

About recurrence time for a semi-Lagrangian discontinuous Galerkin Vlasov solver

Laurent Navoret, Michel Mehrenberger

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About recurrence time for a semi-Lagrangian discontinuous Galerkin Vlasov solver

Michel Mehrenberger

IRMA, Université de Strasbourg & projet TONUS (INRIA), France

CIRM, 31.10.2017

Joint work with Laurent Navoret (IRMA)

Introduction : results from Cheng-Knorr, 1976

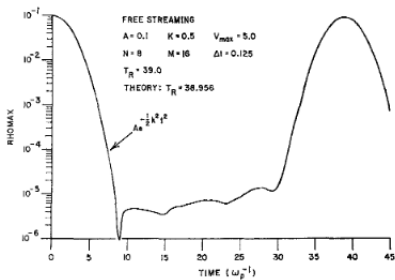
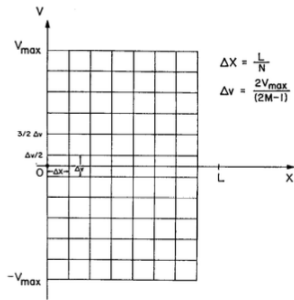
Recurrence time for
SLDG

M. Mehrenberger

Recurrence time $T_R = \frac{2\pi}{k\Delta v}$ is analyzed for free streaming case...

$$\partial_t f + v \partial_x f = 0, \rho = \int f dv, f(t=0) = A \cos(kx) f_0(v), f_0(v) = \frac{1}{\sqrt{2\pi}} e^{-v^2/2}$$

$$\rho(t, x) = A \Delta v \sum_{j=-M}^{M-1} f_0(v_j) \cos(k(x - (j + 1/2)\Delta v t)) \text{ is } T_R \text{ periodic in time}$$



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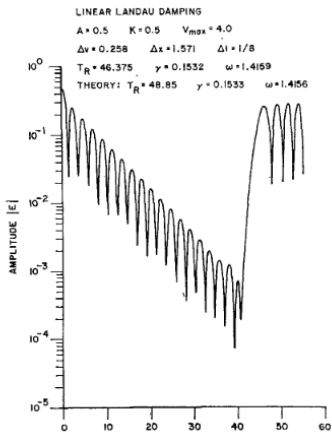
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Introduction : results from Cheng-Knorr 1976

...and observed for the simulation of the linear Landau damping

$$\partial_t f + v \partial_x f + E \partial_v f = 0, \quad \partial_x E = \rho - 1$$

$$f(t=0) = (1 + A \cos(kx)) f_0(v)$$



- ▶ Numerical artefact for several grid-based solvers
- ▶ How to suppress/reduce the phenomenon ?
 - ▶ randomness of the grid
 - ▶ filtering
 - ▶ PIC method
 - ▶ finer grid
- ▶ Recent works : Einkemmer-Ostermann (2014), Cai-Wang (2017)
- ▶ Eliasson (2001, 2010)
- ▶ Objective here : study the behavior of the recurrence for a *Semi-Lagrangian Discontinuous Galerkin* (SLDG) method

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Recurrence time for
SLDG

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- ▶ Rossmanith-Seal → Qiu-Shu → Crouseilles-M-Vecil (2010-2011)
- ▶ already ideas of SLDG in Mangeney-Califano-et-al (2002)
- ▶ variant+analysis : Einkemmer-Ostermann (2014)
- ▶ adaptive : Besse-Deriaz-Madaule (2016)
- ▶ no splitting : Cai-Guo-Qiu (2017)
- ▶ other contexts :
 - ▶ general approach : Restelli-Bonaventuro-Sacco (2006)
 - ▶ Guo-Nair-Qiu (2014)
 - ▶ analysis : Bokanowski-Simarmata, Bokanowski-Cheng-Shu (2016)
- ▶ some ref. for DG for Vlasov :
 - ▶ Heath-Gamba-et-al (2012)
 - ▶ Cheng-Christlieb-Zhong, Madaule-Restelli-Sonnendrücker (2014)
 - ▶ analysis : Ayuso-Carillo-Shu (2012)

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► In 1D for free streaming

- polynomial representation on each cell
- exact transport (constant velocity)
- projection on each cell : piecewise polynomial \rightarrow polynomial
- can be interpreted as advection of Gauss points¹

$$x_{j,0}, \dots, x_{j,d} \in]x_{\min} + j\Delta x, x_{\min} + (j+1)\Delta x[$$

and written as

$$\begin{pmatrix} f(t + \Delta t, x_{j,0}) \\ \dots \\ f(t + \Delta t, x_{j,d}) \end{pmatrix} = A(\alpha) \begin{pmatrix} f(t, x_{j,0}) \\ \dots \\ f(t, x_{j,d}) \end{pmatrix} + B(\alpha) \begin{pmatrix} f(t, x_{j+1,0}) \\ \dots \\ f(t, x_{j+1,d}) \end{pmatrix}$$

with $x_{j,\cdot} - v\Delta t = x_{i,\cdot} + \alpha\Delta x$ and $A(\alpha), B(\alpha) \in \mathbb{R}^{d+1} \times \mathbb{R}^{d+1}$

► In 1D \times 1D phase-space for Vlasov-Poisson

- polynomial represented by Gauss points on each rectangular cell
- splitting : succession of
 - 1D advections of Gauss points in x direction (each 1D advection updates its horizontal line)
 - Poisson
 - 1D advections of Gauss points in v direction (each 1D advection updates its vertical line)

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Free streaming for Gauss type mesh

The mesh in velocity is given by

$$v_{j,\ell} = (j + \bar{v}_\ell)\Delta v, \quad \ell = 0, \dots, d$$

with $0 < \bar{v}_0 < \dots < \bar{v}_d < 1$ and we solve

$$\partial_t f + v_{j,\ell} \partial_x f = 0$$

► Framework not only for SLDG : what counts here is the mesh

► Do we have a recurrence time ?

► is it formula $T_R = 2\pi/(k\Delta v)$?

► OK for $d = 0$ and $\bar{v}_0 = 1/2$.

► but looking at uniform cases :

► $d = 1$ and $\bar{v}_0 = 1/4, \bar{v}_1 = 3/4, \rightarrow T_R = 2\pi/(k\Delta v/2)$

► $d = 2$ and $\bar{v}_0 = 1/6, \bar{v}_1 = 3/6, \bar{v}_2 = 5/6, \rightarrow T_R = 2\pi/(k\Delta v/3)$

► ... $\bar{v}_k = (2\ell + 1)/(2d + 2), \ell = 0, \dots, d$

we can hope formula $T_R = 2\pi(d + 1)/(k\Delta v)$

► We will see that formula is between :

$$T_R \simeq 2\pi(\lfloor d/2 \rfloor + 1)/(k\Delta v)$$

We get

$$\rho(t, x) = A\Delta v \sum_{j=-M}^{M-1} \sum_{\ell=0}^d \omega_{\ell} f_0(v_{j,\ell}) \cos(k(x - (j + \bar{v}_{\ell})\Delta vt))$$

With symmetry of discrete velocity distribution, we obtain

$$\rho(t, x) = A \cos(kx) h(t),$$

with

$$h(t) = \Delta v \sum_{j=-M}^{M-1} \sum_{\ell=0}^d \omega_{\ell} f_0(v_{j,\ell}) \cos(k(j + \bar{v}_{\ell})\Delta vt)$$

We look the expression for $t = t_m = \frac{2\pi m}{k\Delta v}$, $m \in \mathbb{N}^*$

$$h(t_m) = \sum_{\ell=0}^d \omega_{\ell} \cos(2\pi m \bar{v}_{\ell}) \Delta v \sum_{j=-M}^{M-1} f_0(v_{j,\ell})$$

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As $\Delta v \sum_{j=-M}^{M-1} f_0(v_{j,\ell}) \rightarrow \int f_0(v) dv$ exponentially, we have

$$h(t_m) = h_m \int f_0(v) dv + O(\Delta v^p), \quad h_m = \sum_{\ell=0}^d \omega_\ell \cos(2\pi m \bar{v}_\ell)$$

- ▶ Energy restituted at $t_m = \frac{2\pi m}{k\Delta v}$, $m \in \mathbb{N}^*$ is \simeq proportional to h_m .
- ▶ h_m is a quadrature formula for $\int_0^1 \cos(2\pi mv) dv$
- ▶ At continuous level, we have

$$\int_0^1 \cos(2\pi mv) dv = 0 : \text{no recurrence}$$

⇒ link between recurrence effect and quadrature rule

- ▶ For uniform mesh :
 - ▶ $\cos(2\pi m \cdot)$ exactly reproduced by standard quadrature for $m = 0, 1, \dots, d$:

$$\frac{1}{d+1} \sum_{\ell=0}^d \cos(2\pi m \frac{2\ell+1}{2d+2}) = \begin{cases} 0, & \text{if } m = q(d+1) + r, r = 1, \dots, d \\ (-1)^q, & \text{if } m = q(d+1). \end{cases}$$

- ▶ consistent with $T_R = 2\pi(d+1)/(k\Delta v)$.

We look for a quadrature rule such that

$$\sum_{\ell=0}^d \omega_{\ell} \cos(2\pi m \bar{v}_{\ell}) = \int_0^1 \cos(2\pi mv) dv,$$

for all $m \leq n$ with n as high as possible.

- ▶ Possible for $n = \lfloor d/2 \rfloor$ and

$$\omega_{\ell} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \prod_{j=0, j \neq \ell}^{\lfloor d/2 \rfloor} \frac{\cos(v) - \cos(2\pi \bar{v}_j)}{\cos(2\pi \bar{v}_{\ell}) - \cos(2\pi \bar{v}_j)} dv, \quad \ell = 0, \dots, d.$$

- ▶ Optimal points are uniform and lead to $n = d$ (like Gauss integration)
- ▶ for Gauss points, weights remain positive until $d = 18$.
- ▶ recent references : Peherstorfer, 2011 ; Austin, PhD 2016

⇒ Instead of using classical *polynomial* integration quadrature rule,
we can use this *trigonometric* integration

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Numerical results for free streaming

 $d = 2$ (gauss points)

	classical	new
h_1	0.02245	0
h_2	0.52996	0.51916
h_3	0.73652	0.73047

 $d = 3$ (gauss points)

	classical	new
h_1	$-1.0678 \cdot 10^{-3}$	0
h_2	-0.12573	-0.12664
h_3	-0.74097	-0.74155
h_4	-0.33811	-0.33830

 $d = 4$ (gauss points)

	classical	new
h_1	$3.0803 \cdot 10^{-5}$	0
h_2	0.016669	0
h_3	0.30402	0.30657
h_4	0.79870	0.80082
h_5	-0.010458	-0.018835

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$d = 5$ (gauss points)

	classical	new
h_1	$-5.9704 \cdot 10^{-7}$	0
h_2	$-1.4223 \cdot 10^{-3}$	0
h_3	-0.071038	-0.073117
h_4	-0.50843	-0.50769
h_5	-0.67454	-0.67455
h_6	0.30953	0.30852

$d = 6$ (gauss points)

	classical	new
h_1	$8.3166 \cdot 10^{-9}$	0
h_2	$8.5037 \cdot 10^{-5}$	0
h_3	0.010947	0
h_4	0.17907	0.18741
h_5	0.66873	0.66483
h_6	0.40512	0.40575
h_7	-0.41364	-0.41456

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- ▶ Gauss quadrature (classical) already good
- ▶ Gauss quadrature becomes better when d increases
- ▶ small improvement for new trigonometric quadrature
- ▶ do not use when $d > 18$ (negative weights)
- ▶ T_R coherent with result of Madaule-Restelli-Sonnendrücker (2014)

Gauss-Lobatto points

The analysis can be extended to Gauss-Lobatto points

Example : $d = 2$, the mesh is uniform :

$$x_0 = 0, x_1 = 1/2, x_2 = 1$$

- ▶ Trigonometric interpolation corresponds to trapezoidal formula :

$$\omega_0 = \omega_2 = 1/4, \omega_1 = 1/2$$

- ▶ Gauss-Lobatto quadrature is Simpson rule :

$$\omega_0 = \omega_2 = 1/6, \omega_1 = 2/3$$

$d = 2$ (gauss-lobatto points)

	classical	new
h_1	1/3	0
h_2	1	1
h_3	1/3	0
h_4	1	1

Numerical results for free streaming

$d = 4$ (gauss-lobatto points)

	classical	new
h_1	$1.3174 \cdot 10^{-3}$	0
h_2	0.14855	0
h_3	0.79651	0.93393
h_4	0.25734	0.12776
h_5	-0.10028	-0.11932

$d = 6$ (gauss-lobatto points)

	classical	new
h_1	$6.9354 \cdot 10^{-7}$	0
h_2	$1.6300 \cdot 10^{-3}$	0
h_3	0.079357	0
h_4	0.542841	0.663454
h_5	0.64533	0.51383
h_6	-0.34435	-0.22468
h_7	0.15274	$2.1373 \cdot 10^{-3}$

positive weights until $d = 7$.

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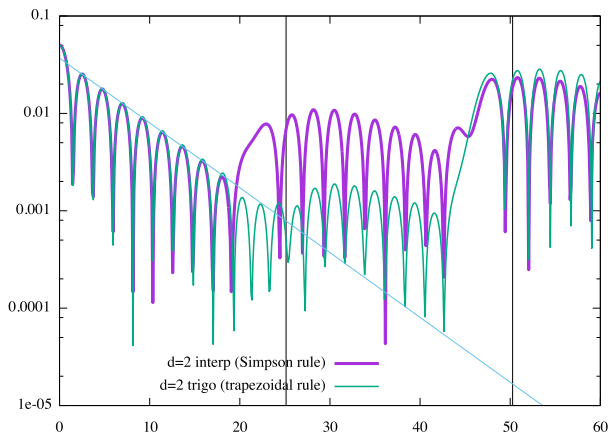
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SLDG

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$$N = 16, v_{\max} = 4, A = 10^{-2}, k = 0.5, 2\pi/(k\Delta v) \simeq 25.13$$



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$N = 64$, $v_{\max} = 8$, $A = 10^{-6}$, $k = 0.5$, $2\pi/(k\Delta v) \simeq 50.26$

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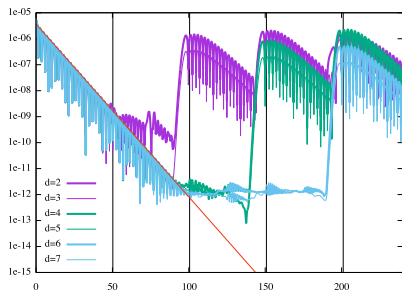
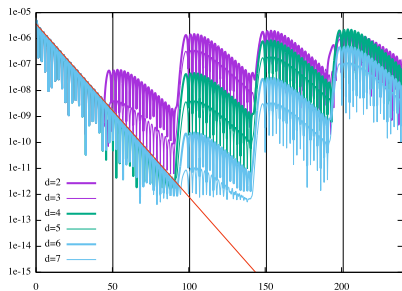
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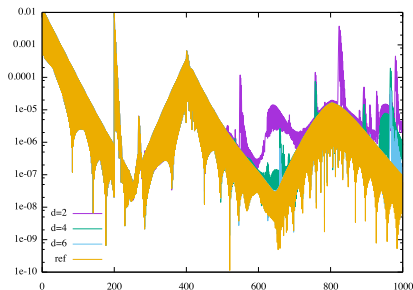
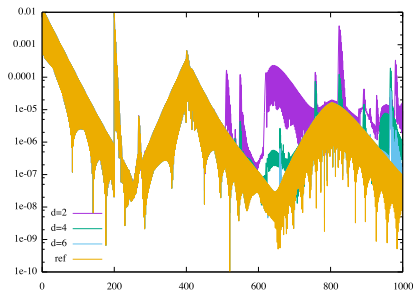
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Numerical results for plasma echo

$N = 600$, $v_{\max} = 8$, $A = 10^{-3}$, $k_1 = \frac{12\pi}{100}$ perturbation with $k_2 = \frac{24\pi}{100}$ at time $t = 200$ is added. $2\pi/(k_1 \Delta v) = 625$



Recurrence time for SLDG

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 - ▶ link between recurrence and quadrature
 - ▶ introduction of trigonometric quadrature for non uniform velocity discretization
 - ▶ analysis for free streaming
 - ▶ numerical results for Landau damping
- ▶ Perspectives/possible applications
 - ▶ velocity semi-discretization method (N. Pham, PhD 2016)
 - ▶ curvilinear grid
 - ▶ DG, NUFFT

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