{w,l^*}(\delta) = \begin{bmatrix} \tilde{C}(\delta) \\ \tilde{C}(\delta)S(\delta)\tilde{A}(\delta) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \delta & 1 & 0 \\ 0 & 0 & 0 \\ \delta-\delta^2 & \delta-1 & 1 \end{bmatrix}$$

which is left unimodular over $\mathbb{R}[\delta]$ since $\mathrm{rank}_{\mathbb{R}[\delta]}\mathcal{O}_{w,l^*}(\delta) = 3$ and $\mathrm{Inv}_S\left[\mathcal{O}_{w,l^*}(\delta)\right] = \{1,1,1\} \subset \mathbb{R}$.

Finally, we obtain

$$\tilde{A}_0 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tilde{C}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

and $\tilde{F}(\delta) = \begin{bmatrix} 0 & 0 \\ 0 & \delta-1 \\ 0 & 0 \\ -\delta^3 & 0 \end{bmatrix}$, $\tilde{T}(\delta) = \begin{bmatrix} 1 & 0 & 0 \\ \delta & 1 & 0 \\ 0 & 0 & 0 \\ -2\delta^2 & 0 & 1 \end{bmatrix}$. According to (11), we have $Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ which gives $T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ \delta & 1 & 0 \\ -2\delta^2 & 0 & 1 \end{bmatrix}$ transforming the studied system into the form (24) with

$$\begin{aligned} A &= \mathrm{diag}\{A_1, A_2\}, A_1 = A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ C &= \mathrm{diag}\{C_1, C_2\}, C_1 = C_2 = [1, 0] \end{aligned}$$

and $B(\delta) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ -1 & 1-\delta \\ \delta^2-\delta+1 & 1-\delta^2 \end{bmatrix}$, $F(\delta) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & \delta-1 \\ 1-\delta^3 & 0 \end{bmatrix}$, $L(\delta) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ \delta+1 & 0 \\ -2\delta^2+3\delta & 0 \end{bmatrix}$.

Then we can choose a matrix K such that $(A-KC)$ is Hurwitz. In this example, we choose $K = \begin{bmatrix} 12 & 0 \\ 32 & 0 \\ 0 & 12 \\ 0 & 32 \end{bmatrix}$ which assigns the eigenvalues of $(A-KC)$ as $[-8,-4,-8,-4]$. With the deduced K and $T_L^{-1}(\delta)$ and the chosen $\tilde{E}(\delta)$ in (30), we can then design a finite-time observer of form (29) to estimate $x(t)$ and the unknown input $w(t)$.

For the simulation setting, the sampling step is set to be $0.001s$ and the basic delay $h = 0.1s$. In order to show the effectiveness of the proposed finite-time observer, we compared our results with respect to that of Zheng et al. (2015) with the same gains. Firstly we plot (in Figure 1) the estimation errors by using the observer proposed in Zheng et al. (2015). From Figure 1, we can see that after the period of transition (the jumps are due to the zero cross of estimation error when calculating the log value), the convergent speed is linear, which corresponds exactly to the asymptotic estimation case.

In Figure 2, we depict the estimation error for $x(t)$ by applying the proposed finite-time observer. Comparing with the asymptotic observer proposed in Zheng et al. (2015), after the transition, the convergent speed of the proposed finite-time observer is nonlinear (much faster than linear one depicted in Figure 1), and finally stays around very small estimation error (around 10^{-20} due to numerical algorithm used to simulate system). Finally, Figure 3 plots the estimation error of the unknown input $w(t)$, which clearly shows that the estimation is finite-time.

6.2 Physical example

Consider a chemical reactor train with delayed recycle streams described in Figure 4. According to Nguang

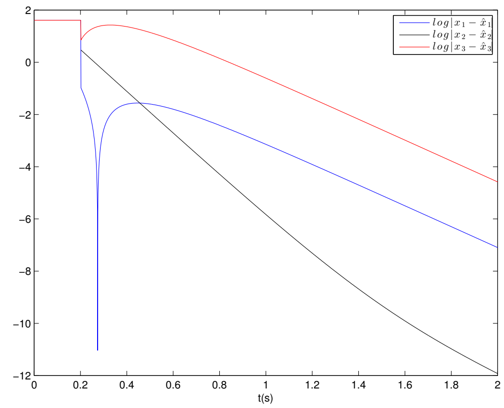

Figure 1: Estimation errors of $x(t)$ using observer proposed in Zheng et al. (2015).

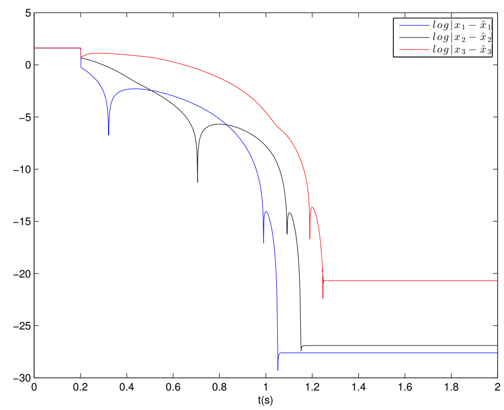

Figure 2: Estimation errors of $x(t)$ using the proposed finite-time observer (29).

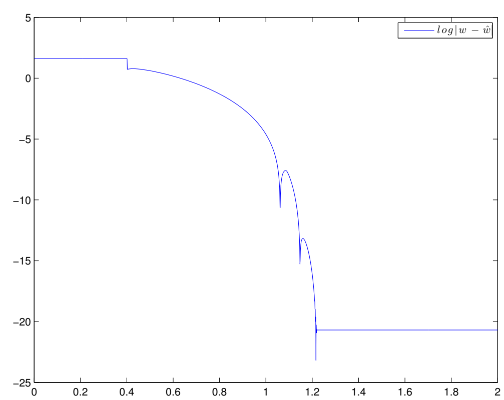

Figure 3: Estimation error of $w(t)$ using the proposed finite-time observer (29).

(2000), the mass balance equation that govern the reactors in the above figure can be modeled as:

$$
\begin{aligned}
&\dot{x}_1(t) = -\tfrac{1}{\theta_1}x_1(t) - \rho_1 x_1(t) + \tfrac{1-R_3}{V_1}x_3(t)\\
&\dot{x}_2(t) = -\tfrac{1}{\theta_2}x_2(t) - \rho_2 x_2(t) + \tfrac{1-R_4}{V_2}x_3(t) + w(t)\\
&\dot{x}_3(t) = -\tfrac{1}{\theta_3}x_3(t) - \rho_3 x_3(t) + \tfrac{R_1}{V_3}x_1(t-2\tau) + \tfrac{R_2}{V_3}x_2(t-2\tau) + \tfrac{R_3+R_4}{V_3}x_3(t-\tau) + \tfrac{F_{rate}}{V_3}u\\
&y_1 = x_1(t)\\
&y_2 = x_2(t)
\end{aligned}
$$

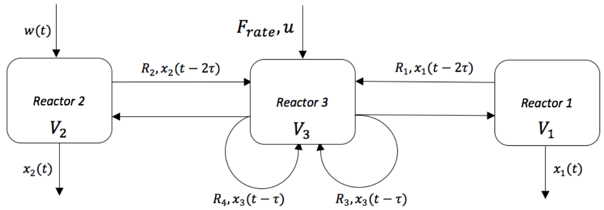

Figure 4: Chemical reactor train with delayed recycle streams.

where $x_1(t)$, $x_2(t)$ and $x_3(t)$ are the compositions, R_1, R_2, R_3 and R_4 are the recycle flow rates, θ_i are the reactor residence times, ρ_i are the reaction constants, F_{rate} is the feed rate and V_i are the reactor volumes. With the stream measurements of reactor 1 and 2 (i.e. $x_1(t)$ and $x_2(t)$), the objective is to achieve a finite-time estimation of the stream in the third reactor (i.e. $x_3(t)$) and the unknown disturbance $w(t)$ presented in the second reactor.

By introducing the delay operator δ, the chemical reactor system can be written in the form of (19) with the following matrices:

$$\tilde{A}(\delta) = \begin{bmatrix} -\frac{1}{\theta_1} - \rho_1 & 0 & \frac{1-R_3}{V_1} \\ 0 & -\frac{1}{\theta_2} - \rho_2 & \frac{1-R_4}{V_1} \\ \frac{R_1}{V_3}\delta^2 & \frac{R_2}{V_3}\delta^2 & -\frac{1}{\theta_3} - \rho_3 + \frac{R_3+R_4}{V_3}\delta \end{bmatrix}, \tilde{B}(\delta) = \begin{bmatrix} 0 \\ 0 \\ \frac{F_{rate}}{V_3} \end{bmatrix}, \tilde{E}(\delta) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

and

$$\tilde{C}(\delta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

It can be checked that there exists $l^* = 2$ such that $\mathrm{rank}_{\mathbb{R}[\delta]} N_2(\delta) = 3$ and $\mathrm{Inv}_S N_2(\delta) = \left\{1, 1, \frac{1-R_3}{V_1}\right\} \subset \mathbb{R}$. Also, we have

$$\mathrm{Inv}_S \begin{bmatrix} \tilde{C}(\delta)\tilde{E}(\delta) \\ \tilde{E}(\delta) \end{bmatrix} = \mathrm{Inv}_S \begin{bmatrix} \tilde{C}(\delta)\tilde{E}(\delta) \end{bmatrix} = \{1\}$$

and $\tilde{E}(\delta)$ is left unimodular over $\mathbb{R}[\delta]$. Therefore, Assumptions 2 and 3 are both satisfied, and all conditions of Theorem 3 and Theorem 4 are satisfied. Thus we can follow the same procedure presented in Section 5 to design a finite-time observer to estimate $x(t)$ and $w(t)$. For the simulation, the following values are chosen: $\theta_i = 2$, $\rho_i = 0.5$, $R_i = 0.5$, $V_i = 0.5$ and $F_{rate} = 0.5$. The corresponding simulation results are depicted in Figure 5.

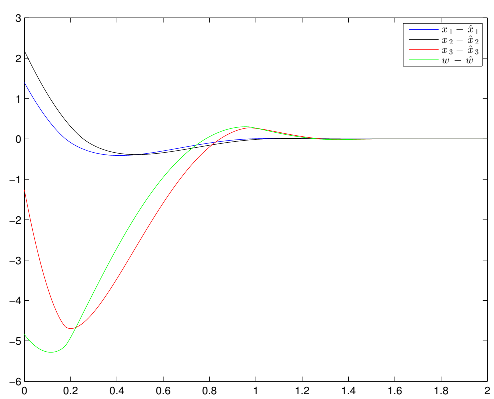

Figure 5: Estimation errors of $x(t)$ and $w(t)$ for the chemical reactor train.

7 Conclusion

In this paper, a finite-time observer for linear time-delay systems has been presented. We adopted the sufficient condition presented in Hou et al. (2002) and Zheng et al. (2015) for a Luenberger-like observer to design a finite-time observer by using the homogeneous technique. We would like to emphasize that the studied system is quite general, since it enables the commensurate delays to be appeared both in the state and in the output. We have proved in this paper that the homogeneous technique provides a finite-time convergence of the estimation. Finally the proposed method is extended to treat as well linear time-delay systems with unknown inputs, and the corresponding finite-time unknown input estimation has been studied.

References

F. J. Bejarano and G. Zheng. Observability of linear systems with commensurate delays and unknown inputs. *Automatica*, 50(8):2077–2083, 2014.

M. Darouach. Linear functional observers for systems with delays in state variables. *IEEE Transactions on Automatic Control*, 46(3):491–496, 2001.

M. Darouach. Full order unknown inputs observers design for delay systems. *in Proceedings of IEEE Mediterranean Conference on Control and Automation*, 2006.

E. Emre and P.P. Khargonekar. Regulation of split linear systems over rings: Coecient-assignment and observers. *IEEE Transactions on Automatic Control*, 27(1):104–113, 1982.

R. Engel and G. Kreisselmeier. A continuous-time observer which converges in finite time. *IEEE Transactions on Automatic Control*, 47(7):1202–1204, 2002.

A. Fattouh, O. Sename, and J.-M. Dion. An unknown input observer design for linear time-delay systems. In *Proceedings of the 38th IEEE Conference on Decision and Control*, pages 4222–4227, 1999.

Z. Gao and S. X. Ding. State and disturbance estimator for time-delay systems with application to fault estimation and signal compensation. *IEEE Transactions on Signal Processing*, 55(12):5541–5551, 2007.

A. Germani, C. Manes, and P. Pepe. An asymptotic state observer for a class of nonlinear delay systems. *Kybernetika*, 37(4):459–478, 2001.

G. Hostetter and J. Meditch. Observing systems with unmeasurable inputs. *IEEE Transactions on Automatic Control*, 18:307–308, 1973.

M. Hou, P. Zitek, and R. J. Patton. An observer design for linear time-delay systems. *IEEE Transactions on Automatic Control*, 47(1):121–125, 2002.

K. Langueh, G. Zheng, and T. Floque. Finite-time unknown input observer for linear time-delay systems. In *Proceedings of the 55th IEEE Conference on Decision and Control*, pages 1435–1440, 2016.

A. Levant. Higher-order sliding modes, differentiation and output-feedback control. *International Journal on Control*, 76(9/10):924–941, 2003.

D. Luenberger. Observer for multivariable systems. *IEEE Transactions on Automatic Control*, 11(2):190–197, 1966.

M. Malek-Zavarei and M. Jamshidi. *Time-Delay Systems: Analysis, Optimization and Applications*. Elsevier Science Inc., New York, NY, USA, 1987.

M. Mboup, C. Join, and M. Fliess. A revised look at numerical differentiation with an application to nonlinear feedback control. *Proceedings of the 15th Mediterranean Conference on Control and Automation*, 2007.

B.P. Molinari. A strong controllability and observability in linear multivariable control. *IEEE Transactions on Automatic Control*, 21(5):761–764, 1976.

S. K. Nguang. Robust stabilization of a class of time-delay nonlinear systems. *IEEE Transactions on Automatic Control*, 45(4):756–762, 2000.

W. Perruquetti, T. Floquet, and E. Moulay. Finite time observers: application to secure communication. *IEEE Transactions on Automatic Control*, 53(1):356–360, 2008.

J.-P. Richard. Time-delay systems: an overview of some recent advances and open problems. *Automatica*, 39(10):1667–1694, 2003.

D. Salamon. Observers and duality between observation and state feedback for time delay systems. *IEEE Transactions on Automatic Control*, 25(6):1187–1192, 1980.

O. Sename. Unknown input robust observer for time delay system. *IEEE Conference on Decision and Control*, 1997.

O. Sename. New trends in design of observers for time-delay systems. *Kybernetica*, 37(4):427–458, 2001.

P. Shi, Y. Zhang, and R. K. Agarwal. Stochastic finite-time state estimation for discrete time-delay neural networks with markovian jumps. *Neurocomputing*, 151, Part 1:168 – 174, 2015.

L. Silverman. Inversion of multivariable linear systems. *IEEE Transactions on Automatic Control*, 14(3): 270–276, 1969.

S. Wang, E. Davison, and P. Dorato. Observing the states of system with unmeasurable disturbances. *IEEE Transaction On Automatic Control*, 20:716–717, 1975.

N. S. Williams and V. Zakian. A ring of delay operators with applications to delay-differential systems. *SIAM Journal on Control and Optimization*, 15(2):247–255, 1977.

D. H. Zhai and Y. Xia. Adaptive finite-time control for nonlinear teleoperation systems with asymmetric time-varying delays. *International Journal of Robust and Nonlinear Control*, 26(12):2586–2607, 2016a.

D. H. Zhai and Y. Xia. Finite-time control of teleoperation systems with input saturation and varying time delays. *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, PP(99):1–13, 2016b.

Z. Zhang, Z. Zhang, H. Zhang, P. Shi, and H. Reza Karimi. Finite-time h_∞ filtering for t-s fuzzy discrete-time systems with time-varying delay and norm-bounded uncertainties. *IEEE Transactions on Fuzzy Systems*, 23(6):2427–2434, 2015.

G. Zheng and H. Wang. A simple finite-time observer for linear time-delay systems. In *Proceedings of the 14th IEEE International Workshop on Variable Structure Systems*, pages 39–42, 2016.

G. Zheng, F. J. Bejarano, W. Perruquetti, and J.-P. Richard. Unknown input observer for linear time-delay systems. *Automatica*, 61:35–43, 2015.