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One-particle reducible contribution to the one-loop spinor propagator in a constant field

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Abstract

Extending work by Gies and Karbstein on the Euler–Heisenberg Lagrangian, it has recently been shown that the one-loop propagator of a charged scalar particle in a constant electromagnetic field has a one-particle reducible contribution in addition to the well-studied irreducible one. Here we further generalize this result to the spinor case, and find the same relation between the reducible term, the tree-level propagator and the one-loop Euler–Heisenberg Lagrangian as in the scalar case. Our demonstration uses a novel world-line path integral representation of the photon-dressed spinor propagator in a constant electromagnetic field background.

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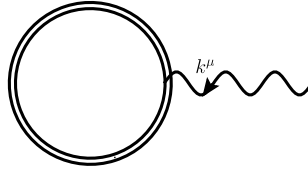


Fig. 1. One-loop one-photon amplitude in a constant external field.

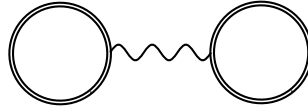


Fig. 2. One-particle reducible contribution to the two-loop EHL.

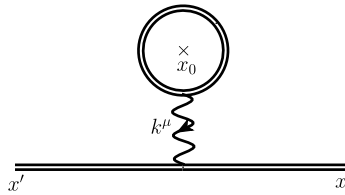


Fig. 3. 1PR contribution to the scalar propagator in a constant field.

1. Introduction

The QED one-loop one-photon amplitude vanishes in vacuum by Furry’s theorem. In the presence of a constant external field this theorem does not imply that the one-photon diagram (Fig. 1) vanishes. However, that diagram is still usually discarded, since it formally vanishes by momentum conservation (as usual, the double-line denotes the electron propagator in a constant field). See, e.g., [1].

Recently, Gies and Karbstein [2] discovered that this diagram can cause non-vanishing contributions when appearing as part of a larger diagram, due to the infrared singularity of the photon propagator connecting it to the rest of the diagram. As their principal example, they analyzed the one-particle reducible (‘1PR’) diagram in spinor QED shown in Fig. 2.

They showed that this diagram leads to a hitherto overlooked contribution $\mathcal{L}^{(2)1PR}$ to the two-loop Euler–Heisenberg Lagrangian [3–6]. Moreover, they found that this “addendum” can be written very simply in terms of the one-loop Euler–Heisenberg Lagrangian $\mathcal{L}^{(1)}$:

$$\mathcal{L}^{(2)1PR} = \frac{\partial \mathcal{L}^{(1)}}{\partial F^{\mu\nu}} \frac{\partial \mathcal{L}^{(1)}}{\partial F_{\mu\nu}}. \tag{1}$$

Two of the present authors in [7] extended this result to scalar QED, and there also to the open line case, i.e. to the scalar propagator in a constant field $D_{\text{scal}}(F)$. For the propagator, a 1PR diagram analogous to Fig. 2 appears already at the one-loop level (Fig. 3).

In [7] it was found that equation (1) generalizes to the propagator in x -space as

$$D_{\text{scal}}^{x'x(1)1PR} = \frac{\partial D_{\text{scal}}^{x'x}}{\partial F_{\mu\nu}} \frac{\partial \mathcal{L}_{\text{scal}}^{(1)}}{\partial F^{\mu\nu}} + \frac{ie}{2} D_{\text{scal}}^{x'x} x'^{\mu} \frac{\partial \mathcal{L}_{\text{scal}}^{(1)}}{\partial F^{\mu\nu}} x^{\nu}. \tag{2}$$

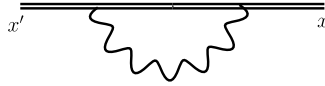


Fig. 4. One-loop propagator in a constant field.

Here Fock–Schwinger gauge centered at x was chosen¹. Of the two terms on the right-hand side, only the first one survives the Fourier transformation to momentum space:

$$D_{\text{scal}}^{(1)1PR}(p) = \frac{\partial D_{\text{scal}}(p)}{\partial F_{\mu\nu}} \frac{\partial \mathcal{L}_{\text{scal}}^{(1)}}{\partial F^{\mu\nu}}. \tag{3}$$

This 1PR addendum is of the same order in α as the standard 1PI diagram shown in Fig. 4.

The purpose of the present paper is to show that the equations (2), (3) generalize to the spinor QED case as they stand, just replacing $\mathcal{L}_{\text{scal}}$, D_{scal} by their spinor QED analogues, denoted by \mathcal{L} and S respectively. As in the scalar case, we will use the sewing relation

$$S^{x'x(1)1PR} = \int \frac{d^D k}{(2\pi)^D k^2} \Gamma_{(1)}^{(1)}[k', \varepsilon'; F] S_{(1)}^{x'x}[k, \varepsilon; F] \Big|_{k' \rightarrow -k, \varepsilon^\mu \varepsilon'^\nu \rightarrow \eta^{\mu\nu}} \tag{4}$$

with the ingredients $\Gamma_{(1)}^{(1)}[k', \varepsilon'; F]$ (the one-loop one-photon amplitude in the constant field) and $S_{(1)}^{x'x}[k, \varepsilon; F]$ (the x -space spinor propagator in the field with one photon attached)².

The former contains a delta function $\delta^D(k)$ and one factor of momentum (see (7) below), so that by itself it vanishes. In the sewing relation (4) it will produce a finite contribution via

$$\int d^D k \delta^D(k) \frac{k^\mu k^\nu}{k^2} = \frac{\eta^{\mu\nu}}{D} \tag{5}$$

when combined with the part of $S_{(1)}^{x'x}[k, \varepsilon; F]$ that is also linear in the momentum (this may seem mathematically dubious, but can be justified by regularizing the delta function as in [2], approaching the constant external field by a slightly space-time dependent one, as is physically appropriate in any case).

In [7], the identities (2), (3) were shown by a direct calculation of all the ingredients, using the worldline approach to QED in a constant field [8–15]. Here, we will proceed in a more efficient manner: rather than explicitly calculating the spinor propagator in the field and with one photon attached, which is the only really new ingredient, we will write down a worldline path integral representation for this object, and show by manipulations under the path integral that its linear part in the photon momentum k^μ – which is all that is required for the sewing – fulfills the identity

¹ We have interchanged x and x' with respect to the conventions of [7].

² By $S_{(N)}^{x'x}$ and $K_{(N)}^{x'x}$ we denote the constant field propagator and its kernel, the latter to be introduced below, with N finite-energy photons attached (in addition to the zero-energy photons exchanged with the constant field background), and we refer to these objects as the “ N -photon propagator” and “ N -photon kernel” respectively. Thus, $N = 0$ corresponds to the constant-field propagator itself, although we will keep writing $S_{(0)}^{x'x}$, $K_{(0)}^{x'x}$ for it rather than $S_{(0)}^{x'x}$, $K_{(0)}^{x'x}$. In this context the constant-field propagator will also be called “photonless,” despite its interaction with the field.

$$\begin{aligned}
 S_{(1)}^{x'x} \Big|_k &= -2i\varepsilon \cdot \left(\frac{\partial S^{x'x}}{\partial F} + \frac{ie}{2} x_- S^{x'x} x_+ \right) \cdot k - \frac{ie}{2} \varepsilon \cdot (\gamma x_- + x_- \gamma) \cdot k K^{x'x} \\
 &= -2i\varepsilon \cdot \left(\frac{\partial S^{x'x}}{\partial F} + \frac{ie}{2} x' S^{x'x} x \right) \cdot k + \varepsilon \cdot L \cdot k,
 \end{aligned}
 \tag{6}$$

where $x_+ \equiv \frac{1}{2}(x + x')$, $x_- \equiv x' - x$, and $L^{\mu\nu}$ is a symmetric tensor that will drop out upon sewing since the contribution from the one-loop one-photon diagram is anti-symmetric in its Lorentz indices (see (17) below). Together with the similar identity for the closed loop [7],

$$\Gamma_{(1)}^{(1)}[k, \varepsilon; F] = -2i(2\pi)^D \delta^D(k) \left[\varepsilon \cdot \frac{\partial \mathcal{L}^{(1)}(F)}{\partial F} \cdot k + O(k^3) \right],
 \tag{7}$$

it is then immediate to obtain, from the relations (4), (5), the spinor QED generalization of (2),

$$S^{x'x(1)1PR} = \frac{\partial S^{x'x}}{\partial F_{\mu\nu}} \frac{\partial \mathcal{L}^{(1)}}{\partial F^{\mu\nu}} + \frac{ie}{2} S^{x'x} x'^\mu \frac{\partial \mathcal{L}^{(1)}}{\partial F^{\mu\nu}} x^\nu,
 \tag{8}$$

and, by Fourier transformation, the momentum space version thereof,

$$S^{(1)1PR}(p) = \frac{\partial S(p)}{\partial F_{\mu\nu}} \frac{\partial \mathcal{L}^{(1)}}{\partial F^{\mu\nu}}.
 \tag{9}$$

The next section will be devoted to the demonstration of the “derivative identity” (6). With the identities (8), (9) in hand, we then use them in section 3 to write the 1PR contribution explicitly in x -space and momentum space, now requiring only the photonless versions of the loop and propagator. In the concluding section we summarize our findings.

2. Worldline derivation of the derivative identity

2.1. The closed loop with zero and one photons

As a warm-up, let us rederive the closed-loop formula (7) in the new approach. In the worldline formalism, the one-loop spinor QED effective action can be written in terms of a double worldline path integral as follows (see [14] and refs. therein):

$$\begin{aligned}
 \Gamma^{(1)}[A] &= -\frac{1}{2} \int_0^\infty \frac{dT}{T} e^{-m^2 T} \int_P Dx \int_A D\psi \\
 &\quad \times \exp \left[- \int_0^T d\tau \left(\frac{1}{4} \dot{x}^2 + \frac{1}{2} \psi \cdot \dot{\psi} + ieA \cdot \dot{x} - ie\psi \cdot F \cdot \psi \right) \right].
 \end{aligned}
 \tag{10}$$

Here the orbital path integral $\int Dx$ is over closed trajectories in space-time, $x(T) = x(0)$, the spin path integral $\int D\psi$ over Grassmann functions $\psi^\mu(\tau)$ obeying antisymmetric boundary conditions, $\psi^\mu(T) = -\psi^\mu(0)$.

For a constant $F_{\mu\nu}$ it is convenient to use Fock–Schwinger gauge centered at the loop center of mass x_0 [9], since this allows one to write A_μ in terms of $F_{\mu\nu}$:

$$A^\mu(x) = \frac{1}{2} (x - x_0)^\nu F_{\nu\mu}.
 \tag{11}$$

Separating off the loop center of mass via $x(\tau) = x_0 + q(\tau)$, one obtains

$$\Gamma^{(1)}(F) = \int d^D x_0 \mathcal{L}^{(1)}(F), \tag{12}$$

where

$$\begin{aligned} \mathcal{L}^{(1)}(F) = & -\frac{1}{2} \int_0^\infty \frac{dT}{T} e^{-m^2 T} \int_P Dq \int_A D\psi \\ & \times \exp \left[- \int_0^T d\tau \left(\frac{1}{4} \dot{q}^2 + \frac{1}{2} \psi \cdot \dot{\psi} + \frac{ie}{2} q \cdot F \cdot \dot{q} - ie \psi \cdot F \cdot \psi \right) \right] \end{aligned} \tag{13}$$

and $q^\mu(\tau)$ now obeys the “string-inspired” constraint $\int_0^T d\tau q^\mu(\tau) = 0$.

The one-loop one-photon amplitude $\Gamma_{(1)}^{(1)}[k, \varepsilon; F]$ is obtained from (10) by the insertion of the photon vertex operator

$$\begin{aligned} -ieV[k, \varepsilon] = & -ie \int_0^T d\tau \left[\varepsilon \cdot \dot{x} + 2i\varepsilon \cdot \psi k \cdot \psi \right] e^{ik \cdot x} \\ = & -ie e^{ik \cdot x_0} \int_0^T d\tau \left[\varepsilon \cdot \dot{q} + 2i\varepsilon \cdot \psi k \cdot \psi \right] e^{ik \cdot q} \end{aligned} \tag{14}$$

under the path integrals in (13). Eventually the prefactor $e^{ik \cdot x_0}$ integrating over x_0 will provide a momentum conserving $\delta(k)$, which means that for the remaining τ – integral we are interested only in the leading, linear term in the momentum expansion (the integrand contains also a momentum-independent term, which however integrates to zero). Projecting that part of the vertex operator correspondingly, we have

$$-ieV[k, \varepsilon] \longrightarrow e^{ik \cdot x_0} \int_0^T d\tau e \left[\varepsilon \cdot \dot{q} k \cdot q + 2\varepsilon \cdot \psi k \cdot \psi \right]. \tag{15}$$

We observe that, when acting on the exponent in (13), we can further replace the integrand in this last expression under the path integral by

$$-2i e^{ik \cdot x_0} \varepsilon^\mu \frac{\partial}{\partial F^{\mu\nu}} k^\nu. \tag{16}$$

Putting things together we obtain eq. (7) (to linear order in k),

$$\begin{aligned} \Gamma_{(1)}^{(1)}[k, \varepsilon; F] \Big|_k = & -2i \int d^D x_0 e^{ik \cdot x_0} \varepsilon^\mu \frac{\partial}{\partial F^{\mu\nu}} k^\nu \mathcal{L}^{(1)}(F) \\ = & -2i (2\pi)^D \delta^D(k) \varepsilon \cdot \frac{\partial}{\partial F} \cdot k \mathcal{L}^{(1)}(F). \end{aligned} \tag{17}$$

2.2. *The open line with zero and one photons*

The worldline representation of the spinor propagator in a constant field, dressed with photons, is a much more complicated issue, and a formalism suitable for practical calculations has been developed only very recently [16], based on [6,17]. For the photonless case, it leads to the representation³

$$S^{x'x}(F) = \left[m + i\gamma \cdot \left(\frac{\partial}{\partial x'} - \frac{ie}{2} F \cdot x_- \right) \right] K^{x'x}(F) \tag{18}$$

with the “kernel” function

$$\begin{aligned} K^{x'x}(F) &= \int_0^\infty e^{-m^2 T} dT \int_{x(0)=x}^{x(T)=x'} Dx e^{-\int_0^T d\tau (\frac{1}{4}\dot{x}^2 + ieA \cdot \dot{x})} \\ &\quad \times \frac{1}{4} \text{symb}^{-1} \left\{ \int_A D\psi e^{-\int_0^T d\tau [\frac{1}{2}\psi \cdot \dot{\psi} - ie(\psi + \frac{1}{2}\eta) \cdot F \cdot (\psi + \frac{1}{2}\eta)]} \right\} \\ &= \int_0^\infty dT e^{-m^2 T} e^{-\frac{x^2}{4T}} \int Dq e^{-\int_0^T d\tau \frac{1}{4}q(-\frac{d^2}{d\tau^2} + 2ieF\frac{d}{d\tau})q + \frac{ie}{T}x_- FQ} \\ &\quad \times \frac{1}{4} \text{symb}^{-1} \int D\psi e^{-\int_0^T d\tau [\frac{1}{2}\psi(\frac{d}{d\tau} - 2ieF)\psi - ie\psi \cdot F \cdot \eta - \frac{ie}{4}\eta \cdot F \cdot \eta]} \\ &= \int_0^\infty dT e^{-m^2 T} (4\pi T)^{-\frac{D}{2}} \det^{-\frac{1}{2}} \left[\frac{\tan Z}{Z} \right] e^{-\frac{1}{4T}x_- \cdot Z \cdot \cot Z \cdot x_-} \text{symb}^{-1} \left[e^{\frac{i}{4}\eta \cdot \tan Z \cdot \eta} \right]. \end{aligned} \tag{19}$$

Here the propagation is from x to x' , η^μ is a constant Grassmann vector and $Z_{\mu\nu} \equiv eF_{\mu\nu}T$. Again we have used Fock–Schwinger gauge, now centered at the initial point x (it is well-known how to convert the propagator from this gauge to a general gauge [19]). In the second line we have introduced the orbital path-integral $\int Dq$ which runs over fluctuations about the straight-line path leading from x to x' ; that is, trajectories $q(\tau)$ obeying Dirichlet boundary conditions in proper-time, $q(0) = q(T) = 0$, and have defined

$$Q^\mu \equiv \int_0^T d\tau q^\mu(\tau). \tag{20}$$

The “symbol map” symb is defined by

$$\text{symb}(\hat{\gamma}^{[\alpha\beta \dots \rho]}) \equiv \eta^\alpha \eta^\beta \dots \eta^\rho \tag{21}$$

where

$$\hat{\gamma}^\mu \equiv i\sqrt{2}\gamma^\mu \tag{22}$$

and $\hat{\gamma}^{[\alpha\beta \dots \rho]}$ denotes the totally antisymmetrized product,

³ Our field theory conventions follow [18], except that we use a different sign of the elementary charge.

$$\hat{\gamma}^{[\alpha_1\alpha_2\cdots\alpha_n]} \equiv \frac{1}{n!} \sum_{\pi \in S_n} \text{sign}(\pi) \hat{\gamma}^{\alpha_{\pi(1)}} \hat{\gamma}^{\alpha_{\pi(2)}} \dots \hat{\gamma}^{\alpha_{\pi(n)}}. \tag{23}$$

By simple algebra, one may verify that

$$\text{symb}^{-1} \left[e^{\frac{i}{4} \eta \cdot \tan \mathcal{Z} \cdot \eta} \right] = \mathbb{1} - \frac{i}{4} (\tan \mathcal{Z})^{\mu\nu} [\gamma^\mu, \gamma^\nu] + \frac{i}{8} \epsilon^{\mu\nu\alpha\beta} (\tan \mathcal{Z})^{\mu\nu} (\tan \mathcal{Z})^{\alpha\beta} \gamma_5 \tag{24}$$

and it is then easy to identify (18) with the form given, e.g., in [19].

Similarly, the fermion propagator with one photon attached can be written as [16]

$$S_{(1)}^{x'x}[k, \varepsilon; F] = \left[m + i\gamma \cdot \left(\frac{\partial}{\partial x'} - \frac{ie}{2} F \cdot x_- \right) \right] K_{(1)}^{x'x}[k, \varepsilon; F] - e \not{\epsilon} e^{ik \cdot x'} K^{x'x}(F). \tag{25}$$

Here the photonless kernel $K^{x'x}(F)$ has been given in (19) above, and the one-photon kernel $K_{(1)}^{x'x}[k, \varepsilon; F]$ has the following path-integral representation:

$$K_{(1)}^{x'x}[k, \varepsilon; F] = -\frac{ie}{4} \int_0^\infty dT e^{-m^2 T} e^{-\frac{x_-^2}{4T}} \int Dq e^{-\int_0^T d\tau \frac{1}{4} q \cdot \left(-\frac{d^2}{d\tau^2} + 2ieF \frac{d}{d\tau} \right) q + \frac{ie}{T} x_- \cdot FQ} \\ \times \text{symb}^{-1} \int D\psi e^{-\int_0^T d\tau \left[\frac{1}{2} \psi \cdot \left(\frac{d}{d\tau} - 2ieF \right) \psi - ie\psi \cdot F \cdot \eta - \frac{ie}{4} \eta \cdot F \cdot \eta \right]} V_\eta^{x'x}[k, \varepsilon]. \tag{26}$$

The photon vertex operator now appears in the form

$$V_\eta^{x'x}[k, \varepsilon] = \int_0^T d\tau \left[\varepsilon \cdot \left(\frac{x_-}{T} + \dot{q} \right) + 2i\varepsilon \cdot \left(\psi + \frac{\eta}{2} \right) k \cdot \left(\psi + \frac{\eta}{2} \right) \right] e^{ik \cdot (x + x_- \frac{\tau}{T} + q)}. \tag{27}$$

Again we are free to restrict this operator to its linear part,

$$V_\eta^{x'x}[k, \varepsilon] \Big|_k = i\varepsilon \cdot \int_0^T d\tau \left[\left(\frac{x_-}{T} + \dot{q} \right) \left(x + \frac{x_-}{T} \tau + q \right) + 2 \left(\psi + \frac{\eta}{2} \right) \left(\psi + \frac{\eta}{2} \right) \right] \cdot k. \tag{28}$$

With suitable integration by parts and application of the boundary conditions this can be written as

$$V_\eta^{x'x}[k, \varepsilon] \Big|_k = i\varepsilon \cdot \left[\frac{1}{2} x_- (x + x') + \int_0^T d\tau \left(\dot{q}q + \frac{x_-}{T} q - q \frac{x_-}{T} + 2 \left(\psi + \frac{\eta}{2} \right) \left(\psi + \frac{\eta}{2} \right) \right) \right] \cdot k. \tag{29}$$

In this form it is easy to see that an insertion of $-ieV_\eta^{x'x}[k, \varepsilon] \Big|_k$ into the $N = 0$ kernel is equivalent to acting on it with the operator

$$(-2i)\varepsilon \cdot \left(\frac{\partial}{\partial F} + \frac{ie}{2} x_- x_+ \right) \cdot k, \tag{30}$$

leading to

$$K_{(1)}^{x'x}[k, \varepsilon; F] \Big|_k = (-2i)\varepsilon \cdot \left(\frac{\partial}{\partial F} + \frac{ie}{2} x_- x_+ \right) \cdot k K^{x'x}(F). \tag{31}$$

Using this derivative identity for $K_{(1)}^{x'x}$ in (25), and expanding the second term of that equation to linear order in k , it is then simple to show the derivative identity for $S_{(1)}^{x'x}$ itself, eq. (6). As described in the introduction this immediately leads to our compact expression for the addendum⁴ in the spinor case, equation (8).

3. Explicit form of the 1PR addendum

After having proven the identities (8), (9), we can now use them to forget about the one-photon amplitudes, and work out the 1PR contribution from the photonless propagator, as given in (18), and the one-loop Euler–Heisenberg Lagrangian $\mathcal{L}^{(1)}(F)$. A representation of the latter suitable for our purposes is [9]

$$\mathcal{L}^{(1)}(F) = -2 \int_0^\infty \frac{dT}{T} (4\pi T)^{-\frac{D}{2}} e^{-m^2 T} \det^{-\frac{1}{2}} \left[\frac{\tan Z}{Z} \right]. \quad (32)$$

3.1. The 1PR addendum in configuration space

In configuration space, an explicit representation of the addendum can be found by simply carrying out the differentiation (note that all matrices are built out of the constant anti-symmetric field strength tensor and so commute with one another) of the photonless propagator and one-loop effective action:

$$\begin{aligned} & S^{x'x(1)1PR} \\ &= e^2 \int_0^\infty dTT e^{-m^2 T} (4\pi T)^{-\frac{D}{2}} \det^{-\frac{1}{2}} \left[\frac{\tan Z}{Z} \right] e^{-\frac{1}{4T} x_- \cdot Z \cdot \cot Z \cdot x_-} \\ & \quad \times \int_0^\infty dT' (4\pi T')^{-\frac{D}{2}} e^{-m^2 T'} \det^{-\frac{1}{2}} \left[\frac{\tan Z'}{Z'} \right] \left\{ \left[m - \frac{i}{2T} \gamma \cdot Z \cdot (\cot Z + i) \cdot x_- \right] \right. \\ & \quad \times \left[\frac{i}{2T} x' \cdot \Xi' \cdot x - \frac{1}{4T} x_- \cdot (\cot Z - Z \cdot \csc^2 Z) \cdot \Xi' \cdot x_- \right. \\ & \quad \left. \left. + \frac{1}{2} \text{tr}(\Xi \cdot \Xi') + \Xi'_{\mu\nu} \frac{\partial}{\partial Z_{\mu\nu}} \right] \right. \\ & \quad \left. - \frac{i}{2T} \gamma \cdot (\cot Z - Z \cdot \csc^2 Z + i) \cdot \Xi' \cdot x_- \right\} \text{symb}^{-1} \left\{ e^{\frac{i}{4} \eta \cdot \tan Z \cdot \eta} \right\}. \quad (33) \end{aligned}$$

Here we have abbreviated

$$\Xi \equiv \frac{1}{Z} - \frac{1}{\sin Z \cdot \cos Z} = \frac{d}{dZ} \text{tr} \ln \left[\frac{\tan Z}{Z} \right], \quad (34)$$

⁴ We remark that $S_{(1)}^{x'x}[k, \varepsilon; F]$ also contains a term independent of k , but it does not contribute in the sewing procedure. The reason is that it leads to a k -integral that vanishes by antisymmetry when matched with the lowest-order term in $\Gamma_{(1)}^{(1)}[k', \varepsilon'; F]$, and, as indicated in (7), the next-higher term in the momentum expansion of $\Gamma_{(1)}^{(1)}[k', \varepsilon'; F]$ is already of order $\delta^D(k')k'^3$ [7].

and accordingly for Ξ' . It remains to apply the inverse symbol map, which requires the identity (24) and one further relation,

$$\begin{aligned} & \Xi'_{\mu\nu} \frac{\partial}{\partial \mathcal{Z}_{\mu\nu}} \text{symb}^{-1} \left\{ e^{\frac{i}{4} \eta \cdot \tan \mathcal{Z} \cdot \eta} \right\} \\ &= \text{symb}^{-1} \left\{ \frac{i}{4} \eta \cdot \sec^2 \mathcal{Z} \cdot \Xi' \cdot \eta e^{\frac{i}{4} \eta \cdot \tan \mathcal{Z} \cdot \eta} \right\} \\ &= -\frac{i}{4} \left[\left(\sec^2 \mathcal{Z} \cdot \Xi' \right)_{\mu\nu} [\gamma^\mu, \gamma^\nu] - \epsilon^{\mu\nu\alpha\beta} \left(\sec^2 \mathcal{Z} \cdot \Xi' \right)_{\mu\nu} (\tan \mathcal{Z})_{\alpha\beta} \gamma_5 \right]. \end{aligned} \tag{35}$$

3.2. The 1PR addendum in momentum space

The momentum space version of the addendum can be obtained either by Fourier transforming our x -space result above (33), or by a direct use of the momentum space derivative identity (9). To give an explicit expression, first we need the photonless propagator in momentum space [16]

$$S(p) = \int_0^\infty dT [m - \gamma \cdot (\mathbb{1} + i \tan \mathcal{Z}) \cdot p] e^{-T(m^2 + p \cdot \frac{\tan \mathcal{Z}}{\mathcal{Z}} \cdot p)} \text{symb}^{-1} \left\{ e^{\frac{i}{4} \eta \cdot \tan \mathcal{Z} \cdot \eta} \right\}. \tag{36}$$

After simple algebra, one finds

$$\begin{aligned} & S^{(1)1PR}(p) \\ &= e^2 \int_0^\infty dT T e^{-T(m^2 + p \cdot \frac{\tan \mathcal{Z}}{\mathcal{Z}} \cdot p)} \int_0^\infty dT' (4\pi T')^{-\frac{D}{2}} e^{-m^2 T'} \det^{-\frac{1}{2}} \left[\frac{\tan \mathcal{Z}'}{\mathcal{Z}'} \right] \\ & \times \left\{ \left[m - \gamma \cdot (\mathbb{1} + i \tan \mathcal{Z}) \cdot p \right] \left[-T p \cdot \frac{\mathcal{Z} - \sin \mathcal{Z} \cdot \cos \mathcal{Z}}{\mathcal{Z}^2 \cdot \cos^2 \mathcal{Z}} \cdot \Xi' \cdot p + \Xi'_{\mu\nu} \frac{\partial}{\partial \mathcal{Z}_{\mu\nu}} \right] \right. \\ & \left. - i \gamma \cdot \sec^2 \mathcal{Z} \cdot \Xi' \cdot p \right\} \text{symb}^{-1} \left\{ e^{\frac{i}{4} \eta \cdot \tan \mathcal{Z} \cdot \eta} \right\}, \end{aligned} \tag{37}$$

where the same relations (24) and (35) are required to apply the inverse symbol map. This concludes the determination of the addendum for the spinor propagator.

4. Summary and outlook

Extending recent results we have shown that there is a one-particle reducible contribution to the spinor propagator in a constant field at one-loop order, given by a Feynman diagram that was previously believed to vanish. This contribution is of the same order in α as the standard 1PI self-energy diagram in QED in a constant field. We have derived formulas for this addendum in terms of the spinor propagator in the field, and the one-loop Euler–Heisenberg Lagrangian. Although these formulas are analogous to the ones obtained for scalar QED in [7], here we have introduced a more efficient approach to their derivation in the framework of the worldline formalism, using formal manipulations under the path integrals rather than an actual calculation of the path integrals as had been done in [7]. We have given explicit but compact expressions for the addendum in configuration space and momentum space. These expressions should make it easy to study the effect of the addendum on such important quantities in external-field QED as the strong-field asymptotics [20] and the Ritus mass shift [21].

As in the case of the Euler–Heisenberg Lagrangian studied in [2], here, too, for possible applications of the addendum it must be stressed that the 1PR diagram, unlike the 1PI one, can involve two different fermions, e.g. we could have an electron in the loop and a quark in the line, or vice versa.

At higher loop order the non-vanishing of these lately uncovered contributions will clearly lead to a proliferation of previously overlooked terms of this type in constant-field QED.

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References

- [1] E.S. Fradkin, D.M. Gitman, S.M. Shvartsman, *Quantum Electrodynamics with Unstable Vacuum*, Springer, 1991.
- [2] H. Gies, F. Karbstein, An addendum to the Heisenberg–Euler effective action beyond one loop, *J. High Energy Phys.* 1703 (2017) 108, arXiv:1612.07251 [hep-th].
- [3] W. Heisenberg, H. Euler, Folgerungen aus der Diracschen Theorie des Positrons, *Z. Phys.* 98 (1936) 714.
- [4] V.I. Ritus, Lagrangian of an intense electromagnetic field and quantum electrodynamics at short distances, *Zh. Èksp. Teor. Fiz.* 69 (1975) 1517; *Sov. Phys. JETP* 42 (1975) 774.
- [5] W. Dittrich, M. Reuter, *Effective Lagrangians in Quantum Electrodynamics*, Springer, 1985.
- [6] M. Reuter, M.G. Schmidt, C. Schubert, Constant external fields in gauge theory and the spin 0, 1/2, 1 path integrals, *Ann. Phys.* 259 (1997) 313, arXiv:hep-th/9610191.
- [7] J.P. Edwards, C. Schubert, One-particle reducible contribution to the one-loop scalar propagator in a constant field, *Nucl. Phys. B* 923 (2017) 339, arXiv:1704.00482 [hep-th].
- [8] M.J. Strassler, Field theory without Feynman diagrams: one-loop effective actions, *Nucl. Phys. B* 385 (1992) 145, arXiv:hep-ph/9205205.
- [9] M.G. Schmidt, C. Schubert, On the calculation of effective actions by string methods, *Phys. Lett. B* 318 (1993) 438, arXiv:hep-th/9309055.
- [10] M.G. Schmidt, C. Schubert, Multiloop calculations in the string-inspired formalism: the single spinor-loop in QED, *Phys. Rev. D* 53 (1996) 2150, arXiv:hep-th/9410100.
- [11] R.Zh. Shaisultanov, On the string-inspired approach to QED in external field, *Phys. Lett. B* 378 (1996) 354, arXiv:hep-th/9512142.
- [12] S.L. Adler, C. Schubert, Photon splitting in a strong magnetic field: recalculation and comparison with previous calculations, *Phys. Rev. Lett.* 77 (1996) 1695, arXiv:hep-th/9605035.
- [13] C. Schubert, Vacuum polarisation tensors in constant electromagnetic fields: part I, *Nucl. Phys. B* 585 (2000) 407, arXiv:hep-ph/0001288.
- [14] C. Schubert, Perturbative quantum field theory in the string-inspired formalism, *Phys. Rep.* 355 (2001) 73, arXiv:hep-th/0101036.
- [15] A. Ahmad, N. Ahmadiniaz, O. Corradini, S.P. Kim, C. Schubert, Master formulas for the dressed scalar propagator in a constant field, *Nucl. Phys. B* 919 (2017) 9, arXiv:1612.02944 [hep-ph].
- [16] N. Ahmadiniaz, F. Bastianelli, O. Corradini, J.P. Edwards, C. Schubert, in preparation.
- [17] E.S. Fradkin, D.M. Gitman, Path integral representation for the relativistic particle propagators and BFV quantization, *Phys. Rev. D* 44 (1991) 3230.
- [18] M. Srednicki, *Quantum Field Theory*, Cambridge University Press, 2007.
- [19] W. Dittrich, H. Gies, *Probing the Quantum Vacuum: Perturbative Effective Action Approach in Quantum Electrodynamics and Its Application*, Springer Tracts Mod. Phys., vol. 166, 2000, p. 1.
- [20] A. Kuznetsov, N. Mikheev, *Electroweak Processes in External Electromagnetic Fields*, Springer Tracts in Modern Physics, Springer, 2004.
- [21] V.I. Ritus, Method of eigenfunctions and mass operator in quantum electrodynamics of a constant field, *Zh. Èksp. Teor. Fiz.* 75 (1978) 1560; *J. Exp. Theor. Phys.* 48 (1978) 788.