Strategic Behaviour in Financial Markets

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To my family

Acknowledgements

Contents

Chapter 1

Introduction

A market typically involves a relatively complex set of interactions between agents over time. They may be buyers and sellers, shareholders and debtholders, rivals or allies. Whatever the role they play in the market, they will always try to make themselves better off by exploiting some advantages they have. Thus, the agents will try to anticipate the effects of their behaviour on the actions of the other participants and on the payoffs they will receive. Many economic issues involve strategic interaction: auctions, R&D races, economic negotiations and there is no reason for the financial markets to make an exception.

Consequently, recent research in finance tries to integrate strategic behaviour of the agents in the existing models. Two important directions of research where the incidence of strategic behavior has been widely exploited are valuation of corporate debt and market microstructure. The modelling of strategic behavior in valuation of corporate debt proved to be very important because it lessened one of the important criticism of previous work: the fact that the spreads produced by the models were very low by comparison with the ones observed from real data. The models using this new approach envisage that strategic debt service can explain a large proportion of spreads of corporate debt and suggest that strategic behaviour plays a significant role in the pricing of various types of corporate debt.

On the other hand, strategic behaviour becomes even more important when imper-

fections exist in the market. In many instances, incomplete and asymmetric information has fundamental effects on the market system since an informational advantage can be exploited strategically. As a result, it became essential to consider issues of strategic behaviour in market microstructure problems. The fact that a strategic trader exploits his informational advantage taking into account the effect the quantity he chooses is expected to have both on prices and the others traders' strategies is essential. One of the most important result driven by strategic behaviour is that in the case of imperfect competition prices are less informative than in the case of perfect competition.

It has been pointed out also that asymmetric information is very important in the design of income tax systems. The models that incorporate asymmetric information are going back to the work of Vickrey (1945) who showed how a tax system could be designed to take into consideration not only goals for the distribution of income after tax, such as equality, but also the fact that taxes should not distort economic behavior. However, the issues of the tax evasion and their implications in the financial markets where mainly considered at international markets level and had as a main concern the government's instruments to fight tax evasion, tax arbitrage and the effects of tax law on the economic growth.

This thesis consists of three papers that deal with strategic issues in these two branches of the financial literature: valuation of corporate debt and market microstructure. The first chapter develops a contingent pricing model that emphasizes the role of maturity and place of the lender's claim in the hierarchy of debt when we allow for strategic behaviour. The second chapter focuses on the effects of strategic behaviour in the presence of different types of information, while the third chapter considers again strategic behaviour issues in a market microstructure setup, this time in connection with a tax compliance problem.

The first chapter is concerned with the effects of debt renegotiation and capital structure of a firm on the prices of bonds. Following the recent trend that has begun the task of integrating contingent claims analysis with modern corporate finance (incorporating strategic behaviour in corporate security valuation), my first paper presents a simple pricing model in which two debtholders decide whether to restructure the firm or not

in the event of default. The novelty of the paper consists of the fact that we consider restructuring of the entire debt, not only of the coupon payment, and also that we allow the debt to be owned by different creditors.

An extensive literature in finance has built on Merton's (1974) model for the valuation of corporate debt that is subject to the risk of default. Merton's original framework considers a firm with equity and zero-coupon debt as claims, and prices these claims under an exogenously given process for the value of the firm's assets. The paper makes some simplifying assumptions that are analytically advantageous, but they come at a steep cost: most importantly, the model generates yield spreads that are too small in relation to observed levels. Motivated by this discrepancy, subsequent research has generalized the Merton model in many ways. Two of the most important limitations signalled by the empirical literature are the fact that default is assumed to occur only when the firm exhausts its assets and that the firm is assumed to have a simple capital structure.

The assumption of default occurring when the firm exhausts its assets was widely criticized. These critics lead to the conclusion that a credit valuation model has to provide a genuine representation of the relationship between the state of the firm and the events that might influence the deterioration of the firm value. Pursuing this goal, a new approach to credit valuation was introduced. This approach combines theory of bankruptcy and default with modern financial theory. The first to use this new approach were Leland (1994) and Leland and Toft (1996) who consider the design of optimal structure and the pricing of debt with credit risk. They allow bankruptcy to be determined endogenously and they also examine the pricing of bonds with arbitrary maturities. Later on, Anderson and Sundaresan (1996) explicitly describe the interaction between bondholders and shareholders in a firm where there is only on type of debt. By allowing strategic behaviour by shareholders and bondholders they overcome the unrealistic approach of financial distressed we find in Merton - the presence of an exogenously given fixed absorbing barrier. Thus, by modelling the interaction between shareholders and bondholders they obtain an endogenous reorganization boundary and default does not lead immediately to liquidation. The shareholders will behave strategically, paying less than the amount stipulated

by the contract but enough to persuade the debtholders to concede rather than to incur the costs of liquidation.

On the other hand, the assumption of simple capital structure received very little attention, mainly because of complexity introduced in the models by relaxing this assumption. A first step in this direction was made by Black and Cox (1976), who developed a model for pricing subordinate debt where both senior and junior debt have the same maturity. They follow Merton's approach (1974), in which risky debt is interpreted as a portfolio containing the safe assets and a short position in a put option written on the value of the firm's assets. Their junior debt could be seen as a portfolio comprising two calls: a long position in a call with a strike price equal to the face value of the senior bond and a short position in a call with a strike price equal to the sum of the face values of the two bonds.

The theory developed till now to overcome these limitations was concerned with the evaluation of credit status for securities with the same time of maturity and from the point of view of a particular lender. However, it is also important which are the maturity time and the place of the lender's claim in the hierarchy of the debt of a firm. It is not enough that the value of the firm is sufficient for paying the debt at its maturity. If the firm cannot fulfill the payment obligations at interim periods, than the payment of the debt that has later maturity will be affected. As a result, claims that have earlier maturity and are junior may trigger default and, therefore, bankruptcy.

To emphasize the above phenomena we develop a contingent valuation model in which we consider a firm that issues equity and two bonds with different seniority and different maturity. We allow the debt to be owned by two different bondholders and in case of default on the earlier maturity bond, we allow for renegotiation. Thus, in case of default, the shareholders transfer the control to the bondholders and restructuring may take place. To understand the effect of debt restructuring and different capital structure, we compare the prices of the bonds in the above mentioned scenario with the prices of bonds in two different scenarios. Thus, to understand the effect of debt restructuring we build a scenario where we consider the case of a firm with a similar capital structure (the bonds have the same seniority and the same maturity). But in this case, we do not allow the

bondholders to step in to rescue the firm in case of default. As a result, default is always followed by liquidation. We obtain that allowing for strategic behaviour of bondholders has significant different effects on the prices of the bonds. We obtain that the price of the short-term bond increases (the spread decreases), while the price of the long-term bond decreases (the spread increases). Consequently, our model suggests that presence of renegotiation possibilities and strategic debt service when there are multiple creditors may lead to qualitatively different implications for pricing.

Finally, we want to explore the consequence of the presence of other bonds with different maturity and different seniorage. To do so we compare the prices of the two bonds obtained in the second scenario with the prices of a short-term bond and a long-term bond, respectively, in a firm where this is the only outstanding debt. Consequently, in the last scenario we consider the case of a firm with a simpler capital structure: equity and a zero-coupon bond. Firstly, the firm has as the only outstanding bond a short-term zero coupon bond. Secondly, the firm has as the only outstanding bond a long-term zero coupon bond. We build this last scenario to study the effect of an additional bond. It is quite intuitive that the presence of a senior bond decreases the value of a junior bond by comparison with the case when the junior bond is the only bond outstanding. However, we obtain also that the presence of a junior bond with earlier maturity can decrease the price of a senior bond with later maturity. There are two cases when this happens. First case is the one when the value of the firm at date 1 is small and the firm cannot pay out its debt obligations. The firm is defaulting and goes bankrupt. Since bankruptcy involves significant costs, the payments due to the senior bond are also endangered. The second case takes place when the firm is not defaulting at date 1, but the value of the firm is not too high. If the value of the firm is low, so it will lead almost surely to default at date 2, the long-term bondholder might be welcoming a liquidation at date 1 which leaves him better off.

In the second chapter we analyze how different types of information existing in the market are revealed through prices. We develop a model of insider trading in the context of an imperfectly competitive market where agents have private information either about

future prices or about supply. This distinction between price-informed traders and supply-informed traders is designed to capture the different types of information that influence the security prices at any point in time. Moreover, we will study how trade affects market liquidity and informational efficiency of prices due to their strategical choices. Our model suggests that considering the effects of different types of information in the market is essential. We obtain that despite of more information being revealed in the market, the presence of a supply informed agent leads to a decrease in market liquidity.

The first paper to address strategic issues of information in a market microstructure context was Kyle (1985). This paper investigates a model of speculative trading in which an informed insider with long-lived private information maximizes profits by exploiting strategically his monopoly power in a dynamic context. His simple model (which involves a single risk neutral informed trader and a group of uninformed liquidity traders submitting orders to a risk neutral market maker) demonstrates how the liquidity characteristics of an efficient, frictionless market can be derived from underlying information asymmetries in a dynamic trading environment which captures some relevant features of trading in organized exchanges. This framework has become a standard one for analyzing strategic noisy rational expectations markets. Our work is closely related to this and to another paper of Kyle. Kyle's (1989) paper proposes an imperfect competition model in which there are noise traders, price informed traders and uninformed traders and they submit limit orders. He shows that a strategic trader acts as he trades against a residual supply curve. This implies lower quantities by comparison with the competitive rational expectations equilibrium and, consequently, in equilibrium prices reveal less information than in the competitive case.

Our model is similar to Kyle's (1989) model in that we have N price informed traders who submit limit orders and act as imperfect competitors. However, instead of modelling noise traders we assume the existence of a random supply. The modelling is equivalent, we make this choice because we would like to consider the case when a supply informed trader exists in the market. The existence of a supply informed agent was used before by Gennotte and Leland (1990) who consider a model were speculators posses private

and diverse information. They consider price takers speculators who gather information either about prices or about supply and show that these informational differences can cause financial markets to be relatively illiquid. We can think of the supply informed agent as being a dealer who can observe the order flow. The problem of losses incurred by dealers to the informed traders because of the latter informational advantage was explained in numerous studies. However, since dealers can observe the order flow they can aggregate the information from trading and use it to earn speculative profits. Thus, the dealers can learn about the liquidation value of the asset from the orders placed by the price informed agents. The information revelation is increased significantly in our setup because the agents are placing limit orders and therefore, they condition their demands on prices and infer in this way a part of others' information. As pointed out by Brown and Zhang (1997) in a competitive market the dealers cannot earn rents on the information on the order flow. However, we will see that in our setup of an imperfect competitive market they can exploit this information in their own advantage.

We are interested to understand the effects of different types of information on market liquidity, informativeness of prices, price volatility, and the ability of informed traders to exploit their private information. Our goal is to see how market liquidity and price efficiency are influenced by strategic interaction between agents with different types of private information. Then, we study the volatility of prices, the informational content of prices (defined as the difference between the prior variance of the payoff and the variance conditional on price) and the expected profits of the traders. We obtain that the presence of the supply informed trader leads to higher volatility of prices, information efficiency and volume traded by the price informed agents. Finally, we study how changes in supply affect the equilibrium price and we obtain that the price informed trader absorb a higher proportion of a shock in the component of supply known by everyone. On the other hand, we have that the supply informed agent acts as a monopolist on his information and it absorbs always half of a shock in the component of supply known only by him.

The implications of the existence of different types of information on the market indicators is somehow similar to the models where we increase the number of informed agents

(but holders of same type of information). More information revealed in the trading process leads to an increased volatility, price efficiency and volume of trade. However, in our case the different type of information leads to a decrease in market liquidity. Unlike in the Kyle-type models (and more in the spirit of Glosten and Milgrom (1985), where more information leads to an increase in the bid-ask spread and therefore, a decrease in the market liquidity) we obtain that the existence of different types of information in the market brings about a decrease in the market liquidity. Still, if we are increasing the number of price informed traders we will still obtain the increase the market liquidity obtained in Kyle (1985,1989). The closest result to ours is the one of Subrahmanyam (1991), who obtained that market liquidity can be decreased by increasing the number of informed traders in the case traders and market maker are risk averse. Despite of the fact that the decrease in the market liquidity is due to the different type of information, our result is very similar to the one of Subrahmanyam (1991). The similitude is caused by the fact that the supply informed agent is risk neutral, but he behaves strategically. Moreover, since he submits limit orders he has a market-making role, the role played by him in the economy being thus similar to the one played by the risk-averse market maker in Subrahmanyam's (1991) model. This result originates in this differential information, but also in the trading mechanism. With asymmetric information, prices play a dual role of information aggregation and market clearing. However, here the role of prices in information transmission is even more important because the traders can infer a part of the different information through prices.

The third chapter is concerned also with microstructure issues, this time in connection with a taxation under uncertainty problem. The problem of tax evasion has been of great interest in the public finance literature, but its implications in the financial markets were not greatly exploited. The main concern of the papers studying this problem was the importance of tax evasion as a disciplinary mechanism for fiscal policy (the link between tax evasion and time inconsistency in capital taxation and the instruments the government should use to avoid tax evasion), the relationship between tax evasion and policies of financial repression, inflation rates, and economic growth. While this approach takes into

consideration macroeconomic issues, we will explore in this paper the consequences the tax evasion has on the trading in financial markets at the firm level. Thus, we would like to get greater insight into the effects tax evasion has on the insider trading profits.

The first models to study the problem of tax evasion at firm level used a portfolio selection approach. Thus, Allingham and Sandmo (1972), Yitzaki (1974) and Polinsky and Shavell (1979) use as portfolio weights the probability to be caught, in the case when all the taxpayers face a constant probability. This assumption was criticized by Reinganum and Wilde (1986) who point out that the payoff report contains information about the true realization of the payoff and consequently, the probability of auditing should depend on the report made by the taxpayers. While the above papers incorporated the uncertainty about the tax liabilities, another strand of research was concerned with the other sources of randomness that alter the interaction between the taxpayers and tax auditing agency. Mainly, they incorporated in their models the fact that tax code is complex and can lead to involuntary mistakes even when the taxpayers want to conform with the law. Thus, Scotchmer and Slemrod (1989) consider the case where the ambiguity of the law gives place to a random auditing policy depending on the interpretation given to the law. Reinganum and Wilde (1998) incorporate in the model the taxpayers's uncertainty about auditing cost, while Caballé and Panadés (2002) allow for both mistakes made by taxpayers and uncertainty about auditing cost.

We develop a model in which we consider the implications of tax report on the profits from insider trading. We model the interaction between the firm and the tax auditing agency as a principal-agent relationship with no commitment. On the other hand, the financial markets are modelled as in Kyle (1985), with the difference that the market maker will set the price conditional on two signals: the total order flow and the tax report received by the tax auditing agency. Modelling the interaction between the tax agency and the firm allows us actually to endogenize the public signal. Our model points out that the effects the interactions between the firm and the tax auditing agency have both on market maker's and manager's behaviour are significant. Thus, there exist several channels through which the tax report affects the profits of the manager. The tax report

affects the liquidation value of the firm traded in financial markets in two ways: through the direct taxes honestly paid and through the auditing effort (which in our model is contingent on the tax report).On its turn, the liquidation value affects the demand and therefore the order flow. Finally, since the market maker uses the tax report as a signal, it directly affects the pricing rule set by the market maker, and therefore, all the market performance.

We show that uncertainty about the realizations of the payoff of the firm together with the errors produced during the reporting stage, have an important effect on the reporting strategy of the firm and the auditing policy of the tax authority. Our results suggest also that the market performance becomes very sensitive to the values of the parameters, the most relevant parameter being the variance of the payoff. We obtain that endogenizing the public signal and the liquidation value of the firm affects the behaviour of profits of different market participants, market depth, informativeness of prices and volatility of prices. Most important, unlike Kyle (1985), the market indicators are not monotonic in the variance of income.

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Chapter 2

Valuation of Defaultable Debt and Debt Restructuring

2.1 Introduction

In the recent years the work of Black and Scholes (1973) and Merton (1974) on option pricing has become an important tool in the valuation of corporate debt. The option-pricing approach has been used extensively in the valuation of stocks, bonds, convertible bonds and warrants. The theoretical insights of this approach are extremely useful, but unfortunately, the predictive power of this model has been widely challenged by the empirical tests. These empirical results signaled possible limitations of the model. Two of the most important limitations are the fact that default is assumed to occur only when the firm exhausts its assets and that the firm is assumed to have a simple capital structure.

The assumption of default occurring when the firm exhausts its assets was widely criticized. These critics lead to the conclusion that a credit valuation model has to provide a genuine representation of the relationship between the state of the firm and the events that might influence the deterioration of the firm value. Pursuing this goal, a new approach to credit valuation was introduced. This approach combines theory of bankruptcy and default with modern financial theory. The first to use this new approach were Leland (1994) and Leland and Toft (1996) who consider the design of optimal structure and the pricing of debt with credit risk. They allow bankruptcy to be determined endogenously and they also examine the pricing of bonds with arbitrary maturities. Later on, Anderson and Sundaresan (1996) explicitly describe the interaction between bondholders and shareholders. They obtain in this way an endogenous reorganization boundary and deviations from the absolute priority rule. Anderson, Sundaresan and Tychon (1996) extend the previous model from a discrete-time to a continuous-time model. Using this continuous-time setup they compute closed-form solutions and perform comparative statics. Mella-Barral and Perraudin (1997) also derive closed form solution for debt and equity modeling explicitly the shutdown condition for a firm. Fries, Miller and Perraudin (1997) price corporate debt in an industry with entry and exit of firms. Allowing for contract negotiation, Mella-Barral (1999) characterizes the dynamics of debt reorganization and endogenizes departures from the absolute priority rule. Fan and Sundaresan (2000)

provide also a framework for debt renegotiation by endogenizing both the reorganization boundary and the optimal sharing rule between equity and debt holders upon default. Finally, Anderson and Sundaresan (2000) perform a comparison among the models of Merton (1974), Leland (1994), Anderson and Sundaresan (1996) and Mella-Barral and Perraudin (1997) showing that the models including endogenous bankruptcy are to some extent superior to Merton's model.

A step forward in surmounting the limitation of a simple capital structure was made by Black and Cox (1976), who developed a model for pricing subordinate debt where both senior and junior debt have the same maturity. They follow Merton's approach (1974), in which risky debt is interpreted as a portfolio containing the safe assets and a short position in a put option written on the value of the firm's assets. Their junior debt could be seen as a portfolio comprising two calls: a long position in a call with a strike price equal to the face value of the senior bond and a short position in a call with a strike price equal to the sum of the face values of the two bonds.

The theory developed till now to overcome these limitations was concerned with the evaluation of credit status for securities with the same time of maturity and from the point of view of a particular lender. However, it is also important which are the maturity time and the place of the lender's claim in the hierarchy of the debt of a firm. It is not enough that the value of the firm is sufficient for paying the debt at its maturity. If the firm cannot fulfill the payment obligations at interim periods, than the payment of the debt that has later maturity will be affected. As a result, claims that have earlier maturity and are junior may trigger default and, therefore, bankruptcy.

In this paper we develop a contingent valuation model for zero-coupon bonds with different seniority and different maturity. We are interested in studying how renegotiation of debt and capital structure of the firm affect the prices of the bonds with default. Since the debt can be held by different bondholders we permit renegotiation in case of default on the early-maturity bond and this leads to strategic behaviour by bondholders. Incorporating strategic behaviour by bondholders in the valuation framework suggests that the presence of renegotiation possibilities when there are multiple creditors may lead

to qualitatively different implications for pricing. Our approach is similar to the one of Anderson and Sundaresan (1996), but differs from it in two important points. First, we concentrate our attention on the effects of strategic behaviour of the bondholders only, the shareholders being in our model the residual claimants. Second, and more important, we consider the renegotiation of the entire amount of debt and not only on the cupon payment. This approach is used also by Christensen et al. (2002) in a single borrower setup, but the problem of renegotiating the entire amount of debt is reinforced in our case by the strategic behaviour of the two bondholders. The presence of two bondholders helps us also to emphasize the important role the bond covenants play in a firm with a reacher capital structure and when we allow for strategic behaviour of bondholders.

The remainder of this paper is organized as follows. Section 2 presents the basic valuation model. We describe directly the more complex model in which we allow for renegotiation. We present here the timing of the events, and the game that takes place between the bondholders in the case the firm is not able to honour its payments at date 1. Section 3 studies the equilibrium of the Bondholders' game. Section 4 proceeds with the valuation of the bonds. We compare the prices of the bonds in the model specified in Section 2, but also in two simpler models, the purpose of this comparison being to detect the effect on the price of bonds the capital structure of the firm and renegotiation bring about. Finally, Section 5 summarizes the results and gives some directions for further research.

2.2   The Model

There are three agents in our economy: two creditors (commercial banks, mutual or pension funds, etc.) and a firm - issuer of debt securities (corporation, commercial bank, government etc.). All three agents are risk neutral.

The creditors live for two periods and have different liquidity preference. We assume that the preferences of the two creditors are represented by the utility function $U_i(c_1, c_2) = c_1 + \delta_i c_2$, where c_1, c_2 represent the consumption of the creditors in period 1 and 2,

respectively, and δ_i represents creditor i's discount factor. To emphasize the fact that the creditors have different liquidity preferences, we assume that the discount factor is very small for the first creditor, and is very high for the second one. Consequently, the first creditor will prefer to consume in the first period and the second creditor will prefer to consume in the second period.

Consider now a simple situation in which the current liabilities of the firm are assumed to be 0. Thus, the firm has a simple capital structure: equity and debt. Let us assume that markets are complete and frictionless, there are no taxes and the agents can borrow at the riskless interest rate r.

We assume that the firm owns a project and issues two zero coupon bonds and equity to raise funds meant to cover the financial needs of this project at date 0. As a result, the initial investment in the project is equal to the total amount raised by issuing debt and equity. There is a junior bond with face value D_1 that matures at date 1, and a senior bond with face value D_2 that is due to mature at date 2. We assume that initial value of the firm is exogenous and equal to the total investment in the project. Since our economy is characterized by 0 corporate taxes, there is no distinction between the value of the assets of the firm and the value of the firm itself. This value is $V = E + B_1 + B_2$, where E is the value of equity, B_1 is the total market value of the junior corporate bond and B_2 is the total market value of the senior one. The project consists of a technology that transforms the initial investment in a random return. We model the technology as a binomial process: the value of the firm V moves up to Vu with probability p and down to Vd with probability $1 - p$, where $u > 1 > d$. In what it follows we will denote by V_i the value of the firm at time i.

At date 0 the firm issues a short-term bond B_1 which is junior and a long-term debt B_2 which is senior.[1] There are two covenants specified in the indenture of the senior bond: limitation on priority and cross-default. The limitation on priority provision restricts the

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[1] The assumption is without loss of generality and is ment to illustrate the point that junior bond with earlier maturity can trigger default on the long-term, senior bond. The case when the short-term bond is senior and the long-term bond is junior is similar with Black and Cox (1976) and it will not involve debt renegotiation.

shareholders to issue additional debt which may dilute the senior bondholder claim on the assets of the firm. In our case it requires that in the process of debt restructuring only junior bond can be issued. The cross-default provision specifies that the firm is in default when it fails to meet its obligations on any of its debt issues, that is in the case of default on the short-term debt, the senior debt becomes payable immediately.

Both bonds are subject to a positive probability of default. The existence of this positive default probability implies that the debt contracts should specify two contingency provisions: the lower reorganization boundary and the compensation to be received by the creditors when this lower reorganization boundary is reached.

The lower reorganization boundary represents the cut-off point where the liquid assets of the firm are not sufficient to meet the obligations of the debt contracts. When this cut-off point is reached, we say that financial distress takes place. As long as they meet the contractual obligations, shareholders have the residual control rights and debtholders cannot force liquidation. However, when the lower reorganization boundary is reached and, consequently, shareholders default on their debt contracts, the bondholders have a choice between allowing liquidation by court appointed trustee (Chapter 7 of U.S. Bankruptcy Code) or renegotiating the debt contracts. In the case of liquidation the firm sells its assets, pays a liquidation cost and what is left is allocated between bondholders. In the case bondholders choose to renegotiate the debt, this can be done either out of court (workout) or in court (Chapter 11 of U.S. Bankruptcy Code). Since we do not intend to model the shareholders specifically and in case of default the control of the firm is transferred from stockholders to bondholders, our renegotiation procedure will mirror the restructuring through out-of-court arrangements.[2]

A very important assumption of our model is that the compensation received by bondholders after bankruptcy follows the absolute priority rule. According to this absolute priority rule the payments to debtholders should be made before any payment is made to shareholders. Also, the payments of the debtholders are made such that the senior claim

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[2]According to Gilson et al. (1990), almost 50% of the companies in financial distress avoid liquidation through out-of-court debt restructuring. The advantage of this procedure is that workouts are usually a lot less expensive than Chapter 11 bankruptcy procedure.

payments should be always made before any payments are made to the junior claims. We also assume that in case of default of the debt contracts the debtholders can use the assets without any loss of value (except the liquidation costs).

2.2.1 Time Structure

We set up the model in discrete time because it allows the modeling of the bankruptcy process to be more transparent. The sequence of events is the following:

Date 0: The firm issues both short-term and long-term debt B_1 and B_2, respectively. The promised final payments are D_1 and D_2, respectively. Creditor 1 buys the bond B_1 and Creditor 2 buys the bond B_2.

Date 1: Maturity date of bond B_1. The stockholders pay off the Bondholder 1 if they can. If they cannot, the ownership of the firm passes to the bondholders. The bondholders decide if the firm enters a liquidation or a restructuring process. In case of liquidation, the firm pays the liquidation costs L and then the bondholders are paid according to the absolute priority rule. In case of restructuring, the firm either changes the maturity of junior debt at $t = 2$, or issues new debt with maturity at $t = 2$. We assume that there is a cost of restructuring K and this cost is smaller than the cost of liquidation L (more precisely, we assume that $K < \dfrac{r}{1+r}L$, and $L < V_0 d$).[3]

Date 2: Maturity date of bond B_2. Conditional on the fact that the firm did not get bankrupt in the previous period, the stockholders pay off the bondholders if they can. If they cannot, the firm enters in a liquidation process. The control of the firm is transferred from stockholders to the bondholders. The firm is liquidated and the bondholders are paid according to the absolute priority rule.

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[3]Empirical studies show that the costs of debt restructuring are significantly lower than the costs of liquidation.

2.2.2   The Game

At date 1, the value of the firm is V_1. The payment obligation of the firm at this moment amounts to D_1. If the value of the firm V_1 exceeds D_1, the stockholders honour the debt obligation by selling out assets that amount to D_1. Otherwise, the firm defaults and the stockholders give up the control in favour of bondholders. Once the firm defaults on one of its payments all the creditors have the right to demand information, and therefore they discover the value of the firm at date 1, V_1. If the value of the firm following restructuring, V_2^*, is expected to be very low (i.e. $E[V_2^*] \leq D_2$) both bondholders realize that issuing additional debt will not make them better off. Due to the existence of the senior bond covenant, the debt issued at date 1 has to be junior to the debt B_2 and therefore, the expected payment to this newly issued debt will be zero, no bondholder being willing to buy this debt. If the value of the firm is such that $E[V_2^*] > D_2$, the bondholders choose between liquidating and rescuing the firm. We consider the case when unanimity it not necessarily for the reorganization to be approved (see Franks and Torous (1989)). Consequently, liquidation occurs only when both bondholders are taking this decision. In the case of liquidation, the assets of the firm are sold and the payments are made to the bondholders. If one of the bondholders wants to rescue the firm, then the debt will be restructured independently of the other's action. There are different ways to restructure the debt: reducing the principal obligations, increasing maturity of the debt or accepting equity of the firm. We assume that the Bondholder 1 restructures the debt by increasing the maturity of the debt. On the other hand, if the Bondholder 2 wants to prevent liquidation he can do so only if the firm issues new debt.[4] The restructuring of the debt can be done only if the firm pays a cost K, which, for simplicity, we assume that it is the same in both cases.

Let us see now what happens at date 2. The situation is very similar, but the allocation

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[4]It does not pay for an outsider to undertake restructuring since the value of the firm is small, $V_1 < D_1$. If a new creditor is willing to invest D_1, the value of the firm at date 2 will be in expected terms $(V_1 - K)(1 + r)$ which is smaller than $D_1(1 + r)$, the amount that should be paid to the new investor. Moreover, the new issued debt has always lower seniorage than the existent debt so he will be paid only after the senior debt is paid.

of payments depends on what happened at date 1. First, if the payments for the bond B_1 where made at date 1, the only payment left to be honoured at date 2 is the senior bond B_2. In this case the value of the firm becomes $\widehat{V_1} = V_1 - D_1$. Therefore, if the value of the firm at time 2, that is $\widehat{V_2}$, exceeds the payment obligation D_2, the stockholders honour the debt obligation. Otherwise, they will liquidate the firm and obtain the assets' value $\widehat{V_2}$ net of liquidation cost. In the case the firm honoured its payment at date 1, we have to take into account that for doing so the firm is liquidating a part of its assets equal to D_1, and the value of the firm decreases therefore by this amount $\widehat{V_1} = V_1 - D_1$.

Second, if at date 1 we had default on the obligation, three possible cases might occur: liquidation, rescue by Bondholder 1, and rescue by Bondholder 2. If liquidation takes place at date 1, the game is already over. The firm sells out the assets, pays a liquidation cost L and makes the payments according to the priority rule. Bondholder 1 owns the senior bond and he will receive $\min\left\{V_1 - L, \dfrac{D_2}{1+r}\right\}$. Bondholder 2 will receive what is left, i.e. $\max\left\{V_1 - \dfrac{D_2}{1+r} - L, 0\right\}$.

When restructuring takes places, the firm is paying the restructuring cost K, and thus, the value of the firm becomes $V_1^* = V_1 - K$. If the restructuring of the firm is made by Bondholder 1, at date 2 he will be entitled to a payment D_1' which is junior to D_2. If the value of the firm at date 2, V_2^* exceeds the total payment obligation $D_1' + D_2$, the stockholders honour the debt obligation. Otherwise, the Bondholder 1 will receive $\max\{0, V_2^* - D_2\}$ and Bondholder 2 will receive $\min\{V_2^*, D_2\}$. If the firm is in default at date 2, we have to subtract the liquidation cost from these payoffs. In order to keep it simple at this point we will write the exact formula for these payoffs later on. Finally, if the rescue of the firm was made by Bondholder 2, at date 2 the Bondholder 2 will own two bonds and he will be entitled to a payment of $D_1'' + D_2$. The payment he receives depends again on the realization of V_2^* and it is $\min\{D_1'' + D_2, V_2^*\}$.

When the Bondholder 2 is willing to pay the debt, the firm will issue new debt which amounts to D_1. If Bondholder 2 is the only one to rescue the firm, the Bondholder 1 will receive exactly the amount he received in case of liquidation $\max\left\{V_1 - \dfrac{D_2}{1+r} - L, 0\right\}$,

the amount $D_1 - \max\left\{V_1 - \dfrac{D_2}{1+r} - L, 0\right\}$ being used for increasing the value of the firm.

Hence, the value of the firm will be in this case $V_1^{**} = V_1 + D_1 - K - \max\left\{V_1 - \dfrac{D_2}{1+r} - L, 0\right\}$.

Finally, in the case both bondholders are willing to rescue the firm, the firm will accept both offers, the new value of the firm becoming in this case $V_1^{***} = V_1 + D_1 - 2K$. The firm will postpone the debt due to Bondholder 1 by changing the face value of the debt to D_1' and also by issuing new debt with face value D_1''. The two new types of debt are junior to the debt B_2 and they have the same seniority.

The payments made at date 2 in the case of restructuring for the new debt D_1' and D_2'' are chosen such that there exist no arbitrage opportunities between the first and second period.

2.3   The Equilibrium of the Bondholders' Game

We study now the case when the firm is not able to meet its payment obligation at date 1, i.e. $V_1 < D_1$, but the value of the firm is still high enough to allow for restructuring, meaning $E[V_2^*] \geq D_2$. This can be written equivalently as

$$V_1 - K \geq \frac{D_2}{pu + (1-p)d}.$$

Let us define $\overline{V}$ as

$$\overline{V} = \frac{D_2}{pu + (1-p)d} + K.$$

As we have already explained, the ownership of the firm passes into the hands of the bondholders and they decide whether to rescue or to liquidate the firm. We assume that the bondholders have complete information, the game is common knowledge, and that they act in their own interest. Moreover, at the beginning of the game, they can observe the realization of the firm value, V_1.

Equilibrium in the bondholders' game consists of the actions of the bondholders that constitute the best response. When making the decision the bondholders have to take into consideration both current period payoff and continuation payoff.

In order to characterize the solution we need to specify the following notations. The actions of Bondholder 1 are $\{L_1, R_1\}$ and the actions of Bondholder 2 are $\{L_2, R_2\}$, where L_i means that bondholder i chooses to liquidate the firm and R_i means that the bondholder i chooses to restructure the firm.

Proposition 2.1 *In the equilibrium Bondholder* 1 *chooses to restructure,* R_1, *and Bondholder 2 chooses to liquidate,* L_1.

The capital structure of the firm and the covenants of the senior debt play a very important role in our model. While the cross-default provision brings about the renegotiation of the debt contracts, the limitation on priority drives the equilibrium of the bondholders game. As we have seen already the value of the firm is utmost when both bondholders are willing to restructure the firm. The Pareto efficient equilibrium consists of bondholders restructuring and invigorate thus the firm through their action. However, in equilibrium Bondholder 2 chooses to liquidate. The grounds of his decision comes from the fact that his overall position in the hierarchy of debt is downgraded. At the beginning he had a senior bond. If both bondholders undertake restructuring Bondholder 2 will have a senior bond as before but also a junior bond. This last bond has actually the same seniority as the seniority of the bond owned by Bondholder 1 and therefore the payments on these two junior bonds will be made at once. Therefore, the payments of the Bondholder 2 are reduced and in consequence he chooses to liquidate the firm.

There are also two other important issues to be taken into account when solving for the equilibrium: Bondholder 1 owns a junior debt and default occurs when the value of the firm is very small. First, Bondholder 1 owns a junior debt and he receives his payment after the senior bond payment is made. Therefore, the smaller the value of the firm, the smaller the amount that is left after senior bond payment. As a result, his best response to any of Bondholder 2 actions is to restructure and increase the value of the firm. Thus, if Bondholder 2 wants to liquidate, Bondholder 1 is obviously better off by restructuring since restructuring gives him at least as high equal expected payoff. This happens because the bondholder will never undertake restructuring when the expected

payoff is smaller than the present value of the debt (see the non-arbitrage condition). If Bondholder 2 wants to rescue, Bondholder 1 is gaining even more because the value of the firm is increased more by the participation of Bondholder 2, but the newly issued debt for both bondholders has the same seniority.

Second, default occurs when the value of the firm is small. Since Bondholder 2 knows that and owns a senior bond, it does not pay for him to reinvest and accumulate debt. He prefers to leave Bondholder 1 to rescue the firm. As a result, Bondholder 2's best response to R_1 is L_2. In the case the value of the firm net of liquidation costs is still high enough to cover the debt due to him $\dfrac{D_2}{1+r}$, we have that the best response of Bondholder 2 when Bondholder 1 chooses to liquidate is to liquidate. We also obtain that, for some small values of the parameters, the best response to L_1 is to restructure. However, for these values we have already argued that the bondholders are not going to invest and accumulate more debt because if they do, they are going to lose. Under these circumstances, we can conclude that the equilibrium of the game is (R_1, L_2).

Corollary 2.1 *The equilibrium of the bondholders' game is preserved even when $K = L = 0$.*

If we substitute the parameters $K = L = 0$ in the proof of Proposition 1, the proof is still valid. The corollary emphasizes the fact that the equilibrium of the bondholders' game is driven by the capital structure of the firm (and the presence of covenants) and not by liquidation costs. This happens again only for the values of the parameters for which restructuring makes sense, i.e. in this case $V_1 \geq \dfrac{D_2}{pu + (1-p)d}$.

Once the bondholders announced their decisions, the shareholders are compelled to follow the decisions of the bondholders. They play a passive role since the ownership was already conceded to the bondholders. Since the cost associated with the restructuring process is the same, independent of who is restructuring the firm, the shareholders are indifferent between changing the maturity of bond B_1 at $t = 2$ and issuing new debt.

2.4   The Valuation of the Bonds

In order to price a bond we have to compute the present value of the expected bond payments. The prices of the bonds are influenced by the characteristics of the project to be undertaken but also by the structure of the firm. We focus on determining the lower reorganization boundary and the compensation to be received by bondholders and shareholders. Once we know the payments received by every agent, we can compute the prices at date 0 for the two bonds and equity by computing the net present value of future payments. We will determine the prices of the bonds in three different setups and compare the corresponding prices.

First, we will compute the prices of the two bonds in the model we presented above. We are interested in finding out how introducing debt renegotiation will affect the value of the bonds. For this purpose, we will compare the prices we obtained, B_1 and B_2 with the prices of two similar bonds (the same maturity date and the same debt face value) B_1'' and B_2''. The bonds B_1'' and B_2'' are issued by a firm with a similar capital structure, but in which the bondholders are not allowed to restructure the firm in case of default.

We will see that changes in the characteristics of the project (which can be seen as caused by changes in the credit quality of the issuer) are inducing different bond prices. However, it is not the case that only the characteristics of the project are influencing the valuation of the bonds. The prices of the bonds can also be influenced by the presence of other bonds with different maturity or different seniority. To isolate this effect we compare the prices of the short-term bond B_1'' with the price of a short-term bond B_1' and the price of the long-term bond B_2'' with the price of a long-term bond B_2'. The bonds B_1' and B_2' are bonds with similar face value to B_1'' and B_2'' and each of them is a bond in a firm where this is the only debt outstanding.

Before proceeding with the valuation, let us first determine the equilibrium market interest rate. We assume that the firm that owns the project V_0 is financed completely with equity. We determine the interest rate from the following non-arbitrage condition: an investor should be indifferent between investing in equity in the firm fully financed by

equity or in a riskless asset. On the one hand, the expected payoff from investing \$1 in equity is the total expected payment of the project divided by the value of equity q, i.e. $\dfrac{pV_0u + (1-p)V_0d}{q}$. Since the firm is financed fully with equity, we obtain that $q = V_0$, and therefore, the expected payoff from investing \$1 in equity is $pu + (1-p)d$. On the other, the expected payment from investing \$1 in the riskless asset is $1 + r$. As a result, our non-arbitrage condition, becomes $pu + (1-p)d = 1 + r$.

2.4.1 Valuation of Bonds in Case of Restructuring

As we already mentioned in the presentation of the model, the payments in the case of restructuring have to be such that there are no arbitrage opportunities. First, the Bondholder 1 should be indifferent between the payment he is entitled to receive this period, D_1 and the expected payment he will obtain next period if he decides to postpone the maturity of the bond D_1'.[5] Thus, we have $D_1 = \dfrac{1}{1+r}E_1\left[\min\{V_2^* - D_2, D_1'\}\right]$, where as explained above, $V_1^* = V_1 - K$. Second, the Bondholder 2 should be indifferent between rescuing the firm by paying D_1 at date 1 and receiving D_1'' next period. However, he is aware of the fact that if he restructures the firm, its value at date 1 will increase at least by $D_1 - \max\left\{V_1 - \dfrac{D_2}{1+r} - L, 0\right\}$. If we define $V_1^{**} = V_1 - K + D_1 - \max\left\{V_1 - \dfrac{D_2}{1+r} - L, 0\right\}$, we can write the arbitrage condition $D_1 = \dfrac{1}{1+r}E_1[\min\{V_2^{**} - D_2, D_1''\}]$.

The price of the two bonds will depend critically on the relationship between the two debt face values D_1 and D_2.

Remark 1 *In case of default none of the bondholders will be willing to rescue the firm if* $D_1 < \dfrac{D_2}{pu + (1-p)d} + K \equiv \overline{V}.$

In case of default we have $V_1 \leq D_1$. However, the bondholders are willing to rescue the firm only if $V_1 \geq \overline{V}$. Since $V_1 \leq D_1 < \overline{V}$, although we allow the bondholders to renegotiate they will not be willing to restructure the firm. Hence, the value of the bonds and equity will be the same as in the case when we do not allow for renegotiation. Therefore, the

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[5]The bondholder is choosing exactly the quantity that makes him indifferent. He will never choose a larger amount because by doing so either he receives $V_2^* - D_2$ or an arbitrage opportunity exists.

interesting case for our analysis is the case when $D_1 \geq \overline{V}$. While looking at the effects of debt renegotiation on the prices of bonds we will concentrate our attention only on this case because this is the case when the strategic behaviour of bondholders might lead to restructuring.

However, when $\overline{V} \leq D_1$ we will have both cases when the bondholders are willing to restructure and cases when they are not. Thus, if $V_1 \leq \overline{V}$ the bondholders will not be willing to rescue the firm since for these values of V_1 the expected value of the firm is less than D_2, the face value of debt due to Bondholder 2 at date 2. Since the debt issued at date 1 is junior to debt D_2 of Bondholder 2, the expected payment is 0, and none of the creditors is willing to buy this debt. If $\overline{V} \leq V_1 \leq D_1$, the expected payment to the newly issued debt is positive and the bondholders play the game described above. The payoffs of the two bondholders (and therefore, the valuation formula of the bonds) depend both on the face values of the debt and on the initial value V_0.

In the case when $D_1 \geq \overline{V}$, the strategic behaviour of the bondholders affects the payoffs of the bonds at date 1, and thus, the valuation formula is changed. In this case liquidation occurs for values of the firm smaller than a new threshold $\overline{V} = \dfrac{D_2}{pu + (1-p)d} + K$, this threshold being smaller than the threshold we had before (D_1). If $V_1 \leq \overline{V}$, we have default; the firm liquidates its assets and the bondholders share the payments. The payoff of Bondholder 1 is $\max\left\{V_1 - L - \dfrac{D_2}{1+r}, 0\right\}$, while the payoff of Bondholder 2 is $\min\left\{V_1 - L, \dfrac{D_2}{1+r}\right\}$. If $\overline{V} \leq V_1 < D_1$, the firm is not able to honour its debt obligation, but it is not liquidated. In this case, the bondholders decide to restructure the debt. In equilibrium, Bondholder 1 rescues the firm the payoffs of the two bondholders being $\dfrac{1}{1+r}E_1\left[\max\{0, \min\{V_2^* - D_2, D_1'\} - L \cdot I_{\{V_1 | D_2 < V_2^* < D_2 + D_1'\}}(V_1)\}\right]$ for Bondholder 1 and $\dfrac{1}{1+r}E_1[\min\{V_2^* - L, D_2\}]$ for Bondholder 2. In case the firm does not default at date 1 the payoff of Bondholder 1 is D_1. The Bondholder 2 waits till date 2, the maturity date of its debt, and he receives then $\min\{\widehat{V}_2, D_2\} - L \cdot I_{\{\widehat{V}_2 | \widehat{V}_2 < D_2\}}(\widehat{V}_2)$. He receives the entitled debt D_2 if the value of the debt is smaller than the value of the firm. Otherwise, he receives the value of the firm net of liquidation costs.
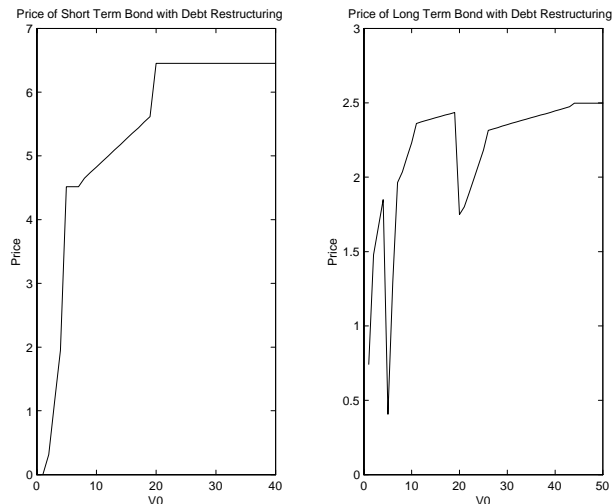
Figure 2.1: **Prices of the short-term bond and the long-term bond when debt restructuring is allowed**. The values of parameters are: $D_1 = 10, D_2 = 6, K = 0.4, L = 0.02, p = 0.7, u = 2, d = 0.5$.

As we can see in Figure 2.1 the price of the short-term bond is increasing in V_0. There are two kinks in the price function. The first one is the result of the upper state value of the firm becoming higher than the face value of the short-term debt D_1 (in our example when $V_0 = 5$) and the second when also the lower state value of the firm exceeds this amount ($V_0 = 20$). However, the price of long-term debt is not anymore an increasing function of V_0. When the value of the firm becomes higher than the face value of the short-term debt D_1, the firm is selling off assets amounting to D_1, and therefore, the value of the firm is decreasing. Consequently, for these values of V_0 we detect a sharp decrease in the price of the long term bond.

The price of equity is computed in the same manner. According to the priority rule the equity owners are the last ones to be paid. So, if we had default at date 1 they would receive nothing. Then, if the value of parameters still allows for restructuring, we have two cases. First, if after restructuring we have default at date 2, the equity owners will be left with nothing. Secondly, if we do not have default at date 2, they will receive the value of the firm net of the payments due to the two bondholders $V_2^* - D_1' - D_2$. If we did not have default at date 1, the equity owners would receive the value of the firm minus the payment

to the Bondholder 2. Thus, if we reach the date 2, the equity owners will receive nothing in case we have default at date 2, and $\widehat{V}_2 - D_2$ in case we do not have default at date 2. Once we have found the valuation formula for the two bonds and for equity, we can also compute the value of the firm V and we notice that we do not obtain the initial value of the project because we have to subtract liquidation and restructuring costs. As expected, in the case where we have these costs the Modigliani-Miller theorem does not hold good. Since in our model we allow for renegotiation, we want to see if this assumption jeopardizes the accomplishment of Modigliani-Miller theorem. For that we assume that the restructuring and liquidation costs are zero, so we can eliminate their disturbing effect. Once all the other assumptions of Modigliani-Miller theorem are fulfilled, we see what happens in our model. The first step is to determine how the behaviour of the agents changes when we set $K = L = 0$. As we already stated in the Corollary 2, the equilibrium of bondholders game is the same when we set $K = L = 0$. Consequently, the payoffs of the two bondholders are exactly the same, except that we substitute $K = L = 0$ in the respective formulas. Since renegotiation does not involve any dissipative cost and the outcome of the project is divided between the agents, we obtain the following result:

Lemma 2.1 *If $K = L = 0$, the Modigliani-Miller theorem holds even when renegotiation is permitted.*

We obtain, hence, that the value of the firm remains the same even when we allow for renegotiation. The allocation of the payoffs is different when we allow renegotiation, but in the absence of liquidation and restructuring costs the value of the firm is unchanged. However, we will see later that the presence of renegotiation will offset the effect of liquidation and restructuring costs on the value of the firm when $\dfrac{r}{1+r}L > K > 0$.

2.4.2 Valuation of Bonds when They Are the Only Outstanding Bonds

As shown above, the payments of the two bonds, and therefore, the values of the bonds depend on the face values of the debt and on the initial value V_0. Let us consider now the following two cases of a similar firm (with a similar project) but with a different capital structure. First, we consider a firm with only one outstanding bond, a bond with maturity date at $t = 1$ and with face value D_1 and equity E'. Second, we consider the case of a firm with only one bond outstanding, a bond with maturity date 2 and with face value D_2 and equity E''.

If we assume that at date 1 the only outstanding debt is B_1', the cash flow depends only on the realization of the value of the firm V_1. If the value of the firms is high enough to pay the debt, $V_1 \geq D_1$, the bondholder receives what he is entitled to (i.e. D_1). Otherwise, he receives the amount that results from liquidating the firm. Since we assume that liquidation is costly, in the case $V_1 \leq D_1$ we have to subtract from the value of the firm the liquidation cost L. We have also computed the price of equity. As expected, the shareholders obtain nothing in case of default at date 1 and they receive $V_1 - D_1$ in case of non-default.

We consider now the second case where the firm issues a bond with maturity date 2 and with face value D_2 and it issues equity E''. When the only outstanding debt is B_2' at date 2, we are interested only if the value of the firm at $t = 2$, V_2, is high enough to pay the debt. If this is the case, i.e. $V_2 \geq D_2$, the bondholder receives what he is entitled to (i.e., D_2). Otherwise, he receives the amount that results from liquidating the firm $V_2 - L$. Similarly to the previous case, we find the price of equity. In case of default, the shareholders do not receive anything. Otherwise, they receive what is left after the payment is done to the bondholder who owns B_2'', $V_2 - D_2$.

2.4.3 Valuation of Bonds without Restructuring

We consider now a firm with the following capital structure: equity, a zero-coupon bond B_1'' with maturity date $t = 1$ and face value D_1, and a zero-coupon bond B_2'' with maturity date $t = 2$ and face value D_2. However, we assume now that the bondholders are not able to rescue the firm in case of default at date 1. We obtain the prices for the two bonds B_1'' and B_2'' in a similar manner to the case when we do allow for debt restructuring.

If $V_1 \leq D_1$, we have default on the junior debt at date 1. Since we do not allow the bondholders to rescue the firm, the default at date 1 will trigger liquidation. The firm sells its assets, pays the liquidation costs L and then the bondholders are paid according to the priority rule. The payoff of Bondholder 2 is $\min\left\{V_1 - L, \dfrac{D_2}{1+r}\right\}$. He is the first one to be paid since his debt is senior to the debt owned by Bondholder 1. Bondholder 1 receives what is left, i.e. $\max\left\{V_1 - L - \dfrac{D_2}{1+r}, 0\right\}$. In case the firm does not default at date 1 the payoff of Bondholder 1 is D_1. The value of the firm decreases by this amount and becomes $\widehat{V_1} = V_1 - D_1$. Bondholder 2 waits till date 2, the maturity date of its debt. If the value of the debt is smaller than the value of the firm, he will receive the entitled debt D_2. Otherwise, he receives the value of the firm net of liquidation costs $\widehat{V_2} - L$. Again, since equity owners are the last ones to receive their payments (according to priority rule), in case of default at date 1 they do not receive anything. Then, if default does not occur at date 1, they will receive at date 2 what is left after payment is made to Bondholder 2. They do not receive anything in case of default at date 2, and they receive $\widehat{V_2} - \min\{\widehat{V_2}, D_2\}$ in case of non-default.

If we compare the values of the firm we obtained in the two cases (in the cases with and without restructuring), we notice that for the parameters values for which restructuring takes place ($\overline{V} < V_1 \leq D_1$) the value of the firm changes from $V_1 - L$ to $\dfrac{1}{1+r}E_1\left[V_2^* - L \cdot I_{\{V_2^* | V_2^* \leq D_2 + D_1'\}}(V_2^*)\right]$. We assumed that $K < \dfrac{r}{1+r}L$ which implies $V_1 - L < (V_1 - K) - \dfrac{1}{1+r}E_1\left[L \cdot I_{\{V_2^* | V_2^* \leq D_2 + D_1'\}}(V_2^*)\right]$. It is interesting to notice that if the liquidation and bankruptcy costs are different from 0, the strategic behaviour induces a change in the value of firm, and therefore, it has an offsetting effect in the violation of

the Modigliani-Miller theorem.

In computing the prices we use the interest rate to discount the payments received. To perform the comparison we need to see if the interest rate is indeed the same in all the cases. We compare the equilibrium interest rate in the case of a firm financed entirely by equity with our two cases when the firm is financed by equity, short-term debt and long-term debt and we allow or not for renegotiation. As the following lemma shows, since the bondholders and shareholders are building their expectations rationally, neither the different structure of the firm nor the presence of renegotiation changes the equilibrium interest rate

Lemma 2.2 *If the firm is financed by equity, short-term debt and long-term debt, the interest rate still satisfies $pu + (1-p)d = 1+r$ independently of the fact that we allow for renegotiation or not.*

2.4.4   Price Comparison

Let us consider now the two cases of the firm with the same capital structure: a short-term bond, a long-term bond and equity, the difference lies in the fact that we allow or not for restructuring in case of default. We compare the prices of the two short-term bonds B_1 and B_1'' and of the two long-term bonds B_2 and B_2'', respectively.

When deriving the equilibrium of the game we obtained that the Bondholder 1 is better off undertaking restructuring independently of the action of the Bondholder 2. Since in equilibrium the Bondholder 1 chooses to restructure, it is obvious that his payoff has to be higher when restructuring takes place than when liquidation occurs. Moreover, by restructuring he postpones or avoids costly liquidation giving the firm the possibility to recover. As a result, we obtain that his expected payments are higher and consequently, that the price at date 0 of the short-term bond is higher if strategic interaction between bondholders is allowed.

In Figure 2.2 we see that there are two ranges for V_0 where the price of the short-term bond is higher in the case we allow for debt restructuring. The values of V_0 for which this
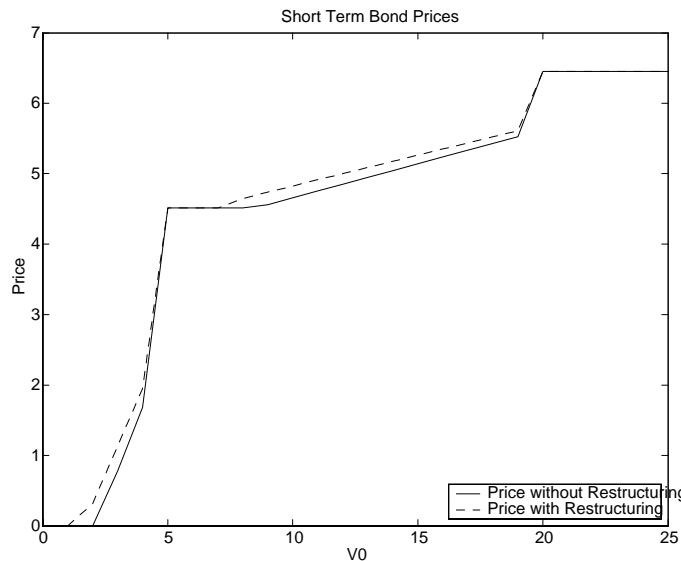
Figure 2.2: **Comparison of the short-term prices.** The values of parameters are: $D_1 = 10$, $D_2 = 6$, $K = 0.4$, $L = 0.02$, $p = 0.7$, $u = 2$, $d = 0.5$.

happens are exactly the two possible cases when the value of the firm is lower than D_1 but higher than $\overline{V}$. Since the price of the short-term bond is higher we have a decrease in the spread of short-term bond when strategic behaviour is allowed.

However, the equilibrium payoff of the Bondholder 2 is lower than in the case both bondholders liquidate. Since the best response of the Bondholder 1 is to restructure when Bondholder 2 liquidates, liquidation by both bondholders will not be an equilibrium for $V_1 > \overline{V}$. So, Bondholder 2 ends up with a payoff lower than in the case we do not allow for debt restructuring. Since in the case we allow for restructuring the expected payoffs are smaller, we will have also that the price at date 0 of the long-term bond is smaller and consequently, the spread is higher. Similarly to the case of short-term bond prices we have two regions where price differ. This can be easily seen in the Figure 2.3.

As we have already explained, the bondholders' payoffs are significantly changed when we allow for debt restructuring. However, this is not the only issue here. We are interested to see how the presence of strategic behaviour is reflected in the payments, and consequently, in the prices of bonds, but also to understand what lies behind these changes. Thus, there are different channels in which strategic behaviour comes into play: through
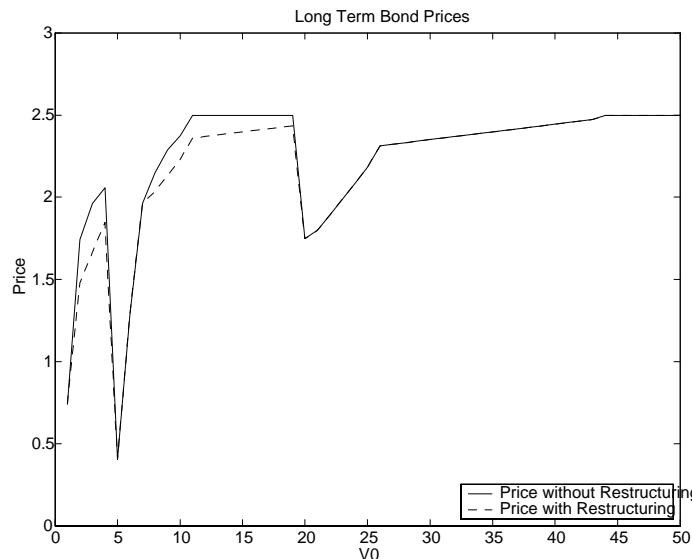
Figure 2.3: **Comparison of the long-term prices.** The values of parameters are: $D_1 = 10, D_2 = 6, K = 0.4, L = 0.02, p = 0.7, u = 2, d = 0.5$.

the change in the value of the firm, through the reallocation of payments, through the possible changes in the hierarchy of debt or avoiding costly liquidation.

We should also emphasize that in our models the bankruptcy and restructuring costs are anticipated by the bondholders and therefore, they are incorporated in prices. The same happens with the bankruptcy procedure. In the case when the bondholders do not restructure the debt, the bankruptcy code predicts liquidation similar to Chapter 7 of U.S. Bankruptcy Code. Our model suggests thus, that the prices of the bonds are also affected by the bankruptcy procedure.

Consider finally, the short-term bonds B_1'' and B_1'. As we have already explained, the two bonds have the same face value D_1 and maturity date 1. Their difference lies in the fact that they are outstanding bonds in firms with different capital structure. We compare the two prices and we obtain that the short-term bond has a higher price when this is the only outstanding bond. The result is very intuitive. Since in case of default the payments are made according to the priority rule, the price of a junior bond is influenced by the presence of another, senior bond. In case of default, the owner of the bond B_1' is paid immediately after the liquidation costs are paid, while the owner of the bond
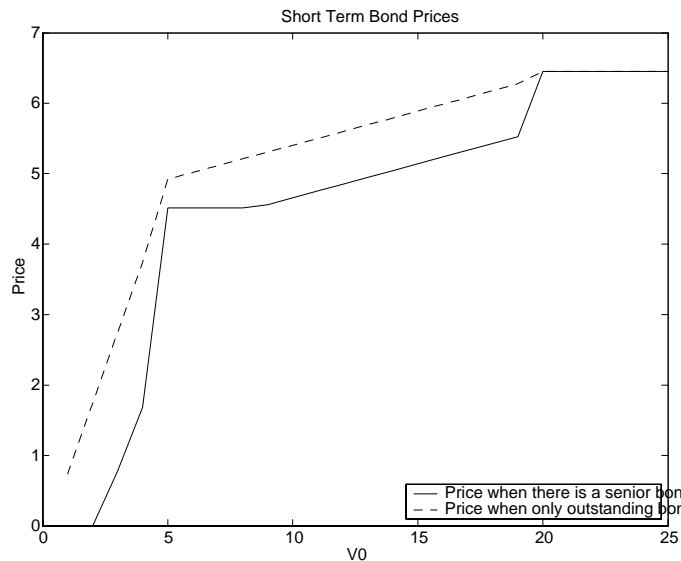
Figure 2.4: **Comparison of the short-term prices in two firms with different capital structure.** The values of parameters are: $D_1 = 10, D_2 = 6, K = 0.4, L = 0.02, p = 0.7, u = 2, d = 0.5$

B_1'' has to wait also for the senior debt to be paid. So, the price of the bond B_1'' with maturity $t = 1$ is strictly lower in the presence of another senior debt (even if this senior debt has later maturity).

We have also compared the price of the long-term bond B_2'' with the price of the bond B_2'. It is quite intuitive that the value of a senior debt is lower or equal in the presence of a junior debt with earlier maturity because the payment done to Bondholder 1 at date 1 decreases the value of the firm, and therefore, may decrease the payment to Bondholder 2 at date 2. However, this does not seem to be always the case. If at date 1, we have default on the obligation D_1, we have liquidation and Bondholder 2 receives $\min\{V_1 - L, \frac{D_2}{1+r}\}$. Assume that the value of the firm is high enough, such that $\min\{V_1 - L, \frac{D_2}{1+r}\} = \frac{D_2}{1+r}$. Consider now what happens with the bond B_2'. If the value of the firm is low enough to give rise to default of the second firm at date 2, the payment to B_2' is going to be lower than $\frac{D_2}{1+r}$ and therefore, the price of the bond B_2'' is going to be higher than the price B_2'. The insight is simple and it is the consequence of the fact that the bond B_2'' is a senior bond. If the value of the firm is low, so it will lead almost surely to default on B_2' in
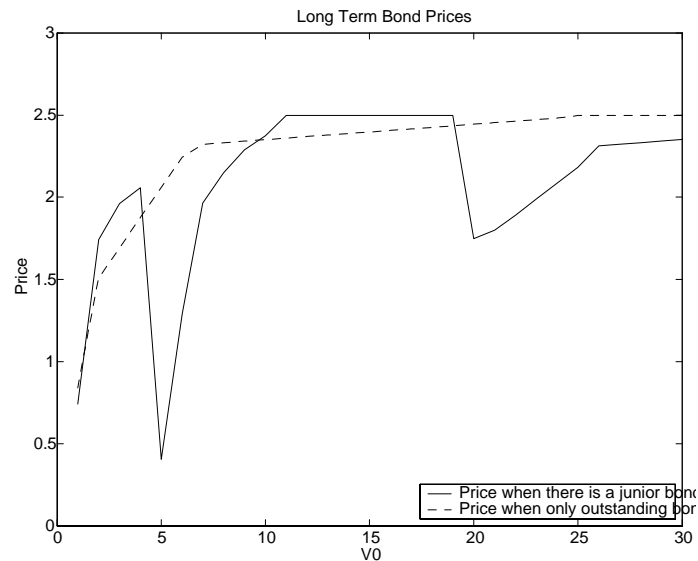
Figure 2.5: **Comparison of the long-term prices in two firms with different capital structure.** The values of parameters are: $D_1 = 10, D_2 = 6, K = 0.4, L = 0.02, p = 0.7, u = 2, d = 0.5$

several states at date 2, Bondholder 2 might be welcoming a liquidation at date 1 which leaves him better off. However, he can be better off by cashing in its payment at date 1, only if the value of the firm is not too low. The two regions in Figure 2.5 where the price of the senior bond is higher when there exists a short-term bond correspond to this case.

2.5   Conclusions

In this paper we attempt to derive the prices of debt and equity and to analyze the implications of strategic behaviour and capital structure of a firm on the prices of bonds. Our main result is that both strategic behaviour and the capital structure of the firm have important effects on the prices of bonds. To study these implications we set our problem in three different backdrops and we compare the prices we obtain. The whole analysis has been conducted focusing on the determination of the lower reorganization boundary and on the payoffs received by agents. We investigate first the inference of strategic behaviour of the agents on the prices of the bonds. For that we compare the prices of the short-term

and long-term bond in two firms with a similar capital structure. The only difference between the two firms lies in the behaviour of the agents in case of default at date 1, in one setting allowing the agents to step in and restructure the debt. The model allows us also to understand the importance of the covenants of the bonds in the case there are multiple creditors. We conclude that allowing for strategic behaviour of bondholders leads to important changes in prices of bonds. In addition, we obtain an increase in the value of the firm, but this increase takes place only when liquidation and restructuring cost are different from zero. We conclude therefore, that strategic behaviour offsets partially the loss provoked by these costs. However, the strategic behaviour by itself does not lead to any detriment in the value of the firm because it just reallocates the present funds. Thus, when there are no liquidation and restructuring costs, we obtain that the Modigliani-Miller theorem holds.

Secondly, we consider the effect of capital structure of the firm on the prices of bonds. We compare the prices of the short and long-term bond in the previous firm (without strategic behaviour) with the prices of a short and a long-term bond, in a firm where the short-term bond and respectively the long-term bond are the only outstanding debt. It is quite intuitive that the presence of a senior bond decreases the value of a junior bond by comparison with the case when the junior bond is the only bond outstanding. However, we obtain also that the presence of a junior bond with earlier maturity can decrease the price of a senior bond with later maturity. There are two cases when this happens. First case is the one when the value of the firm at date 1 is small and the firm cannot pay out its debt obligations. The firm is defaulting and goes bankrupt. Since bankruptcy involves significant costs, the payments due to the senior bond are also endangered. The second case takes place when the firm is not defaulting at date 1, but the value of the firm is not too high. At date 1 the firm is paying the untitled debt D_1 and to do that needs to liquidate a part of its assets. Hence, the value of the firm V_1 decreases by D_1 and this induces a higher likelihood of default at date 2 on the senior bond.

Simple in essence, our model suggests that the presence of multiple creditors and of a reacher capital structure is an important issue to be considered in pricing corporate debt.

Finally, we mention a possible extension of our work. As has been already pointed out by Anderson, Sundaresan and Tychon (1996), the analysis of strategic contingent claims comes at the cost of a substantial calculation time. To overcome this difficulty they recast the Anderson-Sundaresan (1996) model in continuous time. Conditional on the success of the remodeling in continuous time, we could proceed in replicating the work of Anderson *et al.* (1996). As we already mentioned, they compare Merton's model with the one in Anderson and Sundaresan (1996). It will be interesting to make a similar comparison between our model with strategic claims and the one without strategical claims. This last model will be a slightly modified version of Black and Cox (1976), where we have to allow for different maturity dates. However, the task is not trivial in our model because we consider the restructuring of entire debt and therefore we will not be able to use the limit technique used by Anderson et al. (1996).

2.6 Appendix

Proof of Proposition 1. Before proceeding with the computation of the equilibrium of the game, let us write the payoffs in a simpler way. As we already explained, in case of liquidation the bondholders are splitting the assets of the firm according to the priority rule. Then, Bondholder 1 receives

$$P_1(L_1, L_2) = \max\left\{0, V_1 - L - \frac{D_2}{1+r}\right\} \tag{2.1}$$

and Bondholder 2 receives

$$P_2(L_1, L_2) = \min\left\{V_1 - L, \frac{D_2}{1+r}\right\}. \tag{2.2}$$

If Bondholder 1 liquidates and Bondholder 2 rescues Bondholder 1 is paid at date 1. Of course, the shareholders and Bondholder 2 will not be willing to pay him more than he will receive in case of liquidation. Therefore, his payoff is

$$P_1(L_1, R_2) = \max\left\{0, V_1 - L - \frac{D_2}{1+r}\right\}. \tag{2.3}$$

On the other hand, Bondholder 2 will invest D_1 at date 1, and he will be entitled at date 2 to a payment to the senior bond D_2 and also to the payment to the new bond D_1''. Note that we subtract the liquidation costs if at date 2, the firm is not able to honour its payment.

$$P_2(L_1, R_2) = -D_1 + \frac{1}{1+r} E_1 \left[\min\{V_2^{**}, D_2\} + \min\{V_2^{**} - D_2, D_1''\} \right.$$
$$\left. -L \cdot I_{\{V_2^{**} | V_2^{**} < D_2 + D_1''\}}(V_2^{**}) \right]. \tag{2.4}$$

We know that, due to the condition of non-arbitrage opportunities, D_1'' is such that

$$D_1 = \frac{1}{1+r} E_1[\min\{V_2^{**} - D_2, D_1''\}]. \tag{2.5}$$

It can be seen that

$$\min\{V_2^{**}, D_2\} - L \cdot I_{\{V_2^{**} < D_2 + D_1''\}}(V_2^{**}) = \begin{cases} V_2^{**} - L, & if \ V_2^{**} < D_2 \\ V_2^{**} - L, & if \ D_2 \le V_2^{**} < D_2 + L \\ D_2, & if \ D_2 + L \le V_2^{**} < D_2 + D_1'' \\ D_2, & if \ D_2 + D_1'' \le V_2^{**} \end{cases}$$
$$= \min\{V_2^{**} - L, D_2\}.$$

As a result, we can write further that

$$P_2(L_1, R_2) = \frac{1}{1+r} E_1 \left[\min\{V_2^{**}, D_2\} - L \cdot I_{\{V_2^{**} | V_2^{**} < D_2 + D_1''\}} \right] =$$
$$= \frac{1}{1+r} E_1[\min\{V_2^{**} - L, D_2\}]. \tag{2.6}$$

If Bondholder 1 rescues and Bondholder 2 liquidates, Bondholder 1 will be entitled at date 2 to a payment of D_1' which is junior to D_2. As a result, the payoff to Bondholder 1 is

$$P_1(R_1, L_2) = \frac{1}{1+r} E_1[\max\{0, \min\{V_2^* - L - D_2, D_1'\}\}] \tag{2.7}$$

and the payment to Bondholder 2 is calculated as in the previous case

$$P_2(R_1, L_2) = \frac{1}{1+r} E_1[\min\{V_2^* - L, D_2\}]. \tag{2.8}$$

Finally, if both bondholders are willing to rescue, Bondholder 2 will invest D_1 at date 1 and receive at date 2 D_2 (if possible). After this payment is made the bondholders are splitting equally the remained amount. Note that due to the fact that the value of the firm is small, the firm will not be able to pay thoroughly the junior debt[6]. Hence, the payoff of Bondholder 1 is

$$P_1(R_1, R_2) = \frac{1}{1+r} E_1 \left[\max \left\{ 0, \frac{V_2^{***} - L - D_2}{2} \right\} \right] \tag{2.9}$$

while the payoff of Bondholder 2 is

$$P_2(R_1, R_2) = \frac{1}{1+r} E_1 \left[\min\{V_2^{***} - L, D_2\} + \max \left\{ 0, \frac{V_2^{***} - L - D_2}{2} \right\} \right] - D_1. \tag{2.10}$$

In order to construct the equilibrium of the game we determine the best response functions for each action of the bondholders.

Let us say that Bondholder 1 chooses to rescue the firm (R_1). He can do that by postponing the maturity date of his claim till $t = 2$, his claim at date 2 being still junior to the claim of Bondholder 2.

We determine Bondholder $2's$ best response to R_1 by comparing the payoffs obtained by him in case of liquidation and restructuring. As we have already seen, his payment is given by (2.8) if he wants to liquidate and by (2.10) if he wants to rescue. In the description of the game, we have already explained how the value of the firm changes in case of restructuring. If Bondholder 1 restructures and Bondholder 2 liquidates the value of the firm decreases by amount K (the restructuring costs) to $V_1^* = V_1 - K$. If Bondholder 2 wants also to rescue, the firm will take both offers, using the amount D_1 to increase the value of the firm and paying twice the restructuring costs. The new value of the firm becomes now $V_1^{***} = V_1 + D_1 - 2K$.

We compare the two values $P_2(R_1, L_2)$ and $P_2(R_1, R_2)$ for different values of parameters trying to write explicitly the values of min and max in the above formulas.

1. If $(V_1 - K)d < (V_1 + D_1 - 2K)d \leq (V_1 - K)u < (V_1 + D_1 - 2K)u \leq D_2 + L$ the payoff

---

[6]If u is high V_1^*u is high, but also D_1' and D_1'' have to be high to compensate for d being small.

in case of liquidation $P_2(R_1, L_2)$ can be written as

$$P_2(R_1, L_2) = \frac{1}{1+r}[p(V_1 - K)u + (1-p)(V_1 - K)d - L] = (V_1 - K) - \frac{1}{1+r}L.$$

The payoff in case of rescuing will be in this case

$$P_2(R_1, R_2) = \frac{1}{1+r}[p(V_1 + D_1 - 2K)u + (1-p)(V_1 + D_1 - 2K)d - D_1(1+r) - L]$$

$$= \frac{1}{1+r}[pu + (1-p)d](V_1 - 2K) + \frac{1}{1+r}\{[pu + (1-p)d]D_1 - D_1(1+r) - L\}$$

$$= (V_1 - 2K) - \frac{L}{1+r} < (V_1 - K) - \frac{L}{1+r} = P(L_2).$$

2. If $(V_1 - K)d < (V_1 + D_1 - 2K)d \leq (V_1 - K)u \leq D_2 + L < (V_1 + D_1 - 2K)u$ the payoff in case of liquidation $P_2(R_1, L_2)$ can be written again as

$$P_2(R_1, L_2) = (V_1 - K) - \frac{L}{1+r}.$$

The payoff in case of rescuing is in this case the following:

$$P_2(R_1, R_2) = -D_1 + \frac{1}{1+r}\left\{pD_2 + (1-p)[(V_1 + D_1 - 2K)d - L] + \right.$$

$$\left. p\frac{(V_1 + D_1 - 2K)u - L - D_2}{2}\right\} = -D_1 + \frac{1}{1+r}\left\{[pu + (1-p)d](V_1 + D_1 - 2K) - L + \right.$$

$$\left. + pD_2 + pL - pu(V_1 + D_1 - 2K) + p\frac{(V_1 + D_1 - 2K)u - L - D_2}{2}\right\} = -D_1 + V_1 + D_1$$

$$-2K - \frac{L}{1+r} - p\frac{(V_1 + D_1 - 2K)u - L - D_2}{2(1+r)} < V_1 - 2K - \frac{L}{1+r} < P_2(R_1, L_2).$$

We know that in this case $D_2 + L < (V_1 + D_1 - 2K)u$. As a result, we have that the following inequality holds: $p\dfrac{(V_1 + D_1 - 2K)u - L - D_2}{2(1+r)} > 0$. Consequently, the first inequality in the above formula holds too, leading to $P_2(R_1, R_2) < P_2(R_1, L_2)$.

3. If $(V_1 - K)d < (V_1 + D_1 - 2K)d \leq D_2 + L \leq (V_1 - K)u < (V_1 + D_1 - 2K)u$ the payoffs are the following:

$$P_2(R_1, L_2) = \frac{1}{1+r}\{pD_2 + (1-p)[(V_1 - K)d - L]\}$$

and

$$P_2(R_1, R_2) = -D_1 + \frac{1}{1+r} \left\{ pD_2 + (1-p)[(V_1 + D_1 - 2K)d - L] + \right.$$
$$p\frac{(V_1 + D_1 - 2K)u - L - D_2}{2} \right\} = P_2(R_1, L_2) + \frac{1}{1+r} \left[-D_1(1+r) + (1-p)D_1 d + \right.$$
$$p \left. \frac{(V_1 + D_1 - 2K)u - L - D_2}{2} \right] = P_2(R_1, L_2) + \frac{1}{2(1+r)} [p(V_1 - K)u - puD_1 - L$$
$$ - pKu - pD_2] < P_2(R_1, L_2).$$

Note that since always $V_1 \leq D_1$, it results that $pu(V_1 - K) - puD_1 \leq 0$ and therefore, $\frac{1}{2(1+r)} [p(V_1 - K)u - puD_1 - L - pKu - pD_2] < 0$. We can conclude again that $P_2(R_1, R_2) < P_2(R_1, L_2)$.

4. If $(V_1 - K)d < D_2 + L \leq (V_1 + D_1 - 2K)d \leq (V_1 - K)u < (V_1 + D_1 - 2K)u$ the payoffs are the following

$$P_2(R_1, L_2) = \frac{1}{1+r} \left\{ pD_2 + (1-p)[(V_1 - K)d - L] \right\}$$

and

$$P_2(R_1, R_2) = \frac{1}{1+r} \left[pD_2 + (1-p)D_2 + p\frac{(V_1 + D_1 - 2K)u - L - D_2}{2} \right.$$
$$\left. + (1-p)\frac{(V_1 + D_1 - 2K)d - L - D_2}{2} - D_1(1+r) \right] =$$

$$\frac{1}{1+r} \left[D_2 + \frac{(1+r)(V_1 + D_1 - 2K)u - L - D_2}{2} - D_1(1+r) \right] =$$
$$\frac{1}{1+r} \left[\frac{D_2 - D_1(1+r) + (V_1 - 2K)(1+r) - L}{2} \right].$$

We know that $V_1 \leq D_1$ and therefore, we obtain that $(V_1 - 2K)(1+r) < D_1(1+r)$. Consequently, $P_2(R_1, R_2) < \frac{D_2 - L}{2(1+r)}$. In what follows we will prove that

$$P_2(R_1, L_2) > \frac{D_2 - L}{2(1+r)} > P_2(R_1, R_2).$$

First, let us note that $P_2(R_1, L_2) > \frac{D_2 - L}{2(1+r)}$ is equivalent to

$$pD_2 + (1-p)[(V_1 - K)d - L] > \frac{D_2 - L}{2}(*).$$

We know that by assumption $(V_1 - K)d > L$. If $p \geq \frac{1}{2}$, then

$$pD_2 + (1-p)[(V_1 - K)d - L] \geq pD_2 \geq \frac{D_2}{2} > \frac{D_2 - L}{2}.$$

If $p < \frac{1}{2}$, then we can write likewise that $(*)$ is equivalent to $(1-2p)D_2 + 2(1-p)(V_1 - K)d > pL$. But since $p < \frac{1}{2}$, it results $(1-p) > p$. Then together with $(V_1 - K)d > L$ it implies that $(1-2p)D_2 + 2(1-p)(V_1 - K)d > pL$ is true and consequently, $P_2(R_1, L_2) > \dfrac{D_2 - L}{2(1+r)} > P_2(R_1, R_2)$.

5. If $D_2 + L < (V_1 - K)d < (V_1 + D_1 - 2K)d < (V_1 - K)u < (V_1 + D_1 - 2K)u$ then

$$P_2(R_1, L_2) = \frac{D_2}{1+r} \text{ and}$$

$$
\begin{aligned}
P_2(R_1, R_2) &= \frac{1}{1+r}\left\{ D_2 + \frac{[pu + (1-p)d](V_1 + D_1 - 2K) - L - D_2}{2} - D_1(1+r) \right\} \\
&= P_2(R_1, L_2) + \frac{(V_1 + D_1 - 2K) - L - D_2 - 2D_1(1+r)}{2} < P_2(R_1, L_2).
\end{aligned}
$$

The last inequality is due to the fact that $(V_1 + D_1 - 2K) \leq 2(D_1 - K) < 2D_1 \leq 2(1+r)D_1$.

It results that independently on the value of parameters, the best response to R_1 is L_2.

Let us assume now that the Bondholder 1 chooses to liquidate. We compute the best response of Bondholder 2 to this action, $BR(L_1)$, in a similar way.

The payoffs of Bondholder 2 are given by the formulas in (2.2) in case of liquidation by Bondholder 2 and by (2.6) in case he wants to restructure. We discuss on the different values of parameters.

Case 1 $V_1 - L < \dfrac{D_2}{1+r}$

If $V_1 - L < \dfrac{D_2}{1+r}$ then according to above formula we have $P_2(L_1, L_2) = V_1 - L$. We also know that in case Bondholder 1 liquidates and Bondholder 2 rescues the value of the firm increases by D_1 net of the payment to Bondholder 1 and restructuring costs. We observe that in this case $V_1^{**} = V_1 + D_1 - K - \max\{V_1 - L - \dfrac{D_2}{1+r}, 0\} = V_1 + D_1 - K$.

1.1 If $(V_1 + D_1 - K)d < (V_1 + D_1 - K)u \leq D_2 + L$ we can write

$$
\begin{aligned}
P_2(L_1, R_2) &= \frac{1}{1+r}\{p[(V_1 + D_1 - K)u - L] + (1-p)[(V_1 + D_1 - K)d - L]\} \\
&= \frac{1}{1+r}\{(V_1 + D_1 - K)[pu + (1-p)d] - L\} = (V_1 + D_1 - K) \\
&\quad - \frac{L}{1+r} > V_1 + D_1 - \frac{rL}{1+r} - \frac{L}{1+r} = V_1 + D_1 - L > V_1 - L \\
&= P_2(L_1, L_2)
\end{aligned}
$$

the first inequality coming up from the assumption we have made before that $K < \frac{r}{1+r}L$.

1.2 If $(V_1 + D_1 - K)d \leq D_2 + L < (V_1 + D_1 - K)u$ we have that

$$
P_2(L_1, R_2) = \frac{1}{1+r}\{pD_2 + (1-p)[(V_1 + D_1 - K)d - L]\}.
$$

We compare $P_2(L_1, R_2)$ with $P_2(L_1, L_2)$ and we obtain that if $(V_1 + D_1 - K)d \leq D_2 + L < (V_1 + D_1 - K)u - L\frac{r}{p}$, then $P_2(L_1, R_2) < P_2(L_1, L_2)$ and if $(V_1 + D_1 - K)u - L\frac{r}{p} < D_2 + L < (V_1 + D_1 - K)u$, then $P_2(L_1, R_2) < P_2(L_1, R_2)$.

1.3 If $D_2 + L < (V_1 + D_1 - K)d < (V_1 + D_1 - K)u$ the payoff in case of liquidation is $P_2(L_1, R_2) = \frac{D_2}{1+r} > V_1 - L$.

Case 2 $V_1 - L \geq \frac{D_2}{1+r}$

If $V_1 - L \geq \frac{D_2}{1+r}$, then we have that

$$
P_2(L_1, R_2) = \frac{1}{1+r}E_1[\min\{V_2^{**} - L, D_2\}] \leq \frac{D_2}{1+r} = P_2(L_1, L_2).
$$

We define the following set:

$$
\begin{aligned}
\mathcal{A} = &\left\{V_1 | V_1 - L < \frac{D_2}{1+r}, (V_1 + D_1 - K)d < D_2 + L < (V_1 + D_1 - K)u - L\frac{r}{p}\right\} \cup \\
&\left\{V_1 | V_1 - L \geq \frac{D_2}{1+r}\right\}.
\end{aligned}
$$

and with its help with characterize the best response function of L_1

$$
BR(L_1) = \begin{cases} L_2, & \text{if } V_1 \in \mathcal{A} \\ R_2, & \text{otherwise.} \end{cases}
$$

We find now the best response function for the action R_2. If Bondholder 1 his payoff is given by (2.3) if he plays L_1 and by (2.9) if he plays R_1. We discuss again depending on the value of the parameters.

Case 1 $V_1 - L < \dfrac{D_2}{1+r}$.

If $V_1 - L < \dfrac{D_2}{1+r}$ it results that $P_1(L_1, R_2) = 0$. On the other hand, in case both bondholders are willing to rescue the firm, the value of the firm increases to $V_1^{***} = V_1 + D_1 - 2K$. Consequently,

$$P_1(R_1, R_2) = \frac{1}{1+r}\left[p\max\left\{\frac{(V_1 + D_1 - 2K)u - L - D_2}{2}, 0\right\} + \right.$$
$$\left. (1-p)\max\left\{\frac{(V_1 + D_1 - 2K)u - L - D_2}{2}, 0\right\}\right].$$

However, since we assume that the parameters are such that the value of the firm in the worst state at date 2 is higher than the liquidation cost $((V_1 - K)d > L)$ we also have that $(V_1 - K)u > L$. In addition, $(D_1 - K)u > (D_1 - K)(1 + r) > D_2$. Hence, $(V_1 + D_1 - 2K)u > L + D_2$.

Case 2 $V_1 - L \geq \dfrac{D_2}{1+r}$.

If $V_1 - L > \dfrac{D_2}{1+r}$, then $P_1(L_1, R_2) = V_1 - L - \dfrac{D_2}{1+r}$. As we have already explained $(V_1 + D_1 - 2K)u > L + D_2$.

2.1 If $(V_1 + D_1 - 2K)d < L + D_2$ the payoff in case of restructuring is

$$P_1(R_1, R_2) = \frac{p}{2(1+r)}[(V_1 + D_1 - 2K)u - L - D_2] =$$
$$= \frac{1}{1+r}\left[p\frac{(V_1 + D_1 - 2K)u - L - D_2}{2} + (1-p)\frac{(V_1 + D_1 - 2K)d - L - D_2}{2} - \right.$$
$$\left. -(1-p)\frac{(V_1 + D_1 - 2K)d - L - D_2}{2}\right] = \frac{(V_1 + D_1 - 2K)}{2} - \frac{L}{2(1+r)} - \frac{D_2}{2(1+r)}$$
$$- (1-p)\frac{(V_1 + D_1 - 2K)d - L - D_2}{2(1+r)} > (V_1 - L) - \frac{L}{2(1+r)} - \frac{D_2}{2(1+r)}$$
$$- (1-p)\frac{(V_1 + D_1 - 2K)d - L - D_2}{2(1+r)} > V_1 - L - \frac{D_2}{1+r} = P_1(L_1, R_2).$$

The first inequality in the above formula results because $V_1 - L < V_1 - K \leq D_1 - K$ and therefore, $V_1 + D_1 - 2K > 2(V_1 - L)$. The second inequality is due to the fact that $L < D_2$

and $(V_1 + D_1 - 2K)d < L + D_2$.

2.2 If $L + D_2 < (V_1 + D_1 - 2K)d$ the payoff in case of restructuring is

$$P_1(R_1, R_2) = \frac{1}{1+r}\left[p\frac{(V_1 + D_1 - 2K)u - L - D_2}{2} + \right.$$
$$\left.(1-p)\frac{(V_1 + D_1 - 2K)u - L - D_2}{2}\right] = \frac{1}{2(1+r)}\{[pu + (1-p)d)](V_1 + D_1$$
$$- 2K) - D_2 - L\} = \frac{1}{2}\left[V_1 + D_1 - 2K - \frac{D_2}{1+r} - \frac{L}{1+r}\right] > V_1 - L - \frac{D_2}{1+r}$$

the reasons the last inequality holds being exactly the ones as in the previous case $V_1 + D_1 - 2K > 2(V_1 - L)$ and $L < D_2$. We obtain therefore that $BR(R_2) = R_1$ indifferent on the values of the parameters.

Finally, we compute the best response of Bondholder 1 to L_2. The payoff of Bondholder 1 in case he rescues is

$$P_1(R_1, L_2) = \frac{1}{1+r}E_1[\max\{0, \min\{V_2^* - L - D_2, D_1'\}\}] = D_1 \geq V_1 > V_1 - L - \frac{D_2}{1+r} \geq$$

$P_1(L_1, L_2)$. Consequently, the best response to L_2 is R_1.

As a result, we have the following best response functions:

$BR(R_1) = L_2$,

$$BR(L_1) = \begin{cases} L_2, & \text{if } V_1 \in \mathcal{A} \\ R_2, & \text{otherwise} \end{cases} ,$$
$$BR(R_2) = R_1, \quad BR(L_2) = R_1$$

and we can conclude that the equilibrium is (R_1, L_2). An important remark should be made all the interval we are looking at i.e. $V_1 - L > \frac{D_2}{1+r}$ is included in $\mathcal{A}$, so the equilibrium is (R_1, L_2) for the values we specified.

Finally, we remark that by substituting $K = L = 0$ in the above formulas the best response function are exactly the same as before. The only difference appears in determining the best response to R_1. The case 1 from the general discussion gives a different result. However, the condition characterizing this case when we substitute $K = L = 0$, becomes $V_1 d < (V_1 + D_1)d \leq V_1 u < (V_1 + D_1)u \leq D_2$. But this is in contradiction with the case we consider $\frac{D_2}{pu + (1-p)d} \leq V_1$. ∎

Lemma A.1 *The price at date* 0 *of the zero coupon bond maturing at* $t = 1$, *when there exists a positive probability of default is*

$$P_1(D_1, D_2, V_0) = \frac{1}{1+r} E_0 \left[\max \left\{ V_1 - L - \frac{D_2}{1+r}, 0 \right\} \cdot I_{\left\{ V_1 | V_1 \leq K + \frac{D_2}{pu+(1-p)d} \right\}}(V_1) + \right.$$

$$\frac{1}{1+r} E_1 \left[\max\{0, \min\{V_2^* - D_2, D_1'\} - L \cdot I_{\{V_2^* | D_2 < V_2^* < D_2 + D_1'\}}(V_2^*) \} \right]$$

$$\left. \cdot I_{\left\{ V_1 | K + \frac{D_2}{pu+(1-p)d} \leq V_1 < D_1 \right\}}(V_1) + D_1 \cdot I_{\{V_1 | D_1 < V_1\}}(V_1) \right]. \qquad (2.11)$$

The price at date 0 *of the zero coupon bond maturing at* $t = 2$, *is*

$$P_2(D_1, D_2, V_0) = \frac{1}{1+r} E_0 \left[\min \left\{ V_1 - L, \frac{D_2}{1+r} \right\} \cdot I_{\left\{ V_1 | V_1 \leq K + \frac{D_2}{pu+(1-p)d} \right\}}(V_1) + \right.$$

$$\frac{1}{1+r} E_1 \left[\min\{V_2^* - L, D_2\} \right] \cdot I_{\left\{ V_1 | K + \frac{D_2}{pu+(1-p)d} \leq V_1 < D_1 \right\}}(V_1) +$$

$$\left. \frac{1}{1+r} E_1 \left[\min \left\{ \widehat{V}_2, D_2 \right\} - L \cdot I_{\{\widehat{V}_2 | \widehat{V}_2 < D_2\}}(\widehat{V}_2) \right] \cdot I_{\{V_1 | D_1 \leq V_1\}}(V_1) \right]. \qquad (2.12)$$

To write this formula in a concise form we have defined the following indicator function:

$$I_A(x) = \begin{cases} 1, & if \ x \in A \\ 0, & if \ x \notin A. \end{cases}$$

Proof of Lemma A.1. We compute now the prices for the two bonds B_1 and B_2. As we have seen already, in the case the bondholders are allowed to rescue the firm, the default threshold goes down to $\overline{V} \equiv K + \dfrac{D_2}{pu + (1-p)d}$. If $D_1 < \overline{V}$, the firm is liquidating its assets, paying the liquidation cost L. The bondholders are paid according to the priority rule. As a result, the Bondholder 2, who owns the senior debt will be paid first and he will get $\min \left\{ V_1 - L, \dfrac{D_2}{1+r} \right\}$. Then, the Bondholder 1 is paid and he receives what is left i.e. $\max \left\{ V_1 - L - \dfrac{D_2}{1+r}, 0 \right\}$.

If $\overline{V} \leq V_1 \leq D_1$, the firm is not able to honour its debt obligation, but it is not liquidated because the bondholders decide to rescue it. The payments to the two bondholders are the one resulting in the equilibrium of the game we presented before. The payment for Bondholder 1 is $\dfrac{1}{1+r} E_1 \left[\max\{0, \min\{V_2^* - D_2, D_1'\} - L \cdot I_{\{V_2^* | D_2 < V_2^* < D_2 + D_1'\}}(V_2^*) \} \right]$ and the one for Bondholder 2 is $\dfrac{1}{1+r} E_1 [\min\{V_2^* - L, D_2\}]$. A remark has to be made here.

As we have seen in the proof of Proposition 1, by rescuing the firm, it might be the case that Bondholder 1 does not recover his losses completely. He is better off than in the case of liquidation, but he is not always paid back D_1. This happens because the firm has already a low value $V_1 \leq D_1$. As a result, we have also here the possibility of default and in this case from $\frac{1}{1+r} E_1 [\min\{V_2^* - D_2, D_1'\}]$ we subtract the liquidation costs L.

Finally, if $V_1 > D_1$ the firm does not default at date 1 and the payoff of Bondholder 1 is D_1. Bondholder 2 waits till date 2, the maturity date of its debt, and he will receive then $\min\left\{\widehat{V}_2 - L, D_2\right\}$. He will get the entitled debt D_2 if his debt is smaller than the value of the firm net of liquidation costs. Otherwise, he will receive the value of the firm net of liquidation costs. The value of the firm at date 1 decreases because the shareholders are liquidating a part of their assets to pay the debt obligations that amounts to D_1. As a result, the value of the firm at time is $\widehat{V}_2$, where

$$\widehat{V}_2 = \begin{cases} (V_1 - D_1)u, & \text{with probability} \quad p \\ (V_1 - D_1)d, & \text{with probability } 1 - p. \end{cases}$$

Since we know now the payments the bondholders are going to receive at date 1 or $t = 2$, we can compute the prices of the bonds at date0, by computing the net present value of the future payments to each bond. Thus, the prices of the two bonds B_1 and B_2 are, respectively:

$$P_1(D_1, D_2, V_0) = \frac{1}{1+r} E_0 \left[\max\left\{V_1 - L - \frac{D_2}{1+r}, 0\right\} \cdot I_{\left\{V_1 | V_1 \leq K + \frac{D_2}{pu+(1-p)d}\right\}}(V_1) + \right.$$
$$\frac{1}{1+r} E_1 \left[\max\left\{0, \min\left\{V_2^* - D_2, D_1'\right\} - L \cdot I_{\left\{V_2^* | D_2 < V_2^* < D_2 + D_1'\right\}}(V_2^*)\right\} \right]$$
$$\left. \cdot I_{\left\{V_1 | K + \frac{D_2}{pu+(1-p)d} \leq V_1 < D_1\right\}}(V_1) + D_1 \cdot I_{\left\{V_1 | D_1 < V_1\right\}}(V_1) \right].$$

and

$$P_2(D_1, D_2, V_0) = \frac{1}{1+r} E_0 \left[\min\left\{V_1 - L, \frac{D_2}{1+r}\right\} \cdot I_{\left\{V_1 | V_1 \leq K + \frac{D_2}{pu+(1-p)d}\right\}}(V_1) + \right.$$
$$\frac{1}{1+r} E_1 \left[\min\{V_2^* - L, D_2\}\right] \cdot I_{\left\{V_1 | K + \frac{D_2}{pu+(1-p)d} \leq V_1 < D_1\right\}}(V_1) +$$
$$\left. \frac{1}{1+r} E_1 \left[\min\{\{\widehat{V}_2, D_2\} - L \cdot I_{\left\{\widehat{V}_2 | \widehat{V}_2 < D_2\right\}}(\widehat{V}_2) \right] \cdot I_{\left\{V_1 | D_1 < V_1\right\}}(V_1) \right].$$

We obtained thus the general formula for each of the two bonds in the case the bond-holders are allowed to act in order to rescue the firm. ∎

Lemma A.2 *The price of equity at date* 0, *when there is a positive probability of default, is*

$$P_E(D_1, D_2, V_0) = \frac{1}{(1+r)^2} E_0 \left[0 \cdot I_{\{V_1 | V_1 \le D_1, V_2^* \le D_2 + D_1'\}}(V_1) + \right.$$
$$(V_2^* - D_1' - D_2) \cdot I_{\{V_1 | V_1 \le D_1, V_2^* > D_2 + D_1'\}}(V_1) +$$
$$\left. 0 \cdot I_{\{V_1 | D_1 \le V_1, D_2 > \widehat{V_2}\}}(V_1) + (\widehat{V_2} - D_2) \cdot I_{\{V_1 | D_1 \le V_1, D_2 \le \widehat{V_2}\}}(V_1) \right]. \quad (2.13)$$

The value of the firm is

$$V = \frac{1}{1+r} E_0 \left[(V_1 - L) \cdot I_{\{V_1 | V_1 \le K + \frac{D_2}{pu + (1-p)d}\}}(V_1) + \right.$$
$$\frac{1}{1+r} E_1 \left[V_2^* - L \cdot I_{\{V_2^* | V_2^* \le D_2 + D_1'\}}(V_2^*) \right] \cdot I_{\{V_1 | K + \frac{D_2}{pu + (1-p)d} < V_1 \le D_1\}}(V_1) +$$
$$\left. \left(D_1 + \frac{1}{1+r} E_1 [\widehat{V_2} - L \cdot I_{\{\widehat{V_2} | D_2 > \widehat{V_2}\}}(\widehat{V_2})] \right) \cdot I_{\{V_1 | D_1 \le V_1\}}(V_1) \right]. \quad (2.14)$$

Proof of Lemma A.2. We compute now the price of equity. Since the shareholders are the last to be paid in case of default nothing will be left for equity owners. This happens if $V_1 \le D_1$ and restructuring does not take place. However, if restructuring is made, we have to wait till date 2 to see which are the payments to shareholders. If at date 2, the firm honours its debt obligations, the shareholders receive $V_2^* - D_1' - D_2$. Otherwise, we have default and the shareholders will receive nothing. This happens also if we have default at date 2, $V_2 \le D_2$. In this case Bondholder 2 is paid $\min\{\widehat{V_2}, D_2\} - L$ and the shareholders are left with nothing. If $V_1 > D_1$, Bondholder 2 receives D_2 and the shareholders obtain $\widehat{V_2} - D_2$. Once we know the future payments we can compute the price of equity as the net present value of the future payments. We obtain, thus, the following formula:

$$P_E(D_1, D_2, V_0) = \frac{1}{(1+r)^2} E_0 \left[0 \cdot I_{\{V_1, V_2^* | V_1 \le D_1, V_2^* \le D_2 + D_1'\}}(V_1, V_2^*) + \right.$$
$$+ (V_2^* - D_1' - D_2) \cdot I_{\{V_1, V_2^* | V_1 \le D_1, V_2^* > D_2 + D_1'\}}(V_1, V_2^*) +$$
$$\left. 0 \cdot I_{\{V_1, \widehat{V_2} | D_1 \le V_1, D_2 > \widehat{V_2}\}}(V_1, \widehat{V_2}) + (\widehat{V_2} - D_2) \cdot I_{\{V_1 | D_1 \le V_1, D_2 \le \widehat{V_2}\}}(V_1, \widehat{V_2}) \right].$$

We can proceed now to compute the value of the firm by summing up the total value of equity with total value of short-term debt and the total value of long-term debt. Consequently, the value of the firm is:

$$V = P_E(D_1, D_2, V_0) + P_1(D_1, D_2, V_0) + P_2(D_1, D_2, V_0) =$$

$$= \frac{1}{1+r} E_0 \left[(V_1 - L) \cdot I_{\left\{ V_1 | V_1 \leq K + \frac{D_2}{pu + (1-p)d} \right\}}(V_1) + \right.$$

$$\frac{1}{1+r} E_1 \left[V_2^* - L \cdot I_{\{V_2^* | V_2^* \leq D_2 + D_1'\}}(V_2^*) \right] \cdot I_{\left\{ V_1 | K + \frac{D_2}{pu + (1-p)d} < V_1 \leq D_1 \right\}}(V_1) +$$

$$\left. \left(D_1 + \frac{1}{1+r} E_1 \left[\widehat{V}_2 - L \cdot I_{\{\widehat{V}_2 | D_2 > \widehat{V}_2\}}(\widehat{V}_2) \right] \right) \cdot I_{\{V_1 | D_1 \leq V_1\}}(V_1) \right].$$

∎

Proof of Lemma 2.1. According to formula (2.14) the value of the firm is the following

$$V = \frac{1}{1+r} E_0 \left[(V_1 - L) \cdot I_{\left\{ V_1 | V_1 \leq K + \frac{D_2}{pu + (1-p)d} \right\}}(V_1) + \right.$$

$$+ \frac{1}{1+r} E_1 \left[V_2^* - L \cdot I_{\{V_2^* | V_2^* \leq D_2 + D_1'\}}(V_2^*) \right] \cdot I_{\left\{ V_1 | K + \frac{D_2}{pu + (1-p)d} < V_1 \leq D_1 \right\}}(V_1)$$

$$+ \left. \left(D_1 + \frac{1}{1+r} E_1 [\widehat{V}_2 - L \cdot I_{\{\widehat{V}_2 | D_2 > \widehat{V}_2\}}(\widehat{V}_2)] \right) \cdot I_{\{V_1 | D_1 \leq V_1\}}(V_1) \right].$$

If we set $K = L = 0$, in our model, nothing will change. Neither the equilibrium outcome of the game, nor the procedure of valuation. Therefore, we will obtain that the value of the firm is

$$V = \frac{1}{1+r} E_0 \left[V_1 \cdot I_{\left\{ V_1 | V_1 \leq \frac{D_2}{pu + (1-p)d} \right\}}(V_1) + \frac{1}{1+r} E_1 [V_2] \cdot I_{\left\{ V_1 | \frac{D_2}{pu + (1-p)d} < V_1 \leq D_1 \right\}}(V_1) + \right.$$

$$\left. \left(D_1 + \frac{1}{1+r} E_1 [\widehat{V}_2] \right) \cdot I_{\{V_1 | D_1 \leq V_1\}}(V_1) \right].$$

Taking into account that $\dfrac{1}{1+r} E_1[V_2] = \dfrac{pV_1 u + (1-p)V_1 d}{1+r} = V_1$ and $\dfrac{1}{1+r} E_1[\widehat{V}_2] = \dfrac{p\widehat{V}_1 u + (1-p)\widehat{V}_1 d}{1+r} = \widehat{V}_1 = V_1 - D_1$, we can compute the expectations at

date 1 and the formula will become

$$V = \frac{1}{1+r}E_0\left[V_1 \cdot I_{\{V_1|V_1\leq\frac{D_2}{pu+(1-p)d}\}}(V_1) + V_1 \cdot I_{\{V_1|\frac{D_2}{pu+(1-p)d}<V_1\leq D_1\}}(V_1)+\right.$$
$$\left.(D_1+(V_1-D_1))\cdot I_{\{D_1\leq V_1\}}(V_1)\ \right] = \frac{1}{1+r}E[V_1] = V_0$$

$\blacksquare$

Lemma A.3 *The price at date* 0 *of the zero coupon bond maturing at* $t = 1$, *when there is a positive probability of default, is given by*

$$\widetilde{P_1}(D_1, D_2, V_0) = \frac{1}{1+r}E_0\left[\max\left\{V_1 - L - \frac{D_2}{1+r}, 0\right\} \cdot I_{\{V_1|V_1\leq D_1\}}(V_1) + D_1 \cdot I_{\{V_1|D_1<V_1\}}(V_1)\right].$$
$$(2.15)$$

The price at date 0 *of the zero coupon bond maturing at* $t = 2$, *is*

$$\widetilde{P_2}(D_1, D_2, V_0) = \frac{1}{1+r}E_0\left[\min\left\{V_1 - L, \frac{D_2}{1+r}\right\} \cdot I_{\{V_1|V_1\leq D_1\}}(V_1)+\right.$$
$$\left.+\frac{1}{1+r}E_1\left[\min\left\{\widehat{V_2}, D_2\right\} - L \cdot I_{\{\widehat{V_2}|\widehat{V}<D_2\}}(\widehat{V_2})\right] \cdot I_{\{V_1|D_1<V_1\}}(V_1)\right].\quad (2.16)$$

Proof of Lemma A.3. We compute now the prices of the two bonds in the case we do not allow for restructuring. The reasoning is similar with the one in Lemma 1 but simpler. In case we have default at date 1, the value of the firm after liquidation $V_1 - L$ is shared between the two bondholders. Bondholder 2 has priority and he receives $\min\left\{V_1 - L, \frac{D_2}{1+r}\right\}$ while Bondholder 1 receives what is left i.e. $\max\left\{V_1 - L - \frac{D_2}{1+r}, 0\right\}$. Otherwise, Bondholder 1 receives D_1 and Bondholder 2 waits till date 2. Here, if $\widehat{V_2} \leq D_2$, we have default and Bondholder 2 gets $\widehat{V_2} - L$. Otherwise, he receives D_2. We can write now the price of the two bonds by computing the net present value of the payments described above and we obtain

$$\widetilde{P_1}(D_1, D_2, V_0) = \frac{1}{1+r}E_0\left[\max\left\{V_1 - L - \frac{D_2}{1+r}, 0\right\} \cdot I_{\{V_1|V_1\leq D_1\}}(V_1) + D_1 \cdot I_{\{V_1|D_1<V_1\}}(V_1)\right]$$

and

$$\widetilde{P_2}(D_1, D_2, V_0) = \frac{1}{1+r} E_0 \left[\min \left\{ V_1 - L, \frac{D_2}{1+r} \right\} \cdot I_{\{V_1|V_1 \leq D_1\}}(V_1) + \right.$$
$$\left. \frac{1}{1+r} E_1 \left[\min \left\{ \widehat{V_2}, D_2 \right\} - L \cdot I_{\{\widehat{V_2}|\widehat{V_2} < D_2\}}(\widehat{V_2}) \right] \cdot I_{\{V_1|D_1 < V_1\}}(V_1) \right].$$

∎

Lemma A.4 *The price of equity at date 0, when there is a positive probability of default is given by*

$$\widetilde{P_E}(D_1, D_2, V_0) = \frac{1}{1+r} E_0 \left[0 \cdot I_{\{V_1|V_1 \leq D_1\} \cup \{V_1|D_1 < V_1, \widehat{V_2} \leq D_2\}}(V_1) + \right.$$
$$\left. + \left(\max \left\{ \widehat{V_2} - D_2, 0 \right\} \cdot I_{\{\widehat{V_2}|\widehat{V_2} \leq D_2\}}\left(\widehat{V_2}\right) \right) \cdot I_{\{V_1|D_1 \leq V_1\}}(V_1) \right]. \quad (2.17)$$

The value of the firm is

$$\widetilde{V} = \frac{1}{1+r} E_0 \left[(V_1 - L) \cdot I_{\{V_1|V_1 \leq D_1\}}(V_1) + \right.$$
$$\left. \left(D_1 + \frac{1}{1+r} E_1 \left[\widehat{V_2} - L \cdot I_{\{\widehat{V_2}|D_2 > \widehat{V_2}\}}(\widehat{V_2}) \right] \right) \cdot I_{\{V_1|D_1 \leq V_1\}}(V_1) \right]. \quad (2.18)$$

Proof of Lemma A.4. Finally, we compute the equity price in the same way we computed the equity price in Lemma 2.

Since the equity owners are the last to receive their payments, they will not receive anything when the firm defaults. As a result, in both cases $V_1 \leq D_1$ and $\widehat{V_2} \leq D_2$ the payment to equity owners is 0. Finally, when $\widehat{V_2} > D_2$ we obtain that the shareholders receive the value of the firm $\widehat{V_2}$ minus the payment to Bondholder 2, D_2. Now we can write the net present value of the future payments to equity in order to compute the price of equity at date 0. We obtain:

$$\widetilde{P_E}(D_1, D_2, V_0) = \frac{1}{1+r} E_0 \left[0 \cdot I_{\{V_1|V_1 \leq D_1\} \cup \{V_1|D_1 < V_1, \widehat{V_2} \leq D_2\}}(V_1) + \right.$$
$$\left. \left(\max \left\{ \widehat{V_2} - D_2, 0 \right\} \cdot I_{\{\widehat{V_2}|\widehat{V_2} \leq D_2\}}\left(\widehat{V_2}\right) \right) \cdot I_{\{V_1|D_1 \leq V_1\}}(V_1) \right].$$

The value of the firm is computed again by adding up again the total value of equity, the total value of short-term debt and the total value of long-term debt. Consequently,

the value of the firm is

$$V = P_E(D_1, D_2, V_0) + P_1(D_1, D_2, V_0) + P_2(D_1, D_2, V_0) =$$
$$\frac{1}{1+r} E_0 \left[(V_1 - L) \cdot I_{\{V_1|V_1 \le D_1\}}(V_1) + \right.$$
$$\left. \left(D_1 + \frac{1}{1+r} E_1 \left[\widehat{V_2} - L \cdot I_{\{\widehat{V_2}|D_2 > \widehat{V_2}\}} \left(\widehat{V_2} \right) \right] \right) \cdot I_{\{V_1|D_1 \le V_1\}}(V_1) \right].$$

∎

Proof of Lemma 2.2. We consider again the two cases when the firm is financed by equity, short term-debt and long-term debt and we would like to show that the interest rate in this case is the same as the one when the firm is financed fully by equity. Firstly, let us consider the case when the bondholders are not allowed to restructure in case of default at date 1. As we have seen, the prices at date 0 of the bond maturing at $t = 1$, of the bond maturing at $t = 2$ and of the equity are given by (2.15),(2.16) and (2.17), respectively.

Due to the value preservation property we know that the value of the firm is the sum of the prices of equity plus the present value of total expected losses.

$$V_0 = \widetilde{P_1}(D_1, D_2, V_0) + \widetilde{P_2}(D_1, D_2, V_0) + \widetilde{P_E}(D_1, D_2, V_0) + PV(TL).$$

Let us compute the present value of the total expected losses. Since in this case the losses are the one that appear in case of liquidation we can write that

$$PV(TL) = \frac{1}{1+r} \left[L \cdot I_{\{V_1|V_1 \le D_1\}}(V_1) + \frac{1}{1+r} L \cdot I_{\{\widehat{V_2}|\widehat{V_2} < D_2\}} \left(\widehat{V_2} \right) \cdot I_{\{V_1|D_1 < V_1\}}(V_1) \right].$$

i.e. in the two cases when liquidation occurs $\{V_1|V_1 \le D_1\}$ and $\left\{ V_1, \widehat{V_2}|D_1 < V_1, \widehat{V_2} < D_2 \right\}$. Using this and the formula for the equity and bond prices mentioned above we can write

further

$$V_0 = \frac{1}{1+r}E_0[V_1] \cdot I_{\{V_1|V_1\leq D_1\}}(V_1) + E_0\left[\left\{D_1 + \frac{1}{1+r}E_1\left[\widehat{V_2}\right]\right\} \cdot I_{\{\widehat{V_2}|\widehat{V_2}\leq D_2\}}\left(\widehat{V_2}\right)\right]$$

$$= \frac{pu+(1-p)d}{1+r}V_0 \cdot I_{\{V_1|V_1\leq D_1\}}(V_1) + \frac{1}{1+r}E_0\left[\left\{D_1 + \frac{pu+(1-p)d}{1+r}(V_1-\right.\right.$$

$$D_1)\right\} \cdot I_{\{V_1|V_1>D_1\}}(V_1)\right] = \frac{pu+(1-p)d}{1+r}V_0 \cdot I_{\{V_1|V_1\leq D_1\}}(V_1) + \frac{1}{1+r}\left\{D_1 + \frac{pu+(1-p)d}{1+r}\right.$$

$$((pu+(1-p)d)V_0 - D_1)\right\} \cdot I_{\{V_1|V_1>D_1\}}(V_1)\right] = \frac{pu+(1-p)d}{1+r}V_0 \cdot I_{\{V_1|V_1\leq D_1\}}(V_1) +$$

$$\left\{\frac{1}{1+r}D_1(1 - \frac{pu+(1-p)d}{1+r}) + \frac{(pu+(1-p)d)^2}{(1+r)^2}V_0\right\} \cdot I_{\{V_1|V_1>D_1\}}(V_1)\right].$$

We obtain hence, a quadratic equation in $pu+(1-p)d$. We also notice that this equation has only a positive root and this root is exactly $1+r$. Since $pu+(1-p)d \geq 0$, it results that $pu+(1-p)d = 1+r$.

Secondly, we look at the case when the bondholders are allowed to restructure the firm in case of default. As we have seen before the prices of the short-term bond, long-term bond and equity are given by formulas (2.11), (2.12) and (2.13). We write again that the initial value of the firm is equal to the expected payoffs to bonds and equity and expected losses. In this case, we will have less losses due to liquidation, but we also have to take into account the losses that come as costs of restructuring.

$$V_0 = \widetilde{P_1}(D_1, D_2, V_0) + \widetilde{P_2}(D_1, D_2, V_0) + \widetilde{P_E}(D_1, D_2, V_0) + PV(TL) + PV(TR).$$

We denote by TL the total losses due to liquidation and by TR the total losses due to restructuring and we obtain the following present values:

$$PV(TL) = \frac{1}{1+r}\left[L \cdot I_{\left\{V_1|V_1\leq K+\frac{D_2}{pu+(1-p)d}\right\}}(V_1) + \frac{1}{1+r}L \cdot I_{\{V_2^*|D_2<V_2^*<D_2+D_1'\}}(V_2^*)\cdot\right.$$

$$I_{\left\{V_1|K+\frac{D_2}{pu+(1-p)d}\leq V_1<D_1\right\}}(V_1) + \frac{1}{1+r}L \cdot I_{\{\widehat{V_2}|\widehat{V_2}<D_2\}}\left(\widehat{V_2}\right) \cdot I_{\{V_1|D_1\leq V_1\}}(V_1)\right]$$

and

$$PV(TR) = \frac{1}{1+r}K \cdot I_{\left\{V_1|K+\frac{D_2}{pu+(1-p)d}\leq V_1<D_1\right\}}(V_1).$$

Plugging them in the above formula we obtain that

$$V_0 = \frac{pu + (1-p)d}{1+r}V_0 \cdot I_{\left\{V_1 | V_1 \leq K + \frac{D_2}{pu + (1-p)d}\right\}}(V_1) + \frac{pu + (1-p)d}{(1+r)^2}V_0$$

$$\cdot I_{\left\{V_1 | K + \frac{D_2}{pu + (1-p)d} \leq V_1 < D_1\right\}}(V_1) + \frac{1}{1+r}\left[D_1(1 - \frac{pu + (1-p)d}{1+r}) + \right.$$

$$\left.\frac{(pu + (1-p)d)^2}{1+r}V_1\right] \cdot I_{\{V_1 | D_1 < V_1\}}(V_1).$$

Similarly with the previous case we obtain a second degree equation in $pu + (1-p)d$, which has only one positive root namely $1+r$. Consequently, we obtain again that $pu + (1-p)d = 1 + r$. ∎

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Chapter 3

Imperfect Competition and Market Liquidity with a Supply Informed Trader

3.1 Introduction

Agents engaged in trading activities might have access to different sources of information: information about fundamentals or information about the supply. The existence of different types of information might reduce the inefficiencies that appear when agents trade on private information about fundamentals. The purpose of this paper is to study exactly how the existence of different types of information affects market performance. We develop a model of insider trading in the context of an imperfectly competitive market where agents have private information either about future prices or about supply. This distinction between price-informed traders and supply-informed traders is designed to capture the different types of information that influence the security prices at any point in time. Moreover, in an imperfect competition equilibrium prices are less informative than in a competitive rational equilibrium. This is due mainly to the fact that a strategic trader exploits his informational advantage taking into account the effect the quantity he chooses is expected to have on both the price and the other informed traders' strategy. As a result, we study how trading affects market liquidity and informational efficiency of prices due to the strategic choices of the agents in this new setup.

In the Kyle-type models an important assumption is the presence of noise. As it was already explained by Grossman and Stiglitz (1980), noise is needed in the model to prevent prices to be fully revealing. They show that in a model in which agents are price takers and prices are fully revealed no agent will be willing to acquire costly information. To overcome this difficulty several ways to introduce noise were used: adding noise traders, considering uncertainty which has a dimension greater than that of price, or assuming that the aggregate endowment is imperfectly observed. We use this last approach by assuming a random supply. The presence of shocks in supply has a significant price impact. A supply shock leads to a change in prices and this determines the investors to revise their expectations. However, if the supply shock is observable by the supply informed traders, these traders are making use of their informational advantage and therefore, are willing to adjust their demand. Consequently, we assume that there exists a supply informed

trader who receives a signal about supply. This approach was used before by Gennotte and Leland (1990) who consider a model were speculators posses private and diverse information.[1] They consider price takers speculators who gather information either about prices or about supply and show that these informational differences can cause financial markets to be relatively illiquid. Our model builds on the assumption of Gennotte and Leland (1990) about the existence of a random supply and informed supply speculator but we consider an imperfect competition setup with both price informed and supply informed agents ,where the agents submit limit orders. In general dealers observe order flow and collect information from multiple sources. Therefore, we can think of the supply informed agent as being a dealer who can observe the order flow. As pointed out by Brown and Zhang (1997), despite of the fact that dealers may be better informed than other traders, in a competitive market they cannot earn rents on the information on the order flow. This is due to the fact that price informed agents use their informational advantage to make gains on the expense of dealers. However, we will see that in our setup of an imperfect competitive market dealers can aggregate the information from trading and use it to earn speculative profits. Thus, the dealers can learn about the liquidation value of the asset from the orders placed by the price informed agents. The information revelation is increased significantly in our setup since the agents are placing limit orders and therefore, they condition their demands on prices and infer in this way a part of others' information. We assume here that there is only one supply informed trader. Made for simplicity, the assumption is in line with the result obtained by Ellis et al. (2001). They show that in general, one dealer tends to dominate the trading on a stock (executing a little more than half of the day's volume). They also answer the question who is the dominant dealer. Depending on the time passed from the offer day, the dominant dealer might be the underwriter, a wholesaler or a generic market maker.

In the rational expectations paradigm traders understand that prices reveal the in-

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[1]A similar assumption is that market makers have some information about the uninformed order flow and it can be found in Admati and Pfleiderer (1991) and Madhavan (1992). Palomino (2001) considers also a setup where the informed agents have information both about the liquidation value and the quantity traded by one of the noise traders.

formation they have when they choose the quantities to be traded. The link between information and prices via trades provides an explicit mechanism for information transmission between traders. The existence of private information means that a trader may have incentives to act strategically in order to maximize his profits. Therefore, given his private information, a trader maximizes his conditional expected profits taking into account the effect of his trading on prices and taking as given the strategies other traders use to chose their demand schedules. As in the imperfect competition model of Kyle (1989) we assume further that all the speculators choose strategically the amounts they trade. Therefore, the supply informed speculator will also chose his demand taking into account the effect of his trading on prices and revealing a part of information about the shock in supply to the other market participants. As a result, in our model both the information about the value of the asset or about supply is revealed through the quantities to be traded.

In our model we use the framework developed by Kyle (1985, 1989) which have become a standard for analyzing strategic noisy rational expectations markets. Kyle's (1985) model explains how a risk neutral informed trader exploits his informational advantage by behaving strategical and shows that the smoothing behaviour of the trader leads to prices that have constant volatility as the time periods become shorter to approach a continuous auction. An important generalization of the Kyle's model is to allow for multiple informed traders. Since the monopolist trader makes positive profits it follows that other trader might be willing to acquire information. Foster and Wiswanathan (1993) and Holden and Subrahmanyam (1992) explore this restriction of a single informed trader and point out the contrast between the case of a monopolist and the one of multiple traders. Thus, Foster and Wiswanathan (1993) extend Kyle's model to many traders and a larger class of distributions but obtain that Kyle's result that the informed trader can make positive profits does not hold anymore. On the other hand, Holden and Subrahmanyam (1992) conclude that competition between informed traders leads to fully revelation of information. A further extension is proposed by Caballé and Krishnan (1994). They study a multi-security market with risk neutral agents in a correlated setup and they

generalize the result of Kyle (1985) that more noise leads to more aggressive trading. Moreover, an important result is that in their model portfolio diversification arises due to the strategic behavior of the agents and not because of risk considerations.

A different direction of extending this strand of research was to allow for different trading mechanisms. Since traders may have a greater control on their trade behaviour this issue becomes even more important in the context of strategic behaviour. This problem was studied by Kyle (1989), Jackson (1991), Bhattacharya and Spiegel (1991), Caballé (1992), and Rochet and Vila (1994). Kyle (1989), to which our work is closely related, proposes an imperfect competition model in which there are noise traders, price informed traders and uninformed traders. He shows that a strategic trader acts as he trades against a residual supply curve. This implies lower quantities by comparison with the competitive rational expectations equilibrium and, consequently, in equilibrium prices reveal less information than in the competitive case. As it will be emphasized in this paper, in the case we have different types of information the dual role of prices to aggregate information and clear the market is even more important.

We are interested to understand the effects of different types of information on market liquidity, informativeness of prices, price volatility, and the ability of informed traders to exploit their private information. Our goal is to see how market liquidity and price efficiency are influenced by strategic interaction between agents with different types of private information. Allowing the supply informed agent to behave strategically, has an important role in the market-making and in information revelation. Indeed, he decreases the market depth and increases the amount of information revealed in prices but, unlike in the perfect competitive case, he also makes positive profits. Our model suggests that the presence of different types of information in the market decreases market liquidity. The result is line with the one of Glosten and Milgrom (1985) that more information in the market leads to an increase in the bid-ask spread (i.e. a decrease in the market liquidity). The result should be situated in between the one of Kyle (1985, 1989) and the subsequent literature which show that increasing the number of informed traders increases market liquidity, and the one of Subrahmanyam (1991) which shows the opposite. Thus,

Subrahmanyam (1991) also obtains that market liquidity can be decreased by increasing the number of informed traders in the case traders and market maker are risk averse. In our model we obtain that the presence of the supply informed agent and therefore, of a different type of information in the market, leads to a decrease in market liquidity. Still, if we are increasing the number of price informed traders we will still obtain the increase the market liquidity obtained in Kyle (1985, 1989).

We performed comparative statics results for market liquidity measured as market depth and we conclude that if the information received by the supply informed agent is very precise or the one of the price informed agents is very poor the market liquidity is low. Finally, we study how changes in supply affect the equilibrium price. We will consider two cases: a change in supply known to all investors or a change known only to the supply informed investors. We obtain that price informed agents absorb a higher fraction of the known shock, while the supply informed agent absorbs always half of the unknown shock.

The remainder of this paper is organized as follows. Section 2 presents the model. We establish the information structure and define the imperfect competitive rational equilibrium expectations. Section 3 characterizes the equilibrium. We find an unique linear imperfect competitive rational expectations price function together with agents' demand functions in equilibrium. Section 4 proceeds with the calculation of some market indicators: volatility of prices, informativeness of prices and expected profits. Section 5 contains the characterization the equilibrium in the case there is no supply informed trader and then Section 6 compares the market indicators of this economy with the one of the economy with a supply informed agent. Finally, Section 7 summarizes the results and gives some directions for further research. All the proofs appear in the appendix.

3.2   The Model

The framework is similar to the one in Kyle (1989). However, we assume risk neutrality, absence of uniformed traders and random supply with an observable component

for one trader - the supply informed trader. As already pointed out by Kyle (1989), the assumption of existence of uninformed traders does not change the analysis, but their presence leads to an increase in market depth. In what it follows we make the following assumptions:

A.1 There is a single security in the market that trades at market clearing price $\widetilde{p}$ and yields an exogenous liquidation value $\widetilde{v}$ which has a normal distribution with mean $\overline{v}$ and variance σ_v^2.

A.2 There are N price informed traders, indexed $n = 1, ..., N$ and a supply informed trader. The price informed trader n observes a private signal $\widetilde{i}_n = \widetilde{v} + \widetilde{e}_n$. We assume that e_n is distributed $N(0, \sigma_e^2)$ for all $n = 1, ..., N$. We suppose that for any $j \neq n$ $\widetilde{e}_j$ and $\widetilde{e}_n$ are uncorrelated and moreover, they are uncorrelated with all the other random variables in the model. The supply informed trader observes a private signal S which is normal distributed with mean 0 and variance $\sigma_S^2 > 0$.

A.3 The random supply that keeps the traders from perfectly inferring the aggregate information from prices is modelled in a similar manner to the one in Gennotte and Leland (1990). The net supply $\widetilde{m}$ consists of a fixed amount $\overline{m}$ and a random supply $\widetilde{S}$ distributed $N\left(0, \sigma_S^2\right)$. This liquidity shock $\widetilde{S}$ is observed only by the supply informed trader.

A.4 Agents are risk neutral and behave strategically taking into account the effect of their trading on prices.

As in Kyle (1989), the nth price informed trader has a strategy X_n which is a mapping from R^2 (the cartesian product of the set of asset prices and the set of his signals) to R (the set of shares he desires to trade), $X_n(\cdot, \cdot) : \mathsf{R}^2 \longrightarrow \mathsf{R}$. After observing his signal i_n, each price informed trader submits a demand schedule (or generalized limit order) $X_n\left(\cdot, \widetilde{i}_n\right)$, which depends upon his signal. Similarly, the supply informed trader has a strategy Y which is a mapping from R^2 (the cartesian product of the set of asset prices and the set of his signals) to R (the set of shares he wants to trade), $Y(\cdot, \cdot) : \mathsf{R}^2 \longrightarrow \mathsf{R}$. After observing the signal S, the supply informed trader chooses a demand schedule $Y(\cdot, S)$, which depends upon that signal. Notice that since $\overline{m}$ is known by everyone, this implies

that the supply informed agent actually knows $\widetilde{m}$. Given a market clearing price p, the quantities traded by price informed traders and supply informed trader can be written $x_n = X_n(p, i_n)$, $n = 1, ..., N$ and $y = Y(p, S)$. In the above notations a tilde distinguishes a random variable from its realization. Thus, x_n denotes a particular realization of $\widetilde{x}_n$. The assumption that the price informed and the supply informed agents submit limit orders for execution against existing limit orders submitted by the other market participants turns out to be very important (for a detailed discussion see Kyle (1989)). In this context both the price informed and the supply informed agents provide liquidity and therefore, have a market making role in the market.

The price of the asset is set such that the market clears. The traders submit their demand schedules to an auctioneer who aggregates all the schedules submitted, calculates the market clearing price and allocates quantities to satisfy traders' demand. Thus, the market clearing price $\widetilde{p}$ should satisfy with probability one

$$\sum_{n=1}^{N} X_n\left(\widetilde{p}, \widetilde{i}_n\right) + Y\left(\widetilde{p}, \widetilde{S}\right) = \widetilde{m}. \tag{3.1}$$

To emphasize the dependence of the market-clearing price on the strategies of the traders we write

$$p = p(X, Y), \quad x_n = x_n(X, Y), \quad y = y(X, Y),$$

where X is the vector of strategies of price informed traders defined by $X = (X_1, ..., X_N)$ and Y is the strategy of the supply informed trader.

The traders are risk neutral and maximize expected profits. The profits of the price informed trader n and supply informed trader are, respectively, given by

$$\widetilde{\pi}_n^{PI} = (\widetilde{v} - \widetilde{p}(X, Y))\,\widetilde{x}_n(X, Y), \quad \widetilde{\pi}^{SI} = (\widetilde{v} - \widetilde{p}(X, Y))\,\widetilde{y}(X, Y).$$

With these notations, following Kyle (1989) we can proceed to define a rational expectations equilibrium in our setup.

Definition 1 *An imperfectly competitive rational expectations equilibrium is defined as a vector (X, Y, p), where X is a vector of strategies of the price informed agents $X =$*

$(X_1, ..., X_N)$, Y is a strategy of the supply informed agent and p is the equilibrium price such that the following conditions hold:

1. For all $n = 1, ..., N$ and for any alternative strategy vector X' differing from X only in the nth component X_n, the strategy X yields a higher profit than X':

$$E_n \left[(\widetilde{v} - \widetilde{p}(X, Y)) \widetilde{x}_n(X, Y) \mid \widetilde{p}(X, Y) = p, \ \widetilde{i}_n = i \right] \geq$$
$$E_n \left[(\widetilde{v} - \widetilde{p}(X', Y)) \widetilde{x}_n(X', Y) \mid \widetilde{p}(X', Y) = p, \ \widetilde{i}_n = i \right].$$

2. For any alternative strategy Y' the strategy Y yields a higher profit than Y':

$$E \left[(\widetilde{v} - \widetilde{p}(X, Y)) \widetilde{y}(X, Y) \mid \widetilde{p}(X, Y) = p, \ \widetilde{S} = S \right] \geq$$
$$E \left[(\widetilde{v} - \widetilde{p}(X', Y)) \widetilde{y}(X, Y') \mid \widetilde{p}(X, Y') = p, \ \widetilde{S} = S \right].$$

3. The price $p = \widetilde{p}(X, Y)$ clears the market (with probability one) i.e.

$$\sum_{n=1}^{N} X_n \left(\widetilde{p}, \widetilde{i}_n \right) + Y \left(\widetilde{p}, \widetilde{S} \right) = \widetilde{m}.$$

This defines a Nash equilibrium in demand functions. Given their private information, traders maximize their conditional expected profits taking into account the effect of their trading on prices and taking as given the strategies other traders use to choose their demand schedules.

We look for a symmetric linear Bayesian Nash Equilibrium as in Kyle (1989), that is, an equilibrium where the strategies X_n and Y are linear functions:

$$X_n \left(\widetilde{p}, \widetilde{i}_n \right) = \alpha^{PI} + \beta^{PI} \widetilde{i}_n - \gamma^{PI} \widetilde{p}, \text{ for any } n = 1, ..., N \text{ and}$$
$$Y \left(\widetilde{p}, \widetilde{S} \right) = \alpha^{SI} + \beta^{SI} \widetilde{S} - \gamma^{SI} \widetilde{p}, \tag{3.2}$$

where $\alpha^{PI}, \beta^{PI}, \gamma^{PI}, \alpha^{SI}, \beta^{SI}, \gamma^{SI} \in \mathbb{R}$.

With this assumption we can infer from the market clearing condition that the equilibrium price is given by

$$p = \left(N\gamma^{PI} + \gamma^{SI} \right)^{-1} \left(N\alpha^{PI} + \alpha^{SI} + \beta^{PI} \sum_{n=1}^{N} \widetilde{i}_n + \beta^{SI} \widetilde{S} - \widetilde{m} \right). \tag{3.3}$$

3.3 Characterization of the Equilibrium

We describe in the following proposition the equations that characterize the symmetric Bayesian-Nash equilibrium. This equilibrium has linear trading rules and linear pricing rule and is shown to be unique among all linear, symmetric Bayesian-Nash equilibria. As in most Kyle type models, the linearities are not ex-ante imposed in the agents strategy sets: as long as the informed traders use a linear trading strategy, the market maker will use a linear pricing rule and vice versa.

Proposition 3.1 *If $N(N-2) \geq \dfrac{\sigma_e^2}{\sigma_v^2}$ there exists an unique linear symmetric equilibrium defined as:*

$$X_n\left(\widetilde{p}, \widetilde{i}_n\right) = \alpha^{PI} + \beta^{PI}\widetilde{i}_n - \gamma^{PI}\widetilde{p}, \ for \ any \ n = 1, ..., N \ and$$

$$Y\left(\widetilde{p}, \widetilde{S}\right) = \alpha^{SI} + \beta^{SI}\widetilde{S} - \gamma^{SI}\widetilde{p},$$

with $\quad \alpha^{PI}, \beta^{PI}, \gamma^{PI}, \ \alpha^{SI}, \beta^{SI}, \gamma^{SI} \ \ given by$

$$\alpha^{PI} = \frac{\sigma_e^2\left(N\left(3N-2\right)\sigma_v^2 + \left(2N-1\right)\sigma_e^2\right)\delta^{1/2}}{2N^2\sigma_v\left(N^2\sigma_v^2 + \sigma_e^2\right)\left(N\sigma_v^2 + \sigma_e^2\right)}\overline{v} + \frac{N\left(N-2\right)\sigma_v^2 - \sigma_e^2}{N\left(N^2\sigma_v^2 + \sigma_e^2\right)}\overline{m}$$

$$\beta^{PI} = \frac{\delta^{1/2}}{2N\left(N\sigma_v^2 + \sigma_e^2\right)}$$

$$\gamma^{PI} = \frac{\left(N^2\sigma_v + \left(2N-1\right)\sigma_e^2\right)\delta^{1/2}}{2N^2\sigma_v^2\left(N^2\sigma_v^2 + \sigma_e^2\right)}$$

$$\alpha^{SI} = -\frac{\left(N-1\right)\left(N^2\sigma_v^2 + \left(2N-1\right)\sigma_e^2\right)\sigma_e^2\delta^{1/2}}{2N^2\sigma_v^2\left(N\sigma_v^2 + \sigma_e^2\right)\left(N^2\sigma_v^2 + \sigma_e^2\right)}\overline{v} + \frac{N^2\sigma_v^2 + \left(2N-1\right)\sigma_e^2}{N\left(N^2\sigma_v^2 + \sigma_e^2\right)}\overline{m}$$

$$\beta^{SI} = \frac{N^2\sigma_v^2 + \left(2N-1\right)\sigma_e^2}{2N\left(N\sigma_v^2 + \sigma_e^2\right)}$$

$$\gamma^{SI} = -\frac{\left(N-1\right)\sigma_e^2\left(N^2\sigma_v^2 + \left(2N-1\right)\sigma_e^2\right)\delta^{1/2}}{2N^2\sigma_v^2\left(N\sigma_v^2 + \sigma_e^2\right)\left(N^2\sigma_v^2 + \sigma_e^2\right)}, \tag{3.4}$$

where

$$\delta \equiv \frac{\left(N(N-2)\sigma_v^2 - \sigma_e^2\right)\left(N^2\sigma_v^2 + \sigma_e^2\right)\sigma_S^2}{\left(N-1\right)\sigma_e^2}.$$

The condition $N(N-2) \geq \dfrac{\sigma_e^2}{\sigma_v^2}$ is similar to the usual condition $N > 2$ in all Kyle-type models. It tells us that we need competition in order to alleviate the asymmetric

information problem. In our model the asymmetric information problem is even more important than in Kyle (1985, 1989) because we have two different types of information that aggregate in prices. Since the supply informed agent observes the supply he acts as an informational monopolist trading such that he always extracts some rents. However, the price informed agents are competing against him trying to reduce his informational advantage. The worse the quality of the signal of the price informed traders relative to the liquidation value, the more difficult is for them to compete against the supply informed. However, since they are asymmetrically informed, increasing their number it will make it more difficult for the supply informed to infer their information. Consequently, in the case we have a supply informed agent we need more competition in order to reduce his monopoly power and trade aggressiveness and therefore, for the equilibrium to exists.

We would like to understand the effects of different types of information on market liquidity, informativeness of prices, price volatility, and the ability of informed traders to exploit their private information. We are first concerned with market liquidity because it has been recognized as an important determinant of market behaviour. There are different measures of market liquidity used in the literature: market depth, bid-ask spread and price movement after trade. We will use as a measure of liquidity the market depth (as defined by Kyle (1989)), which represents the volume of trading needed to move prices one unit. While solving the above system we have obtained that

$$\gamma = N\gamma^{PI} + \gamma^{SI} = \frac{\left(N^2\sigma_v^2 + (2N-1)\sigma_e^2\right)\delta^{1/2}}{2N^2\sigma_v^2\left(N\sigma_v^2 + \sigma_e^2\right)}.$$

On the other hand, from the price equation (3.3) we can see that an increase (decrease) in the supply by γ induces the price to fall (rise) by one dollar. The trading volume needed to move the price by one unit (market depth) was used by Kyle (1985) as a measure of market liquidity. We use the same measure as Kyle and consequently, γ is our measure of the market liquidity. As we can see the market depth γ has two components that have opposite effect. The first component $N\gamma^{PI}$ is attributed to the price informed agents trading. This is the amount with which they contribute to a change in the price when each of them trades an additional unit. The more priced informed agents are in

the market, the higher the liquidity. Similarly, we have that γ^{SI} is the change in price due to an additional unit of trading by the supply informed agent. The two components have opposite sign and we have here a trade-off: whenever the price informed agents are increasing the market liquidity the supply informed agent will try to reduce it.

The fact that γ^{SI} is negative is a very important result in our model and it is a consequence of the mechanism of information transmission through prices. In general, with asymmetric information prices play a dual role of information aggregation and market clearing. The role of information aggregation played by prices is even more important in our economy with asymmetric and different information. We have thus, two important channels through which we have flow of information (information about the liquidation value from the price informed traders towards the supply informed trader and information about supply from the supply informed trader towards the price informed traders). The supply informed agent puts a positive weight on price ($\gamma^{SI} < 0$) because when he sees an increase in price he associates it with good news about the liquidation value (he knows the supply, so the price increase cannot be due to a decrease in supply). This mechanism of information transmission actually triggers a decrease in market liquidity. For one additional unit demanded by a price informed agent the price goes up. The supply informed agent associates it with good news about the liquidation value and increases his demand leading to a even higher increase in price. Since the same volume will increase the price more we can conclude that we have a decrease in the market liquidity.

Next, let us investigate how the market depth varies with the parameters of the model: the variance of the liquidity shock σ_S^2, the variance of signals σ_e^2, and the variance of the liquidation value σ_v^2.

Corollary 3.1 *(i) Market depth is increasing in the variance of liquidity shock $\widetilde{S}$, σ_S^2.*

(ii) Market depth is decreasing in the variance of the error of the signal received by price informed agents σ_e^2.

(iii) Market depth viewed as a function of the variance of liquidation value σ_v^2 has an inverted U-shaped.

(iv) Market depth is decreasing in the relative quality of the signals $\frac{\sigma_e^2}{\sigma_v^2}$.

As we have seen before, the market depth has two components $\gamma = N\gamma^{PI} + \gamma^{SI}$. The first component is the contribution to the market depth of trades executed by price informed agents while the second one is the contribution to the market depth of trades executed by the supply informed agent. The two components have opposite effect and thus, the final result on market depth due to the market making activity of the agents depends on which of the two components dominates. The first result in the Corollary is similar to the previous ones in the literature (Kyle (1985) and the other imperfect competition models). It tells us that the higher the variance of the supply (in the other papers - the variance of the noise trading), the easier is for the price informed agents to hide and therefore, to make use of their informational advantage (the volume needed to move the price is higher, and this helps them to trade better on their information without being discovered). In our model the same it is true also for the supply informed agents. If the signal of the supply informed agent is very informative he reduces the market liquidity. Otherwise, he might infer wrongly the information embedded in prices and therefore, contribute himself to the increase in the market liquidity. The second result claims that if the signal of the price informed agents is very precise, the market depth is high. This happens because when the price informed agents have poor informational advantage, they trade less aggressively and devote more to market making activities. Notice that these results indicate that the effect on market depth of the trades of price informed agents dominates the effect of the trades of the supply informed agent for all values of σ_S^2 or σ_e^2. What actually happens is that when the difference in the information between the price informed agenst is small, they will compete stronger against the supply informed agent and less among themselves. Once their information become very different, i.e. σ_e^2 increases, they will also start competing more aggressively against each other. The third result is somehow different from the previous results. This difference is triggered exactly by the existence of a supply informed agent. Here we have that the effect on market depth of the trades of supply informed agent may dominate the effect of the trades of the price

informed agents when the variance of liquidation value is high. If the variance σ_v^2 is small, the signal the price informed agents receive is better and the supply informed agent is not able to decrease the market liquidity. However, as the variance of liquidation value σ_v^2 increases, we have more competition in the market and therefore a decrease in the market depth. Finally, we see that the effect of changing σ_e^2 always dominates the one of σ_v^2, the market liquidity being always decreasing in $\dfrac{\sigma_e^2}{\sigma_v^2}$.

We do obtain in our model that the behaviour of the market depth with respect to the variance of the supply and the variance of the error of the signal is very similar to the previous cases in the literature, but overall the quantitative result it is very different. We obtain that the presence of a supply informed decreases the market liquidity. Our result should be interpreted as it follows: if we have different types of information in the market, the liquidity is reduced. The result should be situated in between the one of Kyle (1985, 1989) and the subsequent literature which show that increasing the number of informed traders increases market liquidity, and the one of Subrahmanyam (1991) which shows the opposite. Thus, Subrahmanyam (1991) also obtains that market liquidity can be decreased by increasing the number of informed traders in the case the market maker is risk averse. In our model we obtain that the presence of the supply informed agent and therefore, of a different type of information, leads to a decrease in market liquidity. This result captures the intuition of Glosten and Milgrom (1985), that more information in the market decreases the market liquidity. In their model, they use as a measure of liquidity the bid-ask spread (low liquidity being equivalent to high bid-ask spread), and an increase in the number of informed agents increases the bid-ask spread. Still, if we are increasing the number of price informed traders we will have again the increase the market liquidity obtained in Kyle (1985, 1989). Despite of the fact that the decrease in the market liquidity is due to the different type of information, our result is very similar to the one of Subrahmanyam (1991). The similitude is caused by the fact that the supply informed agent is risk neutral, but he behaves strategically. Moreover, since he submits limit orders he has a market-making role, the role played by him in the economy being thus similar to the one played by the risk-averse market maker in Subrahmanyam's (1991)

model.

Once we have determined the equilibrium demand strategies we can determine also the market clearing price.

Corollary 3.2 *The equilibrium price is given by*

$$
\widetilde{p} = \frac{\sigma_e^2 \left(2N - 1\right)}{N^2\sigma_v^2 + \left(2N - 1\right)\sigma_e^2}\overline{v} + \frac{N\sigma_v^2}{N^2\sigma_v^2 + \left(2N - 1\right)\sigma_e^2}\sum_{n=1}^{N}\widetilde{i}_n
$$
$$
- \frac{N\sigma_v^2 \left(N^2\sigma_v^2 + \sigma_e^2\right)}{\left(N^2\sigma_v^2 + \left(2N - 1\right)\sigma_e^2\right)\delta^{1/2}}\widetilde{S} - \frac{2N\sigma_v^2 \left(N\sigma_v^2 + \sigma_e^2\right)}{\left(N^2\sigma_v^2 + \left(2N - 1\right)\sigma_e^2\right)\delta^{1/2}}\overline{m} \qquad (3.5)
$$

From this corollary we can see that the unconditional expectation of the equilibrium price is

$$
E\left(\widetilde{p}\right) = \overline{v} - \frac{2N\sigma_v^2 \left(N\sigma_v^2 + \sigma_e^2\right)}{\left(N^2\sigma_v^2 + \left(2N - 1\right)\sigma_e^2\right)\delta^{1/2}}\overline{m}
$$

and it depends on the expected supply $\overline{m}$. If $\overline{m} = 0$, the price is an unbiased estimator of $\overline{v}$, but it is biased if $\overline{m} \neq 0$. We also can see that as expected the higher the supply (the expected supply $\overline{m}$, or the realization of the liquidity shock $\widetilde{S}$ observed by the supply informed agent), the lower the price and the higher the signals received by the price informed agents the higher the price.

Notice also that a change in the different components of the supply has a different impact on price. A change in the known part of supply $\overline{m}$ is absorbed by the agents through the quantity demanded in a proportion of $\frac{N-1}{N}$ (we have seen while calculating the strategies that $\alpha = N\alpha^{PI} + \alpha^{SI} = g(N, \sigma_v^2, \sigma_e^2) + \frac{(N-1)}{N}\overline{m}$, where $g(N, \sigma_v^2, \sigma_e^2)$ is the function we had obtained in the Appendix) and only $\frac{1}{N}$ is reflected in price. Similarly, a shock in the component of supply known to supply informed agent $\widetilde{S}$ is absorbed half by this agent through his demand and partly is reflected in price. As I have already explained, the supply informed trader has a monopolist position and extracts rents that amount, as we saw above, to half of the unknown component of supply.

3.4 Market Indicators

In what it follows we study the implications the existence of a supply informed agent have on the market performance. We compute some market indicators: volatility of prices, informativeness of prices and expected profits of different market participants and characterize them with respect to the variance of the liquidation value of the asset.

Corollary 3.3 *The price volatility, measured as the variance of price, is*

$$Var\left(\widetilde{p}\right) = \frac{N^2\left(N-2\right)\left(\sigma_v^2\right)^2 + N\sigma_v^2\sigma_e^2\left(2N^2 - 3N - 1\right) - \left(\sigma_e^2\right)^2}{\left(N(N-2)\sigma_v^2 - \sigma_e^2\right)}\left(\frac{N\sigma_v^2}{N^2\sigma_v^2 + \left(2N-1\right)\sigma_e^2}\right)^2$$

Similar to the case when there exists no supply informed agent we have that the volatility of prices does not depend on the noise in supply. If the noise in supply increases all the agents - both the price informed and the supply informed - trade more aggressively making better use of their particular informational advantage. We can also see that price volatility may decrease or increase with the variance of the liquidation value σ_v^2. We obtain thus that the price volatility has a U shape as respect to σ_v^2. When the variance of the liquidation value σ_v^2 is small there is not too much information revealed. But as we have seen if σ_v^2 increases, the market depth decreases and this leads to more information revelation. Consequently, when σ_v^2 increases the prices become more volatile just because they contain more information. It is interesting to notice that if the competition increases the range in which the volatility of prices is a decreasing function of σ_v^2 shrinks and we recover the result from the case without supply informed trader that the higher the variance of the liquidation value of the asset, the higher the volatility of prices. As a result, in a market where there are enough price informed agents, there is more information revelation and the volatility of prices increases.

Next, we would like to find which is the amount of private information - both about the liquidation value and supply - that is revealed through prices. We define thus, the information content of prices as the difference between the prior variance of the payoff

and the variance conditional on prices. Using the normality assumptions we obtain the expression presented in the following Corollary:

Corollary 3.4 *The information content of prices is*

$$Var\left(\widetilde{v}\right) - Var\left(\widetilde{v}|\,\widetilde{p}\right) = \frac{N\sigma_v^2\left(N(N-2)\sigma_v^2 - \sigma_e^2\right)}{N^2\left(N-2\right)\left(\sigma_v^2\right)^2 + N\sigma_v^2\sigma_e^2\left(2N^2 - 3N - 1\right) - \left(\sigma_e^2\right)^2}.$$

Similarly to the previous Corollary, we obtain here also that price efficiency or the information content of prices does not depend on the variance of supply shock $\widetilde{S}$. Moreover, we obtain that informativenes of prices is increasing the variance of the liquidation value σ_v^2 and decreasing in the variance σ_e^2. These results tells us that when it is difficult to predict the liquidation value or when the signals of price informed agents are poor, the prices play a very important role in information revelation. While these results, are qualitatively similar to the case without supply informed agent, as we will see later they are quantitatively different.

Let us turn to the expected volume traded by the price informed agent and supply informed agent, respectively.

Corollary 3.5 *The expected volume traded by a price informed agent is*

$$E\left(|x_n|\right) = \frac{2\left(N-1\right)\sigma_v^2\overline{m}}{N^2\sigma_v^2 + \sigma_e^2} + \frac{\left(\frac{2}{\pi}\right)^{1/2}}{4N^2}\left(\frac{\left(N^2\sigma_v^2 + \sigma_e^2\right)^2 + N\left(N\sigma_v^2 + \sigma_e^2\right)^2}{\left(N\sigma_v^2 + \sigma_e^2\right)^2\left(N^2\sigma_v^2 + \sigma_e^2\right)^2}\left(\sigma_v^2 + \sigma_e^2\right)\delta + \sigma_S^2\right).$$

The expected volume traded by the supply informed trader is

$$E\left(|y|\right) = \frac{2\left(N\sigma_v^2 + \sigma_e^2\right)\overline{m}}{\left(N^2\sigma_v^2 + \sigma_e^2\right)} + \left(\frac{1}{8\pi}\right)^{1/2}\sigma_S^2\left(1 + \frac{\left(N-1\right)\sigma_e^2\left(N(N-2)\sigma_v^2 - \sigma_e^2\right)\left(\sigma_v^2 + \sigma_e^2\right)}{N\left(N^2\sigma_v^2 + \sigma_e^2\right)\left(N\sigma_v^2 + \sigma_e^2\right)^2}\right).$$

The expected volume traded by price informed agents and supply informed agent depend positively on the expected supply $\overline{m}$ and the variance of the supply shock σ_S^2. However, both the effects of an increase in σ_S^2 and in $\overline{m}$ are stronger in the case of supply informed trader. This is the role we wanted actually the supply informed agent to have.

Since he has information about supply he captures a big part of the shocks. Finally, the comparative statics with respect to the variance of the liquidation value σ_v^2 and the one of the error σ_e^2 are ambiguous. In the case without supply informed we have that the expected volume traded by the informed agents increases when the the variance of liquidation value σ_v^2 increases and the variance the error σ_e^2 decreases. Actually, when the ratio of the variance of the error to the variance of the liquidation value σ_v^2 increases (so the quality of his signal decreases) the expected volume traded increases because the agent has not good informational advantage. However, the presence of a supply informed agent diminishes the informational advantage of the price informed agents and therefore, they are forced to trade more aggresively on their information.

We compute next the unconditional profits for all agents.

Corollary 3.6 *The unconditional expected profit of the nth price informed agent is*

$$
\begin{aligned}
\Pi_n^{PI} &= E\left(\widetilde{\pi}_n^{PI}\right) = \frac{\sigma_v^2 \delta^{1/2}\left(N-1\right)\sigma_e^2}{2N\left(N^2\sigma_v^2 + \left(2N-1\right)\sigma_e^2\right)\left(N\sigma_v^2 + \sigma_e^2\right)}\left(\frac{N\left(N\sigma_v^2 + \sigma_e^2\right)}{\left(N(N-2)\sigma_v^2 - \sigma_e^2\right)} - \right. \\
&\left. \frac{\left(N-1\right)\sigma_e^2}{\left(N^2\sigma_v^2 + \sigma_e^2\right)}\right) + \frac{\left(N-1\right)}{\left(N^2\sigma_v^2 + \sigma_e^2\right)}\frac{2N\sigma_v^2\left(N\sigma_v^2 + \sigma_e^2\right)}{\left(N^2\sigma_v^2 + \left(2N-1\right)\sigma_e^2\right)\delta^{1/2}}\overline{m}^2.
\end{aligned}
$$

The unconditional profit of the supply informed agent is

$$
\begin{aligned}
\Pi^{SI} &= E\left(\widetilde{\pi}^{SI}\right) = \frac{\delta^{1/2}\left(N-1\right)\sigma_e^2\sigma_v^2}{2\left(N^2\sigma_v^2 + \left(2N-1\right)\sigma_e^2\right)}\left(\frac{\left(N-1\right)\sigma_e^2}{\left(N^2\sigma_v^2 + \sigma_e^2\right)\left(N\sigma_v^2 + \sigma_e^2\right)} + \right. \\
&\left. \frac{N}{\left(N(N-2)\sigma_v^2 - \sigma_e^2\right)}\right) + \frac{2N\sigma_v^2\left(N\sigma_v^2 + \sigma_e^2\right)}{\left(N^2\sigma_v^2 + \left(2N-1\right)\sigma_e^2\right)\delta^{1/2}}\frac{\left(N\sigma_v^2 + \sigma_e^2\right)}{\left(N^2\sigma_v^2 + \sigma_e^2\right)}\overline{m}^2.
\end{aligned}
$$

As we expected, allowing the supply informed agent to behave strategically allows him to make positive profits by comparison with the case of perfect competition when he is making zero profits. Notice also that since the price informed traders absorb always $\frac{1}{2N}$ of the shock S, it is actually the different information that they receive the one that makes them have different profits. We want to see also which is the impact of changes in supply on the equilibrium price and the quantity demanded by the different agents. Similar to Gennotte and Leland (1990) we study the two following cases: a supply increase known to all agents $\overline{m}$, and a supply increase known only to supply informed agent $\widetilde{S}$.

Corollary 3.7 *A positive shock in supply known to all the agents $\overline{m}$ leads to an increase in the demand of both type of agents, a decrease in the equilibrium price and therefore, to an increase in the expected profits of both type of agents.*

As expected, an increase in the supply known to all agents determines them to adjust their demands according with the existent supply, and it also leads to a decrease of the equilibrium price. We obtain here that the price informed agents are always absorbing a greater proportion of the shock in supply $\overline{m}$.

Corollary 3.8 *A positive shock in the component of supply $\widetilde{S}$, known to the supply informed agent decreases the equilibrium price and increases the quantities demanded by both price informed and supply informed agents.*

As expected, in the case of a positive shock in the supply $\widetilde{S}$, the supply informed agent increases his demand making use of the private information he has. Moreover, the increase in supply (due to a positive shock in $\widetilde{S}$) absorbed by the supply informed agent is N times higher than the increase of supply absorbed by any price informed agents. An interesting result is that the supply informed agent is always absorbing half of the unobservable shock in supply, the other half being absorbed by the price informed agents.

3.5   Equilibrium without Supply Informed Agent

In order to see which are the effects of different types of information on market liquidity, informativeness of prices, price volatility, and the ability of informed traders to exploit their private information we need to provide a benchmark for making comparison with the equilibrium characterized in the previous section. A first step will be to see how the presence in the market of a supply informed agent affects all these market structure indicators. For that we characterize first, in a similar manner, the equilibrium without a supply informed agent. Notice that this model is a version of Kyle's (1989) model with the difference that we do not have uniformed agents and we replace the noise agents by a random supply.

Proposition 3.2 *There exists a unique linear symmetric equilibrium defined as:*

$$X_{I,n}\left(\widetilde{p},\widetilde{i}_n\right) = \alpha_I + \beta_I \widetilde{i}_n - \gamma_I \widetilde{p}, \ \text{for any } n = 1, ..., N$$

where $\alpha_I, \beta_I, \gamma_I$ *are given by*

$$
\begin{aligned}
\alpha_I &= \frac{2\sigma_e^2}{N\sigma_v^2}\left(\frac{(N-2)\sigma_S^2}{N(N-1)\sigma_e^2}\right)^{1/2}\overline{v} + \frac{(N-2)}{N(N-1)}\overline{m} \\
\beta_I &= \left(\frac{(N-2)\sigma_S^2}{N(N-1)\sigma_e^2}\right)^{1/2} \\
\gamma_I &= \frac{N\sigma_v^2 + 2\sigma_e^2}{N\sigma_v^2}\left(\frac{(N-2)\sigma_S^2}{N(N-1)\sigma_e^2}\right)^{1/2}.
\end{aligned}
$$

Similarly to the case with supply informed agent we proceed with the calculations of the equilibrium price and equilibrium quantities traded by the price informed agent.

Corollary 3.9 *The equilibrium price when there is no supply informed agent is*

$$
\begin{aligned}
\widetilde{p}_I &= \frac{2\sigma_e^2}{N\sigma_v^2 + 2\sigma_e^2}\overline{v} + \frac{\sigma_v^2}{N\sigma_v^2 + 2\sigma_e^2}\sum_{n=1}^{N}\widetilde{i}_n - \frac{\sigma_v^2}{N\sigma_v^2 + 2\sigma_e^2}\left(\frac{N(N-1)\sigma_e^2}{(N-2)\sigma_S^2}\right)\widetilde{S} \\
&\quad - \frac{\sigma_v^2}{(N\sigma_v^2 + 2\sigma_e^2)(N-1)}\left(\frac{N(N-1)\sigma_e^2}{(N-2)\sigma_S^2}\right)^{1/2}\overline{m}.
\end{aligned}
$$

Notice that the price is here also an unbiased estimator of $\widetilde{v}$ if and only if $\overline{m} = 0$. Next we compute the same market indicators we have computed for the economy with a supply informed agent. An interesting remark to be made is that neither the volatility of prices nor the efficiency of prices depend on the shocks in supply.

Corollary 3.10 *The market indicators for an economy without a supply informed agent are the following:*

1) The price volatility, measured as the variance of price, is

$$Var\left(\widetilde{p}_I\right) = N\left(\frac{\sigma_v^2}{N\sigma_v^2 + 2\sigma_e^2}\right)^2\left(\sigma_v^2 + \frac{(2N-3)}{(N-2)}\sigma_e^2\right).$$

2) The information content of prices is

$$Var\left(\widetilde{v}\right) - Var\left(\widetilde{v}|\,\widetilde{p}_I\right) = N\sigma_v^2\left(N-2\right)\left(\left(N-2\right)\sigma_v^2 + \left(2N-3\right)\sigma_e^2\right)^{-1}.$$

3) The expected volume traded by a price informed agent is

$$E\left(|x_{I,n}|\right) = \frac{1}{N}\overline{m} + \left(\frac{2}{\pi}\right)^{1/2}\left(\sigma_S^2\frac{\left(\left(N-2\right)\sigma_v^2 + \left(N-1\right)\sigma_e^2\right)}{N^2\sigma_e^2}\right).$$

4) The expected profit of a price informed agent is

$$\Pi_{I,n}^{PI} = E\left(\widetilde{\pi}_{I,n}^{PI}\right) = E\left(\left(\widetilde{v}-\widetilde{p}_I\right)\widetilde{x}_n\right) = \frac{\sigma_v^2}{N\left(N\sigma_v^2 + 2\sigma_e^2\right)}\left(\frac{N\left(N-1\right)\sigma_e^2}{\left(N-2\right)\sigma_S^2}\right)^{1/2}\left(\frac{\overline{m}^2}{N-1}+\sigma_S^2\right).$$

3.6   Comparison of Market Indicators

We are comparing now the market indicators in the case there exists a supply informed agent with the case there is no supply informed agent. We study first the effect the presence of the supply informed agent brings about on the market depth. We have that

$$\gamma \equiv N\gamma^{PI} + \gamma^{SI} = \frac{\left(N^2\sigma_v^2 + \left(2N-1\right)\sigma_e^2\right)\sigma_e}{2N^2\sigma_v^2\left(N\sigma_v^2+\sigma_e^2\right)\sigma_S}\left(\frac{\left(N(N-2)\sigma_v^2 - \sigma_e^2\right)\left(N^2\sigma_v^2+\sigma_e^2\right)}{\left(N-1\right)}\right)^{1/2}$$

the market depth in the case we have a supply informed agent and

$$\gamma_{PI} \equiv N\gamma_I = \frac{\left(N\sigma_v^2 + 2\sigma_e^2\right)\sigma_S}{\sigma_v^2\sigma_e}\left(\frac{\left(N-2\right)}{N\left(N-1\right)}\right)$$

the market depth in the case we do not have any supply informed agent. The market depth is smaller in the case we have a supply informed agent in the market $\gamma < \gamma_{PI}$. This result is quite intuitive if we think that the supply informed agent plays a dual role in the market. First, he reveals himself a part of his information in the process of trading. Second, by having the information about supply he determines the price informed agents to reveal more of the information they own.

An interesting result that we obtain is that when there exists a supply informed trader in the market the price informed traders are trading more aggressively on their private

information $\left(\beta^{PI} > \beta_I\right)$ but they devote less to the market making activity

$$\omega^{PI} = \frac{\sigma_e^2 \left(N\left(3N-2\right)\sigma_v^2 + \left(2N-1\right)\sigma_e^2\right)\delta^{1/2}}{2N^2\sigma_v^2 \left(N^2\sigma_v^2 + \sigma_e^2\right)\left(N\sigma_v^2 + \sigma_e^2\right)} < \omega_I = \frac{2\sigma_e^2}{N\sigma_v^2}\left(\frac{\left(N-2\right)\sigma_S^2}{N\left(N-1\right)\sigma_e^2}\right).$$

The inside information allows the price informed agents to make gains on the expense of the market makers. However, when there exists a supply informed agent who has the ability to disentangle the order flow originated by price informed agents from a shock in supply, the advantage of the price informed agent diminishes and therefore, his market making gains. A part of the gains that the price informed agents where making from market maker activity are now made by the supply informed agent. As we have seen already the price informed agents still put a higher weight on the maker making activity than the supply informed agent does. This tells us that a dealer although he might have information about supply faces strong competition in market making from the other traders. Moreover, we have that the effect of trading more aggressively on their information dominates the effect of decreasing the market maker activity and this leads to a higher trading volume by price informed agents.

Proposition 3.3 *The presence of the supply informed agent in the market leads to higher volatility of prices, higher informativeness of prices and higher volume of trading by price informed agents.*

The results that the volatility and informativeness of prices increase in the case there exists a supply informed agent is due to two factors. First, the existence in the market of the information about supply forces informed agents to reveal more of their information. But also, the shock in supply affects more the price than in the case there is no supply informed agent because the price informed agents get some information about supply from the action of the supply informed trader.

3.7   Conclusions

In this paper we have presented a model of insider trading where the agents might have information either about prices or about supply. This information is aggregated

and partially revealed through the equilibrium price, so the agents will end up with more information than they initially posses. Our goal is twofold. First we try to understand how the presence in the market of a supply informed agent and the interaction with the price informed agents can change the behaviour of the price informed agents and the structure of the market. Then, we see how the shocks in different components of supply can alter the market structure, the price formation and the behaviour of the agents, and therefore the impact of this shocks in the equilibrium outcome.

We consider an imperfectly competitive rational expectations setup and characterize the Bayesian Nash equilibrium in demand schedules. We characterize in closed form the symmetric linear equilibrium for the case the errors of the signals of the price informed agents are noncorrelated. Allowing the supply informed agent to behave strategically, he makes positive profits (unlike in the perfect competitive case) and increases the amount of information revealed in prices. We see that he has a dual role in inducing information transmission in the market: first because he owns superior information which he reveals in the trading process and second, because he urges the price informed agent to reveal more of their information. Hower, the most important consequence of his presence in the market is that he decreases market liquidity (this outcome being brought about the strategic behavior and the mechanism of information transmission through prices).

We have also studied how the market performance is affected in our model by the quality of information received by the agents. The comparative statics results about market liquidity measured as market depth tell us that it is decreasing in the variance of the error of the signal received by price informed agents, increasing in the variance of the supply shock known only by the supply informed agent and has an inverted U shape as respect to the variance of the liquidation value. Comparing the market indicators in our model with the ones in the benchmark case (where there is no supply informed agent) we conclude that the supply informed agent does indeed have an important effect. We find that the market depth decreases, while the volatility of prices, informativeness of prices and intensity of trading of price informed agents all increase.

We have considered also the case when the supply informed agent has information

only about a component of supply. This setup is similar to the one in Gennotte and Leland (1990), where the supply has three components: a component known by everyone, a component known by the supply informed agent and another one known by nobody. The numerical analysis we have performed for this case suggests a similar pattern. However, in this case the supply informed agent will not put always a positive weight on price. Since he cannot anymore disentangle perfectly the two factors that might affect the prices (the news about the liquidation value of the asset revealed by the price informed agents or a shock in the unknown component of supply), he will not have anymore the same effect on market liquidity. However, for relative high variance of the known component in supply relative to the unknown component, $\dfrac{\sigma_S}{\sigma_L}$ the result we have obtained here will still hold.

Finally, we would like to extend our work in modelling the process of information aquisition in similar way to the work of Froot et al. (1992). They develop a model a la Kyle (1985) were the informed traders have the possibility to acquire information about two different components of the liquidation value of the asset and show that the traders may herd on the same information trying to learn what other traders also know.

3.8   Appendix

Lemma A.1 *In a symmetric linear equilibrium* $N\gamma^{PI} + \gamma^{SI} \neq 0$.

Proof. We look for a symmetric linear equilibrium. Therefore, we use the linear strategies defined in (3.2) and we can write the market clearing condition (3.1) as it follows:

$$N\alpha^{PI} + \beta^{PI}\sum_{n=1}^{N}\widetilde{i}_n - N\gamma^{PI}\widetilde{p} + \alpha^{SI} + \beta^{SI}\widetilde{S} - \gamma^{SI}\widetilde{p} = \overline{m} + \widetilde{S}. \qquad (3.6)$$

We define $\gamma \equiv N\gamma^{PI} + \gamma^{SI}$ and $\alpha \equiv N\alpha^{PI} + \alpha^{SI}$. Using these definitions, the market clearing condition can be written as

$$\alpha + \beta^{PI}\sum_{n=1}^{N}\widetilde{i}_n - \gamma\widetilde{p} - (1 - \beta^{SI})\widetilde{S} = \overline{m}.$$

We want to prove that $\gamma \neq 0$. Let us suppose that $\gamma = 0$. Then, the above condition becomes

$$\alpha + \beta^{PI} \sum_{n=1}^{N} \widetilde{i_n} - (1 - \beta^{SI})\widetilde{S} = \overline{m}.$$

Since $\widetilde{i_n}$, $n = 1, ..., N$ are independent of $\widetilde{S}$, it results that $\beta^{PI} = 0$. Plugging it in the above equation we obtain that

$$\alpha - (1 - \beta^{SI})\widetilde{S} = \overline{m},$$

which cannot be satisfied because α and $\overline{m}$ are real numbers and $\widetilde{S}$ is a random variable. We obtained therefore, a contradiction. ∎

Lemma A.2 *In a symmetric linear equilibrium the optimal demand for the price informed trader n and for the supply informed trader are, respectively,*

$$x_n\left(\widetilde{p}, \widetilde{i_n}\right) = \left((N-1)\gamma^{PI} + \gamma^{SI}\right)\left[E\left(\widetilde{v}\,\middle|\,\widetilde{p}, \widetilde{i_n}\right) - \widetilde{p}\right] \tag{3.7}$$

$$y(\widetilde{p}, \widetilde{S}) = N\gamma^{PI}\left[E\left(\widetilde{v}\,\middle|\,\widetilde{p}, \widetilde{S}\right) - \widetilde{p}\right] \tag{3.8}$$

with $\gamma^{PI} > 0$, and $(N-1)\gamma^{PI} + \gamma^{SI} > 0$.

Proof. Let us first determine the optimal demand for the price informed traders. Price informed trader n considers the other players' strategies as given by (3.2). As a result, he is facing the following residual demand:

$$p = \frac{\alpha - \alpha^{PI} + \beta^{PI} \sum_{j \neq n} \widetilde{i_j} - (1 - \beta^{SI})\widetilde{S} - \overline{m}}{(N-1)\gamma^{PI} + \gamma^{SI}} + \frac{x_n}{(N-1)\gamma^{PI} + \gamma^{SI}} \tag{3.9}$$

and he solves the following maximization problem:

$$\max_{x_n \in \mathbb{R}} E\left((\widetilde{v} - \widetilde{p})\,x_n\,\middle|\,\widetilde{p}, \widetilde{i_n}\right) \Leftrightarrow$$

$$\max_{x_n \in \mathbb{R}} E\left(\left(\widetilde{v} - \frac{\alpha - \alpha^{PI} + \beta^{PI} \sum_{j \neq n} \widetilde{i_j} - (1 - \beta^{SI})\widetilde{S} - \overline{m} - x_n}{(N-1)\gamma^{PI} + \gamma^{SI}}\right)x_n\,\middle|\,\widetilde{p}, \widetilde{i_n}\right).$$

The first order condition for this problem is

$$
E\left(\tilde{v}|\tilde{p}, \tilde{i}_n\right) - E\left(\left.\frac{\alpha - \alpha^{PI} + \beta^{PI} \sum\limits_{j \neq n} \tilde{i}_j - (1 - \beta^{SI})\tilde{S} - \overline{m}}{(N-1)\gamma^{PI} + \gamma^{SI}} \right| \tilde{p}, \tilde{i}_n\right)
$$
$$
- \frac{2x_n}{(N-1)\gamma^{PI} + \gamma^{SI}} = 0. \tag{3.10}
$$

Using (3.9) we can write further (3.10) as

$$
E\left(\tilde{v}\left|\tilde{p}, \tilde{i}_n\right.\right) - p - \frac{x_n}{(N-1)\gamma^{PI} + \gamma^{SI}} = 0,
$$

and from here we find the optimal demand of price informed trader n :

$$
x_n = \left((N-1)\gamma^{PI} + \gamma^{SI}\right)\left(E\left(\tilde{v}\left|\tilde{p}, \tilde{i}_n\right.\right) - p\right).
$$

The second order sufficient condition for this maximization problem is

$$
-\frac{2}{(N-1)\gamma^{PI} + \gamma^{SI}} < 0 \Leftrightarrow (N-1)\gamma^{PI} + \gamma^{SI} > 0.
$$

Similarly, the supply informed trader takes as given the strategies of the price informed traders and in conformity with (3.2). The residual demand faced by him is therefore

$$
p = \frac{N\alpha^{PI} + N\beta^{PI}\tilde{v} + \beta^{PI} \sum\limits_{n=1}^{N} \tilde{e}_n - \overline{m} - \tilde{S}}{N\gamma^{PI}} + \frac{y}{N\gamma^{PI}}. \tag{3.11}
$$

The supply informed trader solves the following maximization problem:

$$
\max_{y \in \mathbb{R}} E\left(\left(\tilde{v} - \tilde{p}\right)y \left|\tilde{p}, \tilde{S}\right.\right) \Leftrightarrow
$$
$$
\max_{y \in \mathbb{R}} E\left(\left.\left(\left(\tilde{v} - \frac{N\alpha^{PI} + N\beta^{PI}\tilde{v} + \beta^{PI} \sum\limits_{n=1}^{N} \tilde{e}_n - \overline{m} - \tilde{S}}{N\gamma^{PI}} + \frac{y}{N\gamma^{PI}}\right)y\right) \right| \tilde{p}, \tilde{S}\right).
$$

The first order condition for this problem is

$$
E\left(\tilde{v}|\tilde{p}, \tilde{S}\right) - E\left(\left.\frac{N\alpha^{PI} + N\beta^{PI}\tilde{v} + \beta^{PI} \sum\limits_{n=1}^{N} \tilde{e}_j - \overline{m} - \tilde{S} - \tilde{L}}{N\gamma^{PI}} \right| \tilde{p}, \tilde{S}\right) - \frac{2y}{N\gamma^{PI}} = 0.
$$
$$\tag{3.12}$$

Using (3.11) we can write further (3.12) as

$$E\left(\widetilde{v}\,\middle|\,\widetilde{p},\widetilde{S}\right) - p - \frac{y}{N\gamma^{PI}} = 0,$$

and from here we find the optimal demand of supply informed trader

$$y = N\gamma^{PI}\left(E\left(\widetilde{v}\,\middle|\,\widetilde{p},\widetilde{S}\right) - p\right).$$

The second order sufficient condition for this maximization problem is

$$-\frac{2}{N\gamma^{PI}} < 0 \Leftrightarrow N\gamma^{PI} > 0.$$

Since $N \geq 1$ it results $\gamma^{PI} > 0$. $\blacksquare$

Lemma A.3 *In a symmetric linear equilibrium for any $n = 1, ..., N$ we have*

$$\begin{aligned}
E\left(\widetilde{v}\,\middle|\,\widetilde{p} = p, \widetilde{i}_n = i_n\right) &= \overline{v}\left(1 - A\left(N-1\right)\beta^{PI} - B\right) - A(\alpha - \overline{m}) \\
&\quad + (B - A\beta^{PI})\widetilde{i}_n + A\gamma\widetilde{p}.
\end{aligned}$$

Proof.   We can rewrite the market clearing condition (3.6) as

$$\widetilde{p}\gamma - \alpha + \overline{m} - \beta^{PI}\widetilde{i}_n = (N-1)\beta^{PI}\widetilde{v} + \beta^{PI}\sum_{j\neq n}\widetilde{e}_j - (1-\beta^{SI})\widetilde{S}. \tag{3.13}$$

From here it results that $\left(\widetilde{p}, \widetilde{i}_n\right)$ is informationally equivalent to $\left(\widetilde{h_n}, \widetilde{i}_n\right)$ where by definition $\widetilde{h_n} \equiv (N-1)\beta^{PI}\widetilde{v} + \beta^{PI}\sum_{j\neq n}\widetilde{e}_j - (1-\beta^{SI})\widetilde{S}$. Consequently, we have $E\left(\widetilde{v}\,\middle|\,\widetilde{p} = p, \widetilde{i}_n = i_n\right) = E\left(\widetilde{v}\,\middle|\,\widetilde{h_n} = h_n, \widetilde{i}_n = i_n\right)$. Applying the projection theorem for normally distributed random variables we obtain that

$$E\left(\widetilde{v}\,\middle|\,\widetilde{h_n} = h_n, \widetilde{i}_n = i_n\right) = \overline{v} + \begin{pmatrix} A & B \end{pmatrix}\begin{pmatrix} \widetilde{h_n} - E\left(\widetilde{h_n}\right) \\ \widetilde{i}_n - E\left(\widetilde{i}_n\right) \end{pmatrix}, \tag{3.14}$$

where $\begin{pmatrix} A & B \end{pmatrix} = cov\left(\widetilde{v}, \left(\widetilde{h_n}, \widetilde{i}_n\right)\right)\left(var\left(\widetilde{h_n}, \widetilde{i}_n\right)\right)^{-1}$, when $\left(var\left(\widetilde{h_n}, \widetilde{i}_n\right)\right)^{-1}$ exists.

We compute $cov\left(\widetilde{v}, \widetilde{h_n}\right) = cov\left(\widetilde{v}, \ (N-1)\beta^{PI}\widetilde{v} + \ \beta^{PI}\sum_{j\neq n}\widetilde{\xi}_j - (1-\beta^{SI})\widetilde{S}\right) = (N-1)\beta^{PI}\sigma_v^2$. Hence, we have that

$$cov\left(\widetilde{v}, \left(\widetilde{h_n, i_n}\right)\right) = \left(cov\left(\widetilde{v}, \widetilde{h_n}\right), \ cov(\left(\widetilde{v}, \widetilde{i_n}\right))\right) = \left((N-1)\beta^{PI}\sigma_v^2, \ \sigma_v^2\right).$$

Then we calculate the variance matrix. We calculate firstly

$$\begin{aligned} var\left(\widetilde{h_n}\right) &= var\left((N-1)\beta^{PI}\widetilde{v} + \beta^{PI}\sum_{j\neq n}\widetilde{e_j} - (1-\beta^{SI})\widetilde{S}\right) = \\ &= (\beta^{PI})^2(N-1)\left((N-1)\sigma_v^2 + \sigma_e^2\right) + (1-\beta^{SI})^2\sigma_S^2. \end{aligned}$$

In order to simplify the notation we define $q \equiv (N-1)\left((N-1)\sigma_v^2 + \sigma_e^2\right)$. Next we see that $cov\left(\widetilde{h_n}, \widetilde{i_n}\right) = (N-1)\beta^{PI}\sigma_v^2$ and consequently, we can write the variance matrix as

$$var\left(\left(\widetilde{h_n, i_n}\right)\right) = \begin{pmatrix} (\beta^{PI})^2 q + (1-\beta^{SI})^2\sigma_S^2 & (N-1)\beta^{PI}\sigma_v^2 \\ (N-1)\beta^{PI}\sigma_v^2 & \sigma_v^2 + \sigma_e^2 \end{pmatrix}.$$

The determinant of the variance matrix is

$$M = (\beta^{PI})^2 (N-1)\left(N\sigma_v^2 + \sigma_e^2\right)\sigma_e^2 + (1-\beta^{SI})^2\sigma_S^2\left(\sigma_v^2 + \sigma_e^2\right).$$

and this is always higher than zero.

Since $M \neq 0$, it exists the inverse of the variance matrix and it equals to

$$\left(var\left(\widetilde{h_n, i_n}\right)\right)^{-1} = \frac{1}{M}\begin{pmatrix} \sigma_v^2 + \sigma_e^2 & -(N-1)\beta^{PI}\sigma_v^2 \\ -(N-1)\beta^{PI}\sigma_e^2 & (\beta^{PI})^2 q + (1-\beta^{SI})^2\sigma_S^2 \end{pmatrix}.$$

Once we have calculated $cov\left(\widetilde{v}, \left(\widetilde{h_n, i_n}\right)\right)$ and $\left(var\left(\widetilde{h_n, i_n}\right)\right)^{-1}$ we can proceed and identify A and B. It results that

$$A = M^{-1}(N-1)\beta^{PI}\sigma_v^2\sigma_e^2 \text{ and}$$

$$B = M^{-1}\left[(\beta^{PI})^2 (N-1)\sigma_v^2\sigma_e^2 + \left(1-\beta^{SI}\right)^2\sigma_S^2\sigma_v^2\right]. \tag{3.15}$$

Since $\widetilde{h_n} \equiv (N-1)\beta^{PI}\widetilde{v} + \beta^{PI}\sum_{j\neq n}\widetilde{e_j} - (1-\beta^{SI})\widetilde{S}$ we have $E\left(\widetilde{h_n}\right) = (N-1)\beta^{PI}\overline{v}$. In addition, we assumed that $E\left(\widetilde{i_n}\right) = E(\widetilde{v} + \widetilde{e_n}) = \overline{v}$. Using the above values for expectations and the formula (3.13) for $\widetilde{h_n}$ the expression (3.14) can be written as

$$E\left(\widetilde{v}\,\Big|\,\widetilde{h_n} = h_n, \widetilde{i_n} = i_n\right) = \overline{v} + A\left(\widetilde{h_n} - (N-1)\beta^{PI}\overline{v}\right) + B\left(\widetilde{i_n} - \overline{v}\right)$$

$$= \overline{v}\left(1 - A(N-1)\beta^{PI} - B\right) - A(\alpha - \overline{m}) + (B - A\beta^{PI})\widetilde{i_n} + A\gamma\widetilde{p}, \tag{3.16}$$

where A and B are given by (3.15). ∎

Lemma A.4 *In a symmetric linear equilibrium we have*

$$E(\widetilde{v}|\widetilde{p} = p, \widetilde{S} = S) = \overline{v}\left(1 - CN\beta^{PI}\right) - C(\alpha - \overline{m}) + (1 - \beta^{SI})C\widetilde{S} + C\gamma\widetilde{p}.$$

Proof. We write again the market clearing condition (3.6) this time in order to find a pair informationally equivalent to $\left(\widetilde{p}, \widetilde{S}\right)$

$$\widetilde{p}\gamma - \alpha + \overline{m} + (1 - \beta^{SI})\widetilde{S} = \beta^{PI}\sum_{n=1}^{N} \widetilde{i_n}. \tag{3.17}$$

We define $\theta \equiv \beta^{PI}\sum_{n=1}^{N} \widetilde{i_n}$. From here it results that $\left(\widetilde{\theta}, \widetilde{S}\right)$ is informationally equivalent to $\left(\widetilde{p}, \widetilde{S}\right)$. Consequently, $E\left(\widetilde{v}\,\middle|\,\widetilde{p} = p, \widetilde{S} = S\right) = E\left(\widetilde{v}\,\middle|\,\widetilde{\theta} = \theta, \widetilde{S} = S\right)$. Applying again the projection theorem for normally distributed random variables we obtain that

$$E\left(\widetilde{v}\,\middle|\,\widetilde{\theta} = \theta, \widetilde{S} = S\right) = \overline{v} + \begin{pmatrix} C, & D \end{pmatrix}\begin{pmatrix} \widetilde{\theta} - E\left(\widetilde{\theta}\right) \\ \widetilde{S} - E\left(\widetilde{S}\right) \end{pmatrix}, \tag{3.18}$$

where $\begin{pmatrix} C, & D \end{pmatrix} = cov\left(\widetilde{v}, \left(\widetilde{\theta}, \widetilde{S}\right)\right)\left(var\left(\widetilde{\theta}, \widetilde{S}\right)\right)^{-1}$.

Let us calculate $cov\left(\widetilde{v}, \left(\widetilde{\theta}, \widetilde{S}\right)\right)$. First we compute the covariance of $\widetilde{v}$ and $\widetilde{\theta}$ $cov\left(\widetilde{v}, \widetilde{\theta}\right) = cov\left(\widetilde{v}, \ N\beta^{PI}\widetilde{v} + \beta^{PI}\sum_{n=1}^{N} \widetilde{e_n}\right) = N\beta^{PI}\sigma_v^2$. Since $\widetilde{v}$ and $\widetilde{S}$ are independent random variables, it results that $cov\left(\widetilde{v}, \left(\widetilde{\theta}, \widetilde{S}\right)\right) = \begin{pmatrix} N\beta^{PI}\sigma_v^2, & 0 \end{pmatrix}$. Similarly we calculate the variance-covariance matrix. First, we calculate

$$cov\left(\widetilde{\theta}, \widetilde{\theta}\right) = cov\left(N\beta^{PI}\widetilde{v} + \beta^{PI}\sum_{n=1}^{N} \widetilde{e_n}, \ N\beta^{PI}\widetilde{v} + \beta^{PI}\sum_{n=1}^{N} \widetilde{e_n}\right) = (\beta^{PI})^2 N\left(N\sigma_v^2 + \sigma_e^2\right).$$

Then notice that $cov\left(\widetilde{\theta}, \widetilde{S}\right) = cov\left(N\beta^{PI}\widetilde{v} + \beta^{PI}\sum_{n=1}^{N} \widetilde{e_n}, \widetilde{S}\right) = 0$. It results that

$$var\left(\widetilde{\theta}, \widetilde{S}\right) = \begin{pmatrix} (\beta^{PI})^2 N\left(N\sigma_v^2 + \sigma_e^2\right) & 0 \\ 0 & \sigma_S^2 \end{pmatrix}.$$

The variance matrix is nonsingular and its inverse is

$$\left(var\left(\tilde{\theta},\tilde{S}\right)\right)^{-1} = \begin{pmatrix} ((\beta^{PI})^2 N\,(N\sigma_v^2 + \sigma_e^2))^{-1} & 0 \\ 0 & (\sigma_S^2)^{-1} \end{pmatrix},$$

and consequently,

$$C = \sigma_v^2 \left(\beta^{PI}\left(N\sigma_v^2 + \sigma_e^2\right)\right)^{-1} \text{ and } D = 0. \tag{3.19}$$

Since $E\left(\tilde{i_n}\right) = \overline{v}$, and $\tilde{\theta} \equiv N\beta^{PI}\tilde{v} + \beta^{PI}\sum_{n=1}^{N}\tilde{e_n}$ we have that $E\left(\tilde{\theta}\right) = N\beta^{PI}\overline{v}$. In addition, we assumed that $E\left(\tilde{S}\right) = 0$. Using the above values for expectations, the fact that $D = 0$ and the formula (3.17) for $\tilde{\theta}$, the expression (3.14) can be written as

$$E\left(\tilde{v}\,\Big|\,\tilde{\theta} = \theta, \tilde{S} = S\right) = \overline{v} + C\left(\tilde{\theta} - N\beta^{PI}\overline{v}\right) + D\tilde{S}$$
$$= \overline{v}\left(1 - CN\beta^{PI}\right) - C(\alpha - \overline{m}) + (1 - \beta^{SI})C\tilde{S} + C\gamma\tilde{p}, \tag{3.20}$$

where C is given by formula (3.19). ∎

Lemma A.5 *The coefficients* $\alpha^{PI}, \beta^{PI}, \gamma^{PI}, \alpha^{SI}, \beta^{SI}, \gamma^{SI}$ *are the solution of the following system of equations:*

$$\begin{cases} \alpha^{PI} = \left((N-1)\gamma^{PI} + \gamma^{SI}\right)\left(\overline{v}\left(1 - A\left(N-1\right)\beta^{PI} - B\right) - A(\alpha - \overline{m})\right) \\ \beta^{PI} = \left((N-1)\gamma^{PI} + \gamma^{SI}\right)\left(B - A\beta^{PI}\right) \\ \gamma^{PI} = \left((N-1)\gamma^{PI} + \gamma^{SI}\right)\left(1 - A\gamma\right) \\ \alpha^{SI} = N\gamma^{PI}(\overline{v}\left(1 - CN\beta^{PI}\right) - C(\alpha - \overline{m})) \\ \beta^{SI} = N\gamma^{PI}((1 - \beta^{SI})C \\ \gamma^{SI} = N\gamma^{PI}\left(1 - C\gamma\right) \\ M = \left(\beta^{PI}\right)^2 (N-1)\left(N\sigma_v^2 + \sigma_e^2\right)\sigma_e^2 + (1 - \beta^{SI})^2\sigma_S^2\left(\sigma_v^2 + \sigma_e^2\right) \\ A = M^{-1}(N-1)\beta^{PI}\sigma_v^2\sigma_e^2 \\ B = M^{-1}\left(\left(\beta^{PI}\right)^2 (N-1)\sigma_v^2\sigma_e^2 + (1 - \beta^{SI})^2\sigma_S^2\sigma_v^2\right) \\ C = \sigma_v^2\left(\beta^{PI}(N\sigma_v^2 + \sigma_e^2)\right)^{-1}. \end{cases}$$

Proof of Lemma A.5. In Lemma A.3 and Lemma A.4 for we have established the expressions for $E\left(\tilde{v}\left|\tilde{p}=p,\tilde{i_n}=i_n\right.\right)$ and $E\left(\tilde{v}\left|\tilde{p}=p,\tilde{S}=S\right.\right)$. We will use them now to find the expressions for the strategies for the price informed agents and for the supply informed agent.

First, since $E\left(\tilde{v}\left|\tilde{p}=p,\tilde{i_n}=i_n\right.\right) = E\left(\tilde{v}\left|\tilde{h_n}=h_n,\tilde{i_n}=i_n\right.\right)$ we plug (3.16) in (3.7) and we obtain that

$$x_n\left(\tilde{p},\tilde{i_n}\right) = ((N-1)\gamma^{PI}+\gamma^{SI})\left(\overline{v}\left(1-A\left(N-1\right)\beta^{PI}-B\right)-A(\alpha-\overline{m})\right.$$
$$+(B-A\beta^{PI})\tilde{i_n}+(A\gamma-1)\tilde{p}\Big).$$

We identify the coefficients in the definition of the strategy of the price informed trader n (3.2) and we get the following equations:

$$\alpha^{PI} = ((N-1)\gamma^{PI}+\gamma^{SI})(\overline{v}\left(1-A\left(N-1\right)\beta^{PI}-B\right)-A(\alpha-\overline{m}))$$
$$\beta^{PI} = ((N-1)\gamma^{PI}+\gamma^{SI})(B-A\beta^{PI})$$
$$\gamma^{PI} = ((N-1)\gamma^{PI}+\gamma^{SI})(1-A\gamma), \tag{3.21}$$

where A and B are given by (3.15).

Second, since $E\left(\tilde{v}\left|\tilde{p}=p,\tilde{S}=S\right.\right) = E\left(\tilde{v}\left|\tilde{\theta}=\theta,\tilde{S}=S\right.\right)$ we plug (3.20) in (3.8) and we obtain in a similar manner

$$y\left(\tilde{p},\tilde{S}\right) = N\gamma^{PI}\left(\overline{v}-C(\alpha-\overline{m})+(1-\beta^{SI})C\tilde{S}+(C\gamma-1)\tilde{p}\right).$$

We identify the coefficients in the definition of the strategy of the supply informed trader (3.2) and we get the following equations:

$$\alpha^{SI} = N\gamma^{PI}(\overline{v}\left(1-CN\beta^{PI}\right)-C(\alpha-\overline{m}))$$
$$\beta^{SI} = N\gamma^{PI}(1-\beta^{SI})C$$
$$\gamma^{SI} = N\gamma^{PI}(1-C\gamma), \tag{3.22}$$

where C is given by (3.19).

Putting together the equations (3.15), (3.21), (3.19) and (3.22) we obtain that $\alpha^{PI}, \beta^{PI}, \gamma^{PI}, \alpha^{SI}, \beta^{SI}, \gamma^{SI}$ are the solution of the above system of equations. ∎

Proof of Proposition 1. We leave apart the equations for α^{PI} and α^{SI} since these variables are not involved in the other equations. Then since by definition $\gamma = N\gamma^{PI} + \gamma^{SI}$ we can write the equation

$$\beta^{PI} = \left((N-1)\gamma^{PI} + \gamma^{SI}\right)\left(B - A\beta^{PI}\right)$$

as

$$\beta^{PI} = \left(\gamma - \gamma^{PI}\right)\left(B - A\beta^{PI}\right)$$

and from here to solve for β^{PI}

$$\beta^{PI} = \frac{\left(\gamma - \gamma^{PI}\right)B}{1 + A\left(\gamma - \gamma^{PI}\right)}. \tag{3.23}$$

Similarly, we have that

$$\gamma^{PI} = \left((N-1)\gamma^{PI} + \gamma^{SI}\right)(1 - A\gamma) = \left(\gamma - \gamma^{PI}\right)(1 - A\gamma)$$

and we obtain from here that

$$\gamma^{PI} = \frac{\gamma(1 - A\gamma)}{2 - A\gamma}. \tag{3.24}$$

By substituting γ^{PI} given by (3.24) in (3.23) we can write further β^{PI} only as a function of γ and A,

$$\beta^{PI} = \frac{\left(\gamma - \dfrac{\gamma(1 - A\gamma)}{2 - A\gamma}\right)B}{1 + A\left(\gamma - \dfrac{\gamma(1 - A\gamma)}{2 - A\gamma}\right)} = \frac{B\gamma}{2}. \tag{3.25}$$

We obtain the coefficients for the supply informed agent in a similar way. We have that

$$\beta^{SI} = N\gamma^{PI}(1 - \beta^{SI})C$$

and from here it results that

$$\beta^{SI} = \frac{N\gamma^{PI}C}{1 + N\gamma^{PI}C}. \tag{3.26}$$

Finally, using the formula (3.24) we obtain that

$$\gamma^{SI} = N\gamma^{PI}(1 - C\gamma) = \frac{N\gamma\,(1 - A\gamma)\,(1 - C\gamma)}{(2 - A\gamma)}. \tag{3.27}$$

By definition $\gamma = N\gamma^{PI} + \gamma^{SI}$. Then by replacing the formulas for γ^{PI} and γ^{SI} obtained before we obtain the following equation in A, C and γ :

$$\frac{2 - A\gamma}{1 - A\gamma} = N\,(2 - C\gamma).$$

In this equation we replace $\gamma = \dfrac{2\beta^{PI}}{B}$ and we obtain further

$$\frac{B - A\beta^{PI}}{B - 2A\beta^{PI}} = N\left(1 - \frac{1}{B}\frac{\sigma_v^2}{N\sigma_v^2 + \sigma_e^2}\right). \tag{3.28}$$

Using the same equation we can rewrite β^{SI} given by (3.26) in a simpler way.

$$\beta^{SI} = \frac{N\gamma^{PI}C}{1 + N\gamma^{PI}C} = \frac{NC\dfrac{\gamma\,(1 - A\gamma)}{2 - A\gamma}}{1 + NC\dfrac{\gamma\,(1 - A\gamma)}{2 - A\gamma}} = \frac{C\dfrac{\gamma}{2 - C\gamma}}{1 + C\dfrac{\gamma}{2 - C\gamma}} = \frac{C\gamma}{2}. \tag{3.29}$$

Next, we define $z \equiv \left(1 - \beta^{SI}\right)^2 \sigma_S^2$. Using (3.25), (3.29) and the equation in the system that defines C we can write

$$z = \left(\frac{2 - C\gamma}{2}\right)^2 \sigma_S^2 = \left(1 - \frac{C\beta^{PI}}{B}\right)^2 \sigma_S^2 = \left(1 - \frac{1}{B}\frac{\sigma_v^2}{N\sigma_v^2 + \sigma_e^2}\right)^2 \sigma_S^2.$$

Further on we compute the expressions for A, B as function only of β^{PI} and z. Thus

$$A = \frac{(N - 1)\beta^{PI}\sigma_v^2\sigma_e^2}{\left(\beta^{PI}\right)^2 (N - 1)\left(N\sigma_v^2 + \sigma_e^2\right)\sigma_e^2 + z\left(\sigma_v^2 + \sigma_e^2\right)}$$

By replacing it in the equation (3.28) we obtain

$$\frac{B\left(\left(\beta^{PI}\right)^2 (N - 1)\left(N\sigma_v^2 + \sigma_e^2\right)\sigma_e^2 + z\left(\sigma_v^2 + \sigma_e^2\right)\right) - (N - 1)\left(\beta^{PI}\right)^2\sigma_v^2\sigma_e^2}{B\left(\left(\beta^{PI}\right)^2 (N - 1)\left(N\sigma_v^2 + \sigma_e^2\right)\sigma_e^2 + z\left(\sigma_v^2 + \sigma_e^2\right)\right) - 2(N - 1)\left(\beta^{PI}\right)^2\sigma_v^2\sigma_e^2}$$
$$= N\left(1 - \frac{1}{B}\frac{\sigma_v^2}{N\sigma_v^2 + \sigma_e^2}\right).$$

Let us define now as $u \equiv \left(\beta^{PI}\right)^2$ and $x = \left(1 - \dfrac{1}{B}\dfrac{\sigma_v^2}{N\sigma_v^2 + \sigma_e^2}\right)$. Then it results that $z = x^2\sigma_S^2$ and the above equation can be written as

$$\frac{B\left(u\left(N-1\right)\left(N\sigma_v^2 + \sigma_e^2\right)\sigma_e^2 + x^2\sigma_S^2\left(\sigma_v^2 + \sigma_e^2\right)\right) - (N-1)u\sigma_v^2\sigma_e^2}{B\left(u\left(N-1\right)\left(N\sigma_v^2 + \sigma_e^2\right)\sigma_e^2 + x^2\sigma_S^2\left(\sigma_v^2 + \sigma_e^2\right)\right) - 2(N-1)u\sigma_v^2\sigma_e^2} = Nx. \qquad (3.30)$$

On the other hand, we have that

$$B = \frac{\left(\beta^{PI}\right)^2(N-1)\sigma_v^2\sigma_e^2 + z\sigma_v^2}{\left(\beta^{PI}\right)^2(N-1)\left(N\sigma_v^2 + \sigma_e^2\right)\sigma_e^2 + z\left(\sigma_v^2 + \sigma_e^2\right)} = \qquad (3.31)$$

$$\frac{u\left(N-1\right)\sigma_v^2\sigma_e^2 + x^2\sigma_S^2\sigma_v^2}{u\left(N-1\right)\left(N\sigma_v^2 + \sigma_e^2\right)\sigma_e^2 + x^2\sigma_S^2\left(\sigma_v^2 + \sigma_e^2\right)}.$$

We can now use this formula for B to rewrite equation (3.30) as

$$\frac{x^2\sigma_S^2\sigma_v^2}{x^2\sigma_S^2\sigma_v^2 - (N-1)u\sigma_v^2\sigma_e^2} = Nx.$$

Since $\sigma_v^2 > 0$ and $\sigma_S^2 > 0$ this equation is equivalent to $x = 0$ or $\dfrac{(Nx-1)x\sigma_S^2}{N} = (N-1)u\sigma_e^2$.

Replacing $(N-1)u\sigma_e^2 = \dfrac{(Nx-1)x\sigma_S^2}{N}$ in (3.31) we get

$$B = \frac{(Nx-1)x\sigma_S^2\sigma_v^2 + Nx^2\sigma_S^2\sigma_v^2}{(Nx-1)x\sigma_S^2\left(N\sigma_v^2 + \sigma_e^2\right) + Nx^2\sigma_S^2\left(\sigma_v^2 + \sigma_e^2\right)}.$$

But also, using the definition of x we can write

$$B = \frac{\sigma_v^2}{(1-x)\left(N\sigma_v^2 + \sigma_e^2\right)}.$$

Note that $1-x$ cannot be 0 since $\dfrac{1}{B}\dfrac{\sigma_v^2}{N\sigma_v^2 + \sigma_e^2} > 0$. So we have two expressions for B and we equalize them obtaining the following equation in x:

$$\frac{\sigma_v^2}{(1-x)\left(N\sigma_v^2 + \sigma_e^2\right)} = \frac{(Nx-1)x\sigma_S^2\sigma_v^2 + Nx^2\sigma_S^2\sigma_v^2}{(Nx-1)x\sigma_S^2\left(N\sigma_v^2 + \sigma_e^2\right) + Nx^2\sigma_S^2\left(\sigma_v^2 + \sigma_e^2\right)},$$

$$\text{or equivalent } \frac{1}{(1-x)\left(N\sigma_v^2 + \sigma_e^2\right)} = \frac{2Nx - 1}{(Nx-1)\left(N\sigma_v^2 + \sigma_e^2\right) + Nx\left(\sigma_v^2 + \sigma_e^2\right)}.$$

This last equation has two solutions

$$x = 0 \text{ and } x = \frac{(N+1)\left(N\sigma_v^2 + \sigma_e^2\right) - N\left(\sigma_v^2 + \sigma_e^2\right)}{2N\left(N\sigma_v^2 + \sigma_e^2\right)}.$$

First, if $x = 0$ then $z = x^2\sigma_S^2 = 0$ and it results $\beta^{SI} = 1$.

Then $B = \dfrac{\sigma_v^2}{(N\sigma_v^2 + \sigma_e^2)}$ and $u = 0$. But $u = 0$ implies $\beta^{PI} = 0$ and from here $A = \gamma = 0$ and the second order condition is not satisfied.

Second, we have that

$$x = \frac{(N+1)\left(N\sigma_v^2 + \sigma_e^2\right) - N\left(\sigma_v^2 + \sigma_e^2\right)}{2N\left(N\sigma_v^2 + \sigma_e^2\right)} = \frac{N^2\sigma_v^2 + \sigma_e^2}{2N\left(N\sigma_v^2 + \sigma_e^2\right)}$$

Using this formula we can compute then

$$\beta^{SI} = \frac{N^2\sigma_v^2 + (2N-1)\sigma_e^2}{2N\left(N\sigma_v^2 + \sigma_e^2\right)}, \tag{3.32}$$

$$B = \frac{2N\sigma_v^2}{N^2\sigma_v^2 + (2N-1)\sigma_e^2} \quad \text{and}$$

$$u = \frac{(Nx-1)x\sigma_S^2}{N(N-1)\sigma_e^2} = \frac{\left(N(N-2)\sigma_v^2 - \sigma_e^2\right)\left(N^2\sigma_v^2 + \sigma_e^2\right)\sigma_S^2}{4\left(N\sigma_v^2 + \sigma_e^2\right)^2 N^2\left(N-1\right)\sigma_e^2}$$

Notice that $\left(\beta^{PI}\right)^2 = u$, so we need $u \geq 0$. If $N(N-2)\sigma_v^2 - \sigma_e^2 > 0$ we have $u \geq 0$ and consequently, we have solution for β^{PI} and it is equal to

$$\beta^{PI} = \frac{1}{2N\left(N\sigma_v^2 + \sigma_e^2\right)}\sqrt{\frac{\left(N(N-2)\sigma_v^2 - \sigma_e^2\right)\left(N^2\sigma_v^2 + \sigma_e^2\right)\sigma_S^2}{(N-1)\sigma_e^2}}. \tag{3.33}$$

Using the last equation in the system we can write also

$$C = \frac{\sigma_v^2}{\beta^{PI}\left(N\sigma_v^2 + \sigma_e^2\right)} = 2N\sigma_v^2\sqrt{\frac{(N-1)\sigma_e^2}{\left(N(N-2)\sigma_v^2 - \sigma_e^2\right)\left(N^2\sigma_v^2 + \sigma_e^2\right)\sigma_S^2}}.$$

Next, since

$$\beta^{SI} = N\gamma^{PI}(1 - \beta^{SI})C,$$

we can find an expression for γ^{PI}. Using the formulas for β^{SI} (3.32) and C it results that

$$\gamma^{PI} = \frac{\beta^{SI}}{N(1 - \beta^{SI})C} = \frac{N^2\sigma_v^2 + (2N-1)\sigma_e^2}{2N^2\sigma_v^2}\sqrt{\frac{\left(N(N-2)\sigma_v^2 - \sigma_e^2\right)\sigma_S^2}{(N-1)\sigma_e^2\left(N^2\sigma_v^2 + \sigma_e^2\right)}}. \tag{3.34}$$

Similarly,

$$\gamma^{SI} = N\gamma^{PI}(1 - C\gamma) = N\gamma^{PI}(1 - 2\beta^{SI}) = -\frac{(N-1)\sigma_e^2}{(N\sigma_v^2 + \sigma_e^2)}\gamma^{PI}. \qquad (3.35)$$

An important remark has to be made. The coefficient $\gamma^{SI} < 0$, however the second order conditions are satisfied since

$$(N-1)\gamma^{PI} + \gamma^{SI} = \frac{N(N-1)\sigma_v^2}{(N\sigma_v^2 + \sigma_e^2)}\gamma^{PI} > 0.$$

We compute now the ratio $\dfrac{\gamma^{PI}}{\beta^{PI}}$ because we will make use of it later on.

$$\frac{\gamma^{PI}}{\beta^{PI}} = \frac{N^2\sigma_v^2 + (2N-1)\sigma_e^2}{N\sigma_v^2}\frac{(N\sigma_v^2 + \sigma_e^2)}{(N^2\sigma_v^2 + \sigma_e^2)}.$$

The only coefficients left to compute are α^{PI} and α^{SI}. We have from the system that

$$\alpha^{PI} = \left((N-1)\gamma^{PI} + \gamma^{SI}\right)\left(\overline{v}\left(1 - A(N-1)\beta^{PI} - B\right) - A(\alpha - \overline{m})\right)$$

$$\text{and } \alpha^{SI} = N\gamma^{PI}(\overline{v}\left(1 - CN\beta^{PI}\right) - C(\alpha - \overline{m}))$$

We will first compute $t \equiv (\alpha - \overline{m})$ and for that we use the definition $\alpha = N\alpha^{PI} + \alpha^{SI}$. Thus, we have

$$(\alpha - \overline{m}) = N\left(\frac{N(N-1)\sigma_v^2}{(N\sigma_v^2 + \sigma_e^2)}\gamma^{PI}\right)(w - A(\alpha - \overline{m})) + N\gamma^{PI}(z - C(\alpha - \overline{m})) - \overline{m}.$$

Solving for $t \equiv (\alpha - \overline{m})$ we obtain

$$t = \frac{N\gamma^{PI}\overline{v}\left(N(N-1)\sigma_v^2\left(1 - A(N-1)\beta^{PI} - B\right) + \left(1 - CN\beta^{PI}\right)(N\sigma_v^2 + \sigma_e^2)\right)}{\left((N\sigma_v^2 + \sigma_e^2)(1 + N\gamma^{PI}C) + (N(N-1)\sigma_v^2)AN\gamma^{PI}\right)}$$

$$-\frac{\overline{m}(N\sigma_v^2 + \sigma_e^2)}{\left((N\sigma_v^2 + \sigma_e^2)(1 + N\gamma^{PI}C) + (N(N-1)\sigma_v^2)AN\gamma^{PI}\right)}.$$

Using that

$$A\beta^{PI} = \frac{(N(N-2)\sigma_v^2 - \sigma_e^2)}{(N-1)(N^2\sigma_v^2 + \sigma_e^2(2N-1))},$$

$$B = \frac{2N\sigma_v^2}{N^2\sigma_v^2 + (2N-1)\sigma_e^2} \quad and$$

$$C\beta^{PI} = \frac{\sigma_v^2}{(N\sigma_v^2 + \sigma_e^2)}$$

we have that the denominator is

$$\frac{N\gamma^{PI}}{\beta^{PI}}\sigma_v^2\left(\frac{2N\left(N\sigma_v^2+\sigma_e^2\right)}{N^2\sigma_v^2+(2N-1)\,\sigma_e^2}+N\frac{\left(N\left(N-2\right)\sigma_v^2-\sigma_e^2\right)}{\left(N^2\sigma_v^2+\sigma_e^2\left(2N-1\right)\right)}\right)=\left(N\sigma_v^2+\sigma_e^2\right)N.$$

and the numerator is

$$\frac{N\gamma^{PI}\overline{v}\left(\left(2N-1\right)\sigma_e^2\left(N^2\sigma_v^2+\sigma_e^2\right)\right)}{N^2\sigma_v^2+\sigma_e^2\left(2N-1\right)}-\overline{m}\left(N\sigma_v^2+\sigma_e^2\right).$$

As a result,

$$t=\frac{\gamma^{PI}\left(\left(2N-1\right)\sigma_e^2\left(N^2\sigma_v^2+\sigma_e^2\right)\right)}{\left(N^2\sigma_v^2+\sigma_e^2\left(2N-1\right)\right)\left(N\sigma_v^2+\sigma_e^2\right)}\overline{v}-\frac{\overline{m}}{N}.$$

We proceed now with the computations of the coefficients α^{PI} and α^{SI}. As we have already seen

$$\alpha^{PI}=\left(\left(N-1\right)\gamma^{PI}+\gamma^{SI}\right)\left(\overline{v}\left(1-A\left(N-1\right)\beta^{PI}-B\right)-At\right),$$

and replacing the formulas we have obtained for A, B and t we obtain that

$$
\begin{aligned}
\alpha^{PI} &= \frac{N\left(N-1\right)\sigma_v^2}{\left(N\sigma_v^2+\sigma_e^2\right)}\gamma^{PI}\left(\frac{\sigma_e^2\left(N\left(3N-2\right)\sigma_v^2+\left(2N-1\right)\right)}{N\left(N-1\right)\sigma_v^2\left(N^2\sigma_v^2+\sigma_e^2\left(2N-1\right)\right)}\overline{v}+\frac{A}{N}\overline{m}\right)\\
&= \frac{\gamma^{PI}}{\left(N\sigma_v^2+\sigma_e^2\right)}\frac{\sigma_e^2\left(N\left(3N-2\right)\sigma_v^2+\left(2N-1\right)\right)}{\left(N^2\sigma_v^2+\sigma_e^2\left(2N-1\right)\right)}\overline{v}+\frac{\left(N-1\right)\sigma_v^2}{\left(N\sigma_v^2+\sigma_e^2\right)}\gamma^{PI}A\overline{m}\\
&= \frac{\sigma_e^2\left(N\left(3N-2\right)\sigma_v^2+\left(2N-1\right)\right)}{2N^2\sigma_v^2\left(N^2\sigma_v^2+\sigma_e^2\right)\left(N\sigma_v^2+\sigma_e^2\right)}\sqrt{\delta}\overline{v}+\frac{N\left(N-2\right)\sigma_v^2-\sigma_e^2}{N\left(N^2\sigma_v^2+\sigma_e^2\right)}\overline{m}
\end{aligned}
$$

Here we have defined δ by

$$\delta\equiv\frac{\left(N(N-2)\sigma_v^2-\sigma_e^2\right)\left(N^2\sigma_v^2+\sigma_e^2\right)\sigma_S^2}{\left(N-1\right)\sigma_e^2}.$$

Similarly,

$$
\begin{aligned}
\alpha^{SI} &= N\gamma^{PI}\left(\overline{v}\left(1-CN\beta^{PI}\right)-Ct\right)=\left(-\frac{\left(N-1\right)\sigma_e^2\gamma^{PI}}{\left(N\sigma_v^2+\sigma_e^2\right)}\right)\overline{v}+C\gamma^{PI}\overline{m}\\
&= \left(-\frac{\left(N-1\right)\sigma_e^2}{\left(N\sigma_v^2+\sigma_e^2\right)}\frac{N^2\sigma_v^2+\left(2N-1\right)\sigma_e^2}{2N^2\sigma_v^2\left(N^2\sigma_v^2+\sigma_e^2\right)}\sqrt{\delta}\right)\overline{v}+\frac{N^2\sigma_v^2+\left(2N-1\right)\sigma_e^2}{N\left(N^2\sigma_v^2+\sigma_e^2\right)}\overline{m}.
\end{aligned}
$$

Consequently, using the definition of δ we can write the coefficients in the following way:

$$\alpha^{PI} = \frac{\sigma_e^2 \left(N\left(3N-2\right)\sigma_v^2 + \left(2N-1\right)\sigma_e^2\right)\delta^{1/2}}{2N^2\sigma_v^2\left(N^2\sigma_v^2+\sigma_e^2\right)\left(N\sigma_v^2+\sigma_e^2\right)}\overline{v} + \frac{N\left(N-2\right)\sigma_v^2-\sigma_e^2}{N\left(N^2\sigma_v^2+\sigma_e^2\right)}\overline{m}$$

$$\beta^{PI} = \frac{\delta^{1/2}}{2N\left(N\sigma_v^2+\sigma_e^2\right)}$$

$$\gamma^{PI} = \frac{\left(N^2\sigma_v^2 + \left(2N-1\right)\sigma_e^2\right)\delta^{1/2}}{2N^2\sigma_v^2\left(N^2\sigma_v^2+\sigma_e^2\right)}$$

$$\alpha^{SI} = \left(-\frac{\left(N-1\right)\sigma_e^2}{\left(N\sigma_v^2+\sigma_e^2\right)}\frac{N^2\sigma_v^2+\left(2N-1\right)\sigma_e^2}{2N^2\sigma_v^2\left(N^2\sigma_v^2+\sigma_e^2\right)}\delta^{1/2}\right)\overline{v} + \frac{N^2\sigma_v^2+\left(2N-1\right)\sigma_e^2}{N\left(N^2\sigma_v^2+\sigma_e^2\right)}\overline{m}$$

$$\beta^{SI} = \frac{N^2\sigma_v^2+\left(2N-1\right)\sigma_e^2}{2N\left(N\sigma_v^2+\sigma_e^2\right)}$$

$$\gamma^{SI} = -\frac{\left(N-1\right)\sigma_e^2}{\left(N\sigma_v^2+\sigma_e^2\right)}\frac{N^2\sigma_v^2+\left(2N-1\right)\sigma_e^2}{2N^2\sigma_v^2\left(N^2\sigma_v^2+\sigma_e^2\right)}\delta^{1/2}.$$

∎

Proof of Corollary 3.1. While solving the above system we have obtained that

$$\gamma = N\gamma^{PI} + \gamma^{SI} = \frac{N^2\sigma_v^2+\left(2N-1\right)\sigma_e^2}{2N^2\sigma_v^2\left(N\sigma_v^2+\sigma_e^2\right)}\left(\frac{\left(N(N-2)\sigma_v^2-\sigma_e^2\right)\left(N^2\sigma_v^2+\sigma_e^2\right)\sigma_S^2}{\left(N-1\right)\sigma_e^2}\right)^{1/2}$$

We study first how market depth varies when the variance of liquidity shock $\widetilde{S}$ varies. We compute the derivative $\dfrac{\partial\gamma}{\partial\sigma_S^2}$ and we obtain

$$\frac{\partial\gamma}{\partial\sigma_S^2} > 0.$$

Then we calculate $\dfrac{\partial\gamma}{\partial\sigma_e^2}$ and after somehow tedious calculations we obtain that

$$\frac{\partial\gamma}{\partial\sigma_e^2} < 0.$$

Finally, we study how the variance of liquidation value, σ_v^2 affects the market depth. We calculate the derivative $\dfrac{\partial\gamma}{\partial\sigma_v^2}$ and we obtain that this expression has the opposite sign to $f\left(\sigma_v^2\right)$, where

$$\begin{aligned}f\left(\sigma_v^2\right) = {} & N^4\left(\sigma_v^2\right)^3\left(N-1\right)\left(N^2-3N+1\right) - 3\sigma_e^2N^2\left(\sigma_v^2\right)^2\left(2N-1\right)\left(N-1\right)\\ & -3\sigma_v^2\left(\sigma_e^2\right)^2 N\left(2N-1\right)\left(N-1\right) - \left(\sigma_e^2\right)^3\left(2N-1\right)\left(N-1\right).\end{aligned}$$

We study this function and we obtain that the equation $f'\left(\sigma_v^2\right) = 0$,

$$f'\left(\sigma_v^2\right) = 3\left(N-1\right)N\left[\left(N^3\left(\sigma_v^2\right)^2\left(N^2-3N+1\right) - 2\sigma_e^2 N\left(2N-1\right)\sigma_v^2 - \left(\sigma_e^2\right)^2\left(2N-1\right)\right)\right],$$

has only one positive solution equal to

$$\sigma_e^2\frac{\left(2N-1\right) + \left(N-1\right)\left(\left(2N-1\right)\left(N-1\right)\right)^{1/2}}{N^2\left(N^2-3N+1\right)} \equiv k_l\left(N\right).$$

We obtain that $k_l\left(N\right) > \dfrac{1}{N\left(N-2\right)}$. So, it results that the function $f\left(\sigma_v^2\right)$ is decreasing for $\dfrac{\sigma_v^2}{\sigma_e^2} \in \left[\dfrac{1}{N\left(N-2\right)}, k_l\left(N\right)\right]$, and is increasing for $\dfrac{\sigma_v^2}{\sigma_e^2} > k_l\left(N\right)$. Since $f\left(0\right) = -\left(\sigma_e^2\right)^3\left(2N-1\right)\left(N-1\right)$, it results that it exists $k^*\left(N,\sigma_e^2\right)$ such that $f\left(k^*\left(N,\sigma_e^2\right)\right) = 0$. Therefore, the function $f\left(\sigma_v^2\right) < 0$ for any $\sigma_v^2 < k^*\left(N,\sigma_e^2\right)$ and is greater than 0 otherwise.

Once we have characterized the behavior of function $f\left(\sigma_v^2\right)$ we can conclude that the market depth is a increasing function of σ_v^2 if $\sigma_v^2 < k^*\left(N,\sigma_e^2\right)$ and is decreasing otherwise.

∎

Proof of Corollary 3.2. From the market clearing condition (3.6) we obtain that the equilibrium price is

$$\widetilde{p} = \left(N\gamma^{PI} + \gamma^{SI}\right)^{-1}\left(\alpha + \beta^{PI}\sum_{n=1}^{N}\widetilde{i}_n - \left(1 - \beta^{SI}\right)\widetilde{S} - \overline{m}\right).$$

We had obtained in the proof of Proposition 1 that

$$\alpha = \frac{\sigma_e^2\left(2N-1\right)}{2N^2\sigma_v^2\left(N\sigma_v^2 + \sigma_e^2\right)}\delta^{1/2}\overline{v} + \frac{\left(N-1\right)}{N}\overline{m}$$

$$N\gamma^{PI} + \gamma^{SI} = \frac{\left(N^2\sigma_v^2 + \left(2N-1\right)\sigma_e^2\right)\delta^{1/2}}{2N^2\sigma_v^2\left(N\sigma_v^2 + \sigma_e^2\right)}.$$

Using these formulas and the ones for β^{PI} and β^{SI} we can write that the equilibrium price equals to

$$\widetilde{p} = \frac{\sigma_e^2\left(2N-1\right)}{N^2\sigma_v^2 + \left(2N-1\right)\sigma_e^2}\overline{v} + \frac{N\sigma_v^2}{N^2\sigma_v^2 + \left(2N-1\right)\sigma_e^2}\sum_{n=1}^{N}\widetilde{i}_n$$

$$- \frac{N\sigma_v^2\left(N^2\sigma_v^2 + \sigma_e^2\right)}{\left(N^2\sigma_v^2 + \left(2N-1\right)\sigma_e^2\right)\delta^{1/2}}\widetilde{S} - \frac{2N\sigma_v^2\left(N\sigma_v^2 + \sigma_e^2\right)}{\left(N^2\sigma_v^2 + \left(2N-1\right)\sigma_e^2\right)\delta^{1/2}}\overline{m}.$$

Notice that since $\widetilde{i}_n = \widetilde{v} + \widetilde{e}_n$ we can write

$$\widetilde{p} = \frac{\sigma_e^2 \left(2N - 1\right)}{N^2\sigma_v^2 + \left(2N - 1\right)\sigma_e^2}\overline{v} + \frac{N^2\sigma_v^2}{N^2\sigma_v^2 + \left(2N - 1\right)\sigma_e^2}\widetilde{v} + \frac{N\sigma_v^2}{N^2\sigma_v^2 + \left(2N - 1\right)\sigma_e^2}\sum_{n=1}^{N}\widetilde{e}_n$$

$$- \frac{N\sigma_v^2 \left(N^2\sigma_v^2 + \sigma_e^2\right)}{\left(N^2\sigma_v^2 + \left(2N - 1\right)\sigma_e^2\right)\delta^{1/2}}\widetilde{S} - \frac{2N\sigma_v^2 \left(N\sigma_v^2 + \sigma_e^2\right)}{\left(N^2\sigma_v^2 + \left(2N - 1\right)\sigma_e^2\right)\delta^{1/2}}\overline{m}.$$

Taking the expectations it results that $E\left(\widetilde{p}\right) = \overline{v} - \dfrac{2N\sigma_v^2 \left(N\sigma_v^2 + \sigma_e^2\right)}{\left(N^2\sigma_v^2 + \left(2N - 1\right)\sigma_e^2\right)\delta^{1/2}}\overline{m}$ ∎

Proof of Corollary 3.3. We have seen that the equilibrium price is given by (3.5). As a result, we can compute the variance

$$Var\left(\widetilde{p}\right) = Var\left(\frac{\sigma_e^2 \left(2N - 1\right)}{N^2\sigma_v^2 + \left(2N - 1\right)\sigma_e^2}\overline{v} + \frac{N\sigma_v^2}{N^2\sigma_v^2 + \left(2N - 1\right)\sigma_e^2}\sum_{n=1}^{N}\widetilde{i}_n\right.$$

$$\left. - \frac{N\sigma_v^2 \left(N^2\sigma_v^2 + \sigma_e^2\right)}{\left(N^2\sigma_v^2 + \left(2N - 1\right)\sigma_e^2\right)\delta^{1/2}}\widetilde{S} - \frac{2N\sigma_v^2 \left(N\sigma_v^2 + \sigma_e^2\right)}{\left(N^2\sigma_v^2 + \left(2N - 1\right)\sigma_e^2\right)\delta^{1/2}}\overline{m}\right) =$$

$$\frac{N^2 \left(N - 2\right)\left(\sigma_v^2\right)^2 + N\sigma_v^2\sigma_e^2 \left(2N^2 - 3N - 1\right) - \left(\sigma_e^2\right)^2}{\left(N(N - 2)\sigma_v^2 - \sigma_e^2\right)}\left(\frac{N\sigma_v^2}{N^2\sigma_v^2 + \left(2N - 1\right)\sigma_e^2}\right)^2.$$

∎

Proof of Corollary 3.4. We compute now $Var\left(\widetilde{v}\right) - Var\left(\widetilde{v}|\,\widetilde{p}\right)$. Due to the normality assumptions we have that

$$Var\left(\widetilde{v}\right) - Var\left(\widetilde{v}|\,\widetilde{p}\right) = \left(Var\left(\widetilde{p}\right)\right)^{-1}\left(Cov\left(\widetilde{v},\widetilde{p}\right)\right)^2$$

We calculate the covariance

$$Cov\left(\widetilde{v},\widetilde{p}\right) = \frac{\left(N\sigma_v^2\right)^2}{N^2\sigma_v^2 + \left(2N - 1\right)\sigma_e^2},$$

and together with the formula for variance $Var\left(\widetilde{p}\right)$ we obtained before, we plug them above to obtain

$$Var\left(\widetilde{v}\right) - Var\left(\widetilde{v}|\,\widetilde{p}\right) = \frac{N\sigma_v^2 \left(N(N - 2)\sigma_v^2 - \sigma_e^2\right)}{N^2 \left(N - 2\right)\left(\sigma_v^2\right)^2 + N\sigma_v^2\sigma_e^2 \left(2N^2 - 3N - 1\right) - \left(\sigma_e^2\right)^2}.$$

∎

Proof of Corollary 3.5. Since the demand of the price informed agent x_n can be written as the sum of normal variables it results that x_n is also a normal variable. The mean of x_n is $\mu_n = \dfrac{(N-1)\,\sigma_v^2}{(N^2\sigma_v^2 + \sigma_e^2)}\,\overline{m}$ while the variance σ_{x_n} is

$$\sigma_{x_n} = Var\left(x_n\right) = \left(\frac{(N-1)\,\sigma_e^2\delta^{1/2}}{2N\left(N^2\sigma_v^2 + \sigma_e^2\right)\left(N\sigma_v^2 + \sigma_e^2\right)}\,\overline{v} + \frac{(N-1)\,\sigma_v^2}{(N^2\sigma_v^2 + \sigma_e^2)}\,\overline{m}\right.$$

$$\left. +\frac{\delta^{1/2}}{2N\left(N\sigma_v^2 + \sigma_e^2\right)}\,\widetilde{i}_n - \frac{\delta^{1/2}}{2N\left(N^2\sigma_v^2 + \sigma_e^2\right)}\sum_{n=1}^{N}\widetilde{i}_n + \frac{1}{2N}\,\widetilde{S}\right) =$$

$$\left(\frac{\delta^{1/2}}{2N\left(N\sigma_v^2 + \sigma_e^2\right)}\right)^2\left(\sigma_v^2 + \sigma_e^2\right) + \left(\frac{\delta^{1/2}}{2N\left(N^2\sigma_v^2 + \sigma_e^2\right)}\right)^2 N\left(\sigma_v^2 + \sigma_e^2\right)$$

$$+\frac{1}{4N^2}\sigma_S^2 = \frac{\left(\sigma_v^2 + \sigma_e^2\right)\delta}{4N^2}\left(\frac{1}{\left(N\sigma_v^2 + \sigma_e^2\right)^2} + \frac{N}{\left(N^2\sigma_v^2 + \sigma_e^2\right)^2}\right) + \frac{\sigma_S^2}{4N^2}.$$

Then, since x_n is $\mathcal{N}(\mu_n, \sigma_{x_n})$ it results that the expected volume of trade

$$E\left(|x_n|\right) = \int_{-\infty}^{\infty} |x_n|\frac{1}{\sigma_{x_n}\sqrt{2\pi}}\exp\left(-\frac{(x_n - \mu_n)^2}{2\sigma_{x_n}^2}\right)dx_n = 2\mu_n + \left(\frac{2}{\pi}\right)^2\sigma_{x_n} =$$

$$\frac{2\left(N-1\right)\sigma_v^2}{\left(N^2\sigma_v^2 + \sigma_e^2\right)}\,\overline{m} + \left(\frac{2}{\pi}\right)^2\left(\frac{\left(\sigma_v^2 + \sigma_e^2\right)\delta}{4N^2}\left(\frac{1}{\left(N\sigma_v^2 + \sigma_e^2\right)^2} + \frac{N}{\left(N^2\sigma_v^2 + \sigma_e^2\right)^2}\right) + \frac{1}{4N^2}\sigma_S^2\right).$$

Similarly, the quantity demanded by the supply informed agent is a normal variable with mean $\mu_y = \dfrac{(N\sigma_v^2 + \sigma_e^2)}{(N^2\sigma_v^2 + \sigma_e^2)}\,\overline{m}$ and variance

$$\sigma_y = Var(y) = \left(\frac{(N-1)\,\sigma_e^2\delta^{1/2}}{2N\left(N^2\sigma_v^2 + \sigma_e^2\right)\left(N\sigma_v^2 + \sigma_e^2\right)}\right)^2 N\left(\sigma_v^2 + \sigma_e^2\right) + \frac{1}{4}\sigma_S^2$$

$$= \frac{1}{4}\sigma_S^2\left(1 + \frac{(N-1)\,\sigma_e^2\left(N(N-2)\sigma_v^2 - \sigma_e^2\right)\left(\sigma_v^2 + \sigma_e^2\right)}{N\left(N^2\sigma_v^2 + \sigma_e^2\right)\left(N\sigma_v^2 + \sigma_e^2\right)^2}\right).$$

Then since y is $\mathcal{N}(\mu_y, \sigma_y)$ it results that the expected volume of trade of the supply informed agent is

$$E\left(|y|\right) = \int_{-\infty}^{\infty} |y|\frac{1}{\sigma_y\sqrt{2\pi}}\exp\left(-\frac{(y - \mu_y)^2}{2\sigma_y^2}\right)dy = 2\mu_y + \sqrt{\frac{2}{\pi}}\sigma_y$$

$$= 2\frac{(N\sigma_v^2 + \sigma_e^2)}{(N^2\sigma_v^2 + \sigma_e^2)}\,\overline{m} + \left(\frac{2}{\pi}\right)^{1/2}\frac{1}{4}\sigma_S^2\left(1 + \frac{(N-1)\,\sigma_e^2\left(N(N-2)\sigma_v^2 - \sigma_e^2\right)\left(\sigma_v^2 + \sigma_e^2\right)}{N\left(N^2\sigma_v^2 + \sigma_e^2\right)\left(N\sigma_v^2 + \sigma_e^2\right)^2}\right).$$

■

Proof of Corollary 3.6. Let us compute first the unconditional expected profit of the nth price informed trader.

$$\Pi_n^{PI} = E\left(\widetilde{\pi_n^{PI}}\right) = E\left((\widetilde{v} - \widetilde{p})\,\widetilde{x_n}\right).$$

Using the formulas we have obtained for $\widetilde{p}$ and $\widetilde{x_n}$ we can write further

$$
\Pi_n^{PI} = E\left(\left(\frac{\sigma_e^2\,(2N-1)}{N^2\sigma_v^2 + (2N-1)\,\sigma_e^2}\,(\widetilde{v} - \overline{v}) - \frac{N\sigma_v^2}{N^2\sigma_v^2 + (2N-1)\,\sigma_e^2}\sum_{n=1}^{N}\widetilde{e}_n + \right.\right.
$$
$$
\frac{N\sigma_v^2\,(N^2\sigma_v^2 + \sigma_e^2)}{(N^2\sigma_v^2 + (2N-1)\,\sigma_e^2)\,\delta^{1/2}}\widetilde{S} + \frac{2N\sigma_v^2\,(N\sigma_v^2 + \sigma_e^2)}{(N^2\sigma_v^2 + (2N-1)\,\sigma_e^2)\,\delta^{1/2}}\overline{m}\right)
$$
$$
\left(-\frac{\delta^{1/2}\,(N-1)\,\sigma_e^2}{2N\,(N^2\sigma_v^2 + \sigma_e^2)\,(N\sigma_v^2 + \sigma_e^2)}\,(\widetilde{v} - \overline{v}) + \frac{\delta^{1/2}}{2N\,(N\sigma_v^2 + \sigma_e^2)}\widetilde{e}_n\right.
$$
$$
\left.\left.-\frac{1}{2N\,(N^2\sigma_v^2 + \sigma_e^2)}\delta^{1/2}\sum_{k=1}^{N}\widetilde{e}_k + \frac{1}{2N}\widetilde{S} + \frac{(N-1)\,\sigma_v^2}{(N^2\sigma_v^2 + \sigma_e^2)}\overline{m}\right)\right) =
$$
$$
\frac{\sigma_v^2\delta^{1/2}\,(N-1)\,\sigma_e^2}{2N\,(N^2\sigma_v^2 + (2N-1)\,\sigma_e^2)\,(N\sigma_v^2 + \sigma_e^2)}\left(\frac{N\,(N\sigma_v^2 + \sigma_e^2)}{(N(N-2)\sigma_v^2 - \sigma_e^2)} - \frac{(N-1)\,\sigma_e^2}{(N^2\sigma_v^2 + \sigma_e^2)}\right) +
$$
$$
+\frac{(N-1)\,\sigma_v^2}{(N^2\sigma_v^2 + \sigma_e^2)}\frac{2N\sigma_v^2\,(N\sigma_v^2 + \sigma_e^2)}{(N^2\sigma_v^2 + (2N-1)\,\sigma_e^2)\,\delta^{1/2}}\overline{m}^2.
$$

Let us compute now the unconditional expected profit of the supply informed trader. Using the formulas we have obtained for $\widetilde{p}$ and $\widetilde{y}$ we can write further

$$\Pi^{SI} = E\left(\widetilde{\pi^{SI}}\right) = E\left((\widetilde{v} - \widetilde{p})\,\widetilde{y}\right).$$

$$
\Pi^{SI} = E\left(\left(\frac{\sigma_e^2\,(2N-1)}{N^2\sigma_v^2 + (2N-1)\,\sigma_e^2}\,(\widetilde{v} - \overline{v}) - \frac{N\sigma_v^2}{N^2\sigma_v^2 + (2N-1)\,\sigma_e^2}\sum_{n=1}^{N}\widetilde{e}_n + \right.\right.
$$
$$
\frac{N\sigma_v^2\,(N^2\sigma_v^2 + \sigma_e^2)}{(N^2\sigma_v^2 + (2N-1)\,\sigma_e^2)\,\delta^{1/2}}\widetilde{S} + \frac{2N\sigma_v^2\,(N\sigma_v^2 + \sigma_e^2)}{(N^2\sigma_v^2 + (2N-1)\,\sigma_e^2)\,\delta^{1/2}}\overline{m}\right)
$$
$$
\left(\frac{(N-1)\,\sigma_e^2\delta^{1/2}}{2\,(N^2\sigma_v^2 + \sigma_e^2)\,(N\sigma_v^2 + \sigma_e^2)}\,(\widetilde{v} - \overline{v}) + \frac{(N\sigma_v^2 + \sigma_e^2)}{(N^2\sigma_v^2 + \sigma_e^2)}\overline{m}\right.
$$
$$
\left.\left.+\frac{(N-1)\,\sigma_e^2\delta^{1/2}}{2N\,(N^2\sigma_v^2 + \sigma_e^2)\,(N\sigma_v^2 + \sigma_e^2)}\sum_{n=1}^{N}\widetilde{e}_n + \frac{1}{2}\widetilde{S}\right)\right)
$$

$$= \frac{\delta^{1/2}\,(N-1)\,\sigma_e^2\sigma_v^2}{2\left(N^2\sigma_v^2+(2N-1)\,\sigma_e^2\right)}\left(\frac{(N-1)\,\sigma_e^2}{\left(N^2\sigma_v^2+\sigma_e^2\right)\left(N\sigma_v^2+\sigma_e^2\right)}+\frac{N}{\left(N(N-2)\sigma_v^2-\sigma_e^2\right)}\right)$$
$$+\frac{2N\sigma_v^2\left(N\sigma_v^2+\sigma_e^2\right)}{\left(N^2\sigma_v^2+(2N-1)\,\sigma_e^2\right)\delta^{1/2}}\frac{\left(N\sigma_v^2+\sigma_e^2\right)}{\left(N^2\sigma_v^2+\sigma_e^2\right)}\overline{m}^2.$$

The total profits in the market are

$$\Pi = N\Pi^{PI} + \Pi^{SI} = E\left((\widetilde{v}-\widetilde{p})\left(\sum_{n=1}^{N}\widetilde{x_n}+\widetilde{y}\right)\right).$$

But from the market clearing condition it results that

$$\Pi = N\Pi^{PI} + \Pi^{SI} = E\left((\widetilde{v}-\widetilde{p})\left(\overline{m}+\widetilde{S}\right)\right) =$$
$$E\left(\frac{\sigma_e^2\,(2N-1)}{N^2\sigma_v^2+(2N-1)\,\sigma_e^2}\,(\widetilde{v}-\overline{v})-\frac{N\sigma_v^2}{N^2\sigma_v^2+(2N-1)\,\sigma_e^2}\sum_{n=1}^{N}\widetilde{e}_n\right.$$
$$\left.+\frac{N\sigma_v^2\left(N^2\sigma_v^2+\sigma_e^2\right)}{\left(N^2\sigma_v^2+(2N-1)\,\sigma_e^2\right)\delta^{1/2}}\widetilde{S}+\frac{2N\sigma_v^2\left(N\sigma_v^2+\sigma_e^2\right)}{\left(N^2\sigma_v^2+(2N-1)\,\sigma_e^2\right)\delta^{1/2}}\overline{m}\right)\left(\overline{m}+\widetilde{S}\right)$$
$$= \frac{N\sigma_v^2}{\left(N^2\sigma_v^2+(2N-1)\,\sigma_e^2\right)\delta^{1/2}}\left(\left(N^2\sigma_v^2+\sigma_e^2\right)\sigma_S^2+2\left(N\sigma_v^2+\sigma_e^2\right)\overline{m}^2\right).$$

We can check and see that indeed the profits we have obtained sum up to this amount.

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Chapter 4

Tax Evasion and Insider Trading

4.1 Introduction

The economic approach to the problem of tax evasion is founded on the analysis of the behavior of individual tax evaders. The way in which the individual perceives his economic opportunities to be affected by the tax code and by the instruments of tax enforcement is extremely important. The system of taxation and its enforcement may induce the taxpayer to hide or misrepresent some of his activities.

The taxpayer may perceive certain choices with regard to tax declaration, financial transactions, or economic activity to be potentially costly in that they are subject to the threat of exposure and penalty. If so, then this perception will influence such choices and these choice on their turn will affect in different ways the functioning of the economy. Although tax evasion has been a problem that received a lot of attention in the public finance literature, its consequences on the performances of financial markets have not been carefully considered. The importance of the tax evasion and of the penalties enforced by the tax authorities in order to alleviate this problem is crucial because they both affect the net payoff after taxes and therefore, might perturb the other activities of firms. Evasion activities typically induces firms either in making decisions under uncertainty (concerning his eventual liability to taxes and penalties) or in trying to eliminate that uncertainty by more thorough concealment. These decisions together with the uncertainty about the inspection policy and the errors committed when filling the tax report alter the net payoff after taxes and therefore, the liquidation value of the firm when traded in the financial markets. The fact that the complexity of tax code can induce even honest taxpayers to evade their taxes has been already explained in the literature. The errors made by the taxpayers bring about considerations of strategic model of reporting and auditing. The effect of false detection and costly error becomes in this setup an important problem. In this paper we study the strategic interactions between a tax enforcement agency and a firm, when there is an insider trader (possibly, a member of the firm's board) who has the possibility of trading in financial markets. In considering this possibility, the tax report chosen by the firm has an effect both on the strategy of the tax enforcement agency and

on the trading strategy of the insider. Our study focuses on the link between the tax report and the profit per share from insider trading. More specifically, we are interested to understand how do the tax report disclosure affects stock valuations and how the interaction between the tax auditing agency and the firm will change the behaviour of the insider while trading in the financial markets.

The literature that analyzes the taxpayer compliance has as a starting point the papers that use a portfolio approach using as weights the probability to be caught and paying a penalty. Thus, Allingham and Sandmo (1972), Yitzaki (1974) and Polinsky and Shavell (1979) consider the case of the individual decision to evade payoff taxes when all the taxpayers face a constant probability of auditing by the tax authority. This assumption was criticized by Reinganum and Wilde (1986) who point out that the payoff report contains information about the true realization of the payoff and consequently, the probability of auditing should depend on the report made by the taxpayers. They model tax compliance as a game with incomplete information where first the tax payer reports his payoff, and then the tax auditing agency chooses the auditing probability depending on the payoff reported by the taxpayer.

While the above papers incorporated the uncertainty about the tax liabilities, another strand of research was concerned with the other sources of randomness that alter the interaction between the taxpayers and tax auditing agency. Mainly, they incorporated in their models the fact that tax code is complex and can lead to involuntary mistakes even when the taxpayers want to conform with the law. Thus, Scotchmer and Slemrod (1989) consider the case where the ambiguity of the law gives place to a random auditing policy depending on the interpretation given to the law. Reinganum and Wilde (1998) incorporate in the model the taxpayers's uncertainty about auditing cost, while Caballé and Panadés (2002) allow for both mistakes made by taxpayers and uncertainty about auditing cost.

On the other hand, insider trading has been extensively studied in the market microstructure literature. Kyle (1985) has become a standard for analyzing strategic noisy rational expectations markets. This model explains how a risk neutral informed trader

exploits his informational advantage by behaving strategical. Nevertheless, most of the theoretical research in market microstructure focuses on the financial markets only. There are few papers concerned with the effect of the insider trading on the real investment and other real decisions: Manove (1989), Leland (1992), Jain and Mirman (2000), Bhattacharya and Nicodano (2001), Medrano and Vives (2002). However, the literature paid little attention to other factors that can affect the liquidation value of the firm traded in financial markets. We direct our attention to one of this factors and consequently, we try to explain the link between the tax report made by a firm and the profit an insider makes from insider trading.

We analyze the optimal strategies of the board and of the insider who have private information about the realization of the payoff of the firm, and the one of the tax auditing agency who has the right to audit the realized payoff of the firm. The board of the firm has private information about the payoff of the firm and uses this information strategically to increase its value. In addition, there exists an insider who chooses to trade in financial markets and makes use of this private information to increase his profits from insider trading. The fact that agents behave strategically becomes even more important in our model because the private information is used both in the interaction with the tax authority and for trading in the stock market. Moreover, the report the board is making to the tax authority is going to influence the auditing effort and this on its turn will affect the payoff after taxes. As a result, the uncertainty induced by the errors appeared in the tax report will influence the liquidation value of the firm traded in financial markets. Since the report becomes accessible to the market maker after the first stage, he can actually infer a part of the information about the liquidation value of the firm.

In the first stage we model the interactions between the board and the tax auditing agency as a principal-agent relationship with no commitment. Once the auditing takes place, and the tax report is revealed, an insider wants to trade in financial markets. We follow Kyle (1985) in modelling the behaviour of the insider in the financial markets. Both the insider and the noise traders submit market orders and the market maker sets the price such that to satisfy the strong-efficiency condition. However, the information

structure in our model is different. The tax report acts as a public signal. However, since the report is the outcome of the strategic interaction between the firm and the tax auditing agency in the first stage, the public signal will be endogenous in our model. As we will see endogenizing the public signal and the liquidation of the value has significant consequences on market performance. Moreover, we show that the tax report affects the profits made from insider trading through three channels. Firstly, the report affects the net payoff after taxes through the taxes paid out (honestly reported). Secondly, the report affects the auditing effort chosen by the tax authority and therefore, affects the total penalty paid for the not reported payoff. These are the two channels through which the tax report affects the net payoff after taxes and consequently, the liquidation value of the firm when traded in the financial markets. Nevertheless, there is one more channel through which the tax report affects the price set by the market maker in the financial markets, and therefore, the demand and the profits of the insider. Modelling the interaction between the firm and the tax auditing agency allows us to endogenize the public signal received by the market maker by emphasizing the different channels mentioned above. While in the other models that consider the existence of a public signal that is aggregated in the prices, the precision and the mean of the signal are exogenous, in our model, they are determined in the first stage through the interaction between the firm and the tax auditing agency. Consequently, our model points out that the performance of the financial markets might be determined by interactions of the agents outside the financial markets - in our case tax evasion - and suggests that the implications of these interactions are very important both at quantitative and qualitative level. We obtain that unlike in Kyle (1985), endogenizing the public signal and the liquidation value induces non monotonicity of profits of different market participants, market depth, informativeness of prices and volatility of prices. Furthermore, allowing for errors in the tax report has important effects both for the firm and for the insider. In spite of using the same information, the welfare of the insider and the one of the firm is differently affected. The main reason for this difference lies in the way the other participants react to the actions of the firm and insider, respectively. In the reporting stage the action of

the firm is very well counterbalanced by the one of the tax authority. However, at the trading stage, the insider has more freedom in using the private information. The market maker tries to embed the information in prices, but given the signals he has, he cannot disentangle the effect of the noise trading and of the reporting error.

The remainder of this paper is organized as follows. Section 2 presents the model. We establish here the information structure and characterize the equilibrium. We find an unique equilibrium in which the price function, the insider's demand, the firm's report and tax auditing agency's inspection policy are linear. Section 3 proceeds with the calculation of some market indicators: volatility of prices, informativeness of prices and expected profits and performs comparative statics for the market indicators. Finally, Section 4 summarizes the results. All the proofs appear in the appendix.

4.2   The Model

We consider a firm which owns a project with an uncertain payoff $\widetilde{y}$.[1] The payoff of the project is normally distributed with mean $\overline{y} > 0$ and variance V_y, and it is observable by the board of directors of the firm. Based on this information, the board has to submit a report about the payoff to the tax authority. Furthermore, an insider (possibly one of the members of the board), will use this private information about the payoff while trading in the financial markets.

We assume that the tax report is exposed to some sources of randomness during the process of filling the reports. As in Rhoades (1999), we can think that a project has usually multiple lines of activities and therefore, of reporting. While the board observes the total payoff $\widetilde{y}$ and chooses the amount to report $\widetilde{x}$, when the reports are made for each component some misstatement can be introduced. This misstatement in the report might be due to improper division of the amount to be reported by each line of activity, to complexity and ambiguity of the tax code or just to human errors. Consequently, in

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[1]A tilde distinguishes a random variable from its realization. Thus, y denotes a particular realization of $\widetilde{y}$.

the report received finally by the tax agency we will have some randomness $\widetilde{z} = \widetilde{x} + \widetilde{\varepsilon}$. The error introduced in the report, $\widetilde{\varepsilon}$ is normally distributed with mean 0 and variance V_ε and is uncorrelated with any other random variables in the model.

The timing of the model is at follows. In the first stage, after observing the payoff $\widetilde{y}$, the board of directors chooses optimally the amount x to be voluntarily reported to the tax authority. On its turn, the tax auditing agency chooses conditional on the effective report - the report it receives - how much effort wants to put forth in auditing the firm. The auditing effort π is chosen such that to maximize the total expected net revenue. After both the report and auditing are made, in the second stage, the insider who wants to trade in the financial market chooses the quantity to trade such that to maximize his profit from insider trading. The price in the financial markets is set by the market maker such that to satisfy the strong-efficiency condition.

In the first stage we model the interaction between the board and auditing agency as a principal-agent relationship with no commitment. First, the board chooses the intended report x such that to maximize the expected payoff of the firm net of taxes. We assume that, in case of evading taxes, the firm will pay for the amount evaded, $\widetilde{y} - \widetilde{z}$, taxes at the penalty rate f. Moreover, the penalty is proportional to the auditing effort. Therefore, the penalty revenue is $\pi f (\widetilde{y} - \widetilde{z})$ and if the reported income z coincides with the true taxable income y, or no effort is devoted to inspection, no additional revenues are going to be collected by the tax authority. We assume that the tax law establishes a tax rate $\tau \in (0,1)$ and a penalty rate $f > 1$. In order to obtain the payoff net of taxes we substract from the payoff of the project both the voluntarily paid taxes $\tau\widetilde{z}$ and the penalties paid after inspection took place $\pi f (\widetilde{y} - \widetilde{z})$. Consequently, the net payoff after taxes is $V = \widetilde{y} - \tau\widetilde{z} - \pi(\widetilde{z})f(\widetilde{y} - \widetilde{z})$. When the board takes his decision he has rational expectations about the strategy of the tax authority, so he bears in mind that the auditing effort depends on the report he makes. Accordingly, the board chooses the amount to report x such that

$$x(y) = \arg \max_{x} E\left[\widetilde{y} - \tau\widetilde{z} - \pi(\widetilde{z})f(\widetilde{y} - \widetilde{z})\,|\, y\right].$$

Next, the tax authority is choosing the auditing strategy such that to maximize the net revenue conditional on the report they receive from the firm. The auditing effort π is contingent upon the report observed by the tax authority z. The total expected net revenue consists of total tax revenue plus the penalty revenue and net of auditing costs. The auditing takes place at a fixed cost c, this cost being observable both by the auditor and board. We also assume that the tax authority has quadratic auditing costs $\frac{1}{2}c\pi^2$, with $c > 0$. As a result, we have that the net revenue of the tax authority is

$$\tau\widetilde{z} + \pi\left(\widetilde{z}\right) f\left(\widetilde{y} - \widetilde{z}\right) - \frac{1}{2}c\pi^2\left(\widetilde{z}\right).$$

The tax authority is choosing the auditing strategy such that to maximize the net revenue conditional on the report they receive from the firm i.e.

$$\pi\left(z\right) = \arg\max_{\pi} E\left[\tau z + \pi f\left(\widetilde{y} - z\right) - \frac{1}{2}c\pi^2\,\middle|\, z\right].$$

The first order condition for this problem is

$$E\left[f\left(\widetilde{y} - z\right) - c\pi\,\middle|\, z\right] = 0,$$

or equivalent the auditing strategy $\pi = \pi\left(z\right)$ is

$$\pi = \frac{f}{c}\left[E\left[\widetilde{y}\middle| z\right] - z\right].$$

The second order condition for this problem is $c > 0$, and is satisfied by assumption (positive costs of auditing).

The tax report of the firm affects the strategy of the auditing agency and consequently, we have two channels through which the report affects the net payoff of the firm (the direct channel through which the payoff is affected by paying out the taxes corresponding to the tax report and the link between the tax report and the auditing effort). Since the insider has to trade in the financial markets on the liquidation value of the firm (the payoff of the project net of taxes), he has to take into account both these effects.

In the second stage, after the board chooses the report and the tax authority chooses the auditing effort, the report received by the agency $\widetilde{z}$ it is learned both by the insider

and the market maker. As a result, the insider is choosing the amount he trades in the financial markets, d, knowing both the realization of the payoff $\widetilde{y}$ and the report received by the agency $\widetilde{z}$, so he actually learns the error produced at the reporting stage $\widetilde{\varepsilon}$. We model the financial markets as in Kyle (1985). However, our model departs from Kyle (1985) in two ways. First, the liquidation value of the firm depends on the choices undertaken by the board and the tax authority, and is not exogenously given. Second, the market maker learns the report which he knows that it depends on the true realization of the payoff. Consequently, he sets the price comprising both the information about the total order flow and the information revealed by the report. As in the other models with public information the announcement of the report lessens the information asymmetry that prevailed in the pre-trading stage between informed and uninformed traders. Nevertheless, since the public information (the tax report) is endogenous in our model, we expect different behaviour. Finally, as in Kyle (1985), the total order flow consists of the order of the insider and the order of the noise traders. We assume that the order of the noise traders $\widetilde{\omega}$ is a random variable normally distributed with mean 0 and variance V_{ω}. The amount traded by the insider is contingent on the firm's payoff and depends on the report the firm made previously. The insider has to take into account the fact the report is perceived erroneously by the auditing agency, and therefore, his optimal choice is

$$d\left(y, z\right) = \arg\max_{d} E\left[\left(\left(\widetilde{y} - \tau\widetilde{z} - \pi(\widetilde{z})f(\widetilde{y} - \widetilde{z})\right) - p\left(d + \widetilde{\omega}\right)\right) d\right| y, z].$$

We assume that in the financial market there exists a market maker who sets the price such that to satisfy the semi-strong efficiency condition

$$p\left(u, z\right) = E\left[\left.\widetilde{V}\right| u, z\right] = \mu + \nu u + \eta z, \tag{4.1}$$

where $u = d(z) + \omega$ is the total order flow.

An equilibrium with rational expectations is thus a report strategy of the board $x\left(y\right)$, a trading strategy of the insider $d\left(y, z\right)$, an auditing strategy of the tax authority $\pi\left(z\right)$, and a pricing strategy of the market maker $p\left(u, z\right)$. We are looking for a linear equilibrium and

therefore, we restrict our attention to strategies that are linear. The report strategy of the of the board of the firm $x(y) = \alpha + \beta y$, and the demand of the insider $d(y, z) = \theta + \rho y + \kappa z$, are linear in the signals they receive. The auditing effort policy of the tax authority, $\pi(z) = \delta + \gamma z$ is linear in the report and the price policy, $p(u, z) = \mu + \nu u + \eta z$ is linear in the total order flow and the report. We solve for the linear equilibrium and the second order condition for the board's problem in the first stage implies that the equilibrium exists only if $4V_\varepsilon > V_y$. This condition requires that the auditing effort is decreasing in the amount of reported payoff. If this had not been true, the auditing effort would increase with the reported income, and the taxpayers would find it optimal to report an infinite negative payoff. The second order condition for the insider problem requires that the tax rate is $\tau < \tau^*$, where

$$\tau^* \equiv \frac{4V_\varepsilon \left[2(V_y + 4V_\varepsilon) + \frac{f^2}{c}\overline{y}(V_y - 4V_\varepsilon)\right]}{(V_y + 4V_\varepsilon)^2}.$$

As a result, when the tax rate increases above this threshold the equilibrium fails to exists.

Proposition 4.1 *There exists a unique linear equilibrium where the board's tax report is*

$$x(y) = \alpha + \beta y$$

where

$$\alpha = \frac{1}{2}\overline{y} - \frac{\tau c}{8f^2}\left(\frac{V_y + 4V_\varepsilon}{V_\varepsilon}\right)$$

$$\beta = \frac{1}{2};$$

the inspection policy of the tax auditing agency is

$$\pi(z) = \delta + \gamma z$$

where

$$\delta = -\frac{f}{c}\left(\frac{V_y - 4V_\varepsilon}{V_y + 4V_\varepsilon}\right)\overline{y} + \frac{1}{4}\left(\frac{\tau V_y}{fV_\varepsilon}\right)$$

$$\gamma = \frac{f}{c}\left(\frac{V_y - 4V_\varepsilon}{V_y + 4V_\varepsilon}\right);$$

the demand of the insider is

$$d\left(y\right) = \theta + \rho y + \kappa z,$$

where

$$\theta = -\rho \overline{y} - \frac{R\rho}{U}$$

$$\rho = \left(\frac{V_\omega \left(V_y + 4V_\varepsilon\right)}{4V_y V_\varepsilon}\right)^{1/2}$$

$$\kappa = -\frac{R\rho}{U};$$

and the equilibrium price

$$p(u) = \mu + \nu u + \eta z,$$

where

$$\mu = U\overline{y} + Q - \alpha^2 \gamma f + (\alpha + \beta \overline{y}) \left(SU + \frac{R}{2}\right)$$

$$\nu = \frac{U}{2\rho}$$

$$\eta = US - \frac{R}{2} + \alpha \gamma f$$

with R, S, U, Q as defined by (4.8) in the Appendix.

We obtain that the demand of the insider in the trading stage is

$$d\left(y, z\right) = \rho \left(\left(y - \overline{y}\right) - \frac{R}{U}\left(z - \alpha - \beta \overline{y}\right)\right) = \rho \left(1 - \frac{R}{2U}\right)\left(y - \overline{y}\right) - \frac{R\rho}{U}\varepsilon.$$

Unlike in Kyle (1985), the insider does not base his decision of buying or selling only by comparing the realization of the payoff with the mean. Here, the insider has to take also into account the effect the erroneous report brings about. Notice that the insider puts a positive weight on the public signal (the report z), because the constant R is always negative. This is different from the models with exogenous public signal where the insiders put a negative weight on the public information when determining the optimal trading

strategy. The intuition is simple. As the error ε increases, the value of the firm decreases and therefore, the price set by the market maker decreases. However, at the trading stage the insider knows both the payoff y and the error ε, so he knows the liquidation value. Consequently, he corrects his demand for the error induced in price by the error in the report.

Using the coefficients we have obtained before we can say that the tax report of the firm is

$$x\left(y\right) = \frac{y + \overline{y}}{2} - \frac{\tau c}{8f^2}\left(\frac{V_y + 4V_\varepsilon}{V_\varepsilon}\right).$$

The expected report received by the agency is

$$E\left(\widetilde{z}\right) = E\left(\widetilde{x} + \widetilde{\varepsilon}\right) = E\left(\frac{\widetilde{y} + \overline{y}}{2} - \frac{\tau c}{8f^2}\left(\frac{V_y + 4V_\varepsilon}{V_\varepsilon}\right) + \widetilde{\varepsilon}\right) = \overline{y} - \frac{\tau c}{8f^2}\left(\frac{V_y + 4V_\varepsilon}{V_\varepsilon}\right).$$

As can be easily seen, we obtain that both the real report and the expected report are increasing with the penalty rate f and decreasing with the cost of auditing c and tax rate τ. It is quite intuitive that if the tax auditing agency faces high costs, the tax payer will report less because the tax auditing agency will monitor him less often. We also recover here the result of Allingham and Sadmo (1972), who obtain a decreasing relationship between the tax report and the tax rate in the case of a risk averse taxpayer displaying non-decreasing absolute risk aversion. The implications of having a risk-neutral taxpayer in our case are similar to the case of risk aversion since we allow the taxpayer to behave strategic. Also, as the ratio of the penalty rate relative to the tax rate decreases, the taxpayer reports less because he will prefer to take the risk and pay the lower penalty rate. We also obtain that the report is increasing with the variance of V_ε, which suggests that in the case the variance of errors is high, the taxpayer auto-disciplines himself and reports closer to the real realization. Finally, we obtain that the report is decreasing with V_y, and this is due to the fact that when the variance of payoff is high the use of the private information by the taxpayer makes it more difficult for tax auditing agency to understand which is the realization of the payoff and therefore, the tax auditing agency has to devote more resources in monitoring the firm. Moreover, we have that $\dfrac{\partial E\left(\widetilde{z}\right)}{\partial \overline{y}} = 1$,

so that an increase in the mean payoff results in an increase of reported payoff of identical amount.

Finally, the tax agency is going to put forth the following auditing effort

$$\pi\left(z\right) = \frac{f}{c}\left(\frac{V_y - 4V_\varepsilon}{V_y + 4V_\varepsilon}\right)\left(z - \overline{y}\right) + \frac{1}{4}\frac{\tau V_y}{fV_\varepsilon}.$$

We can compute then the expected auditing effort and it equals to

$$E\left(\widetilde{\pi}\right) = E\left[\frac{f}{c}\left(\frac{V_y - 4V_\varepsilon}{V_y + 4V_\varepsilon}\right)\left(\widetilde{z} - \overline{y}\right) + \frac{1}{4}\frac{\tau V_y}{fV_\varepsilon}\right] = \frac{1}{4}\frac{\tau V_y}{fV_\varepsilon}.$$

Notice that the expected auditing effort is increasing with the variance of the payoff V_y. As we have already pointed out, when the variance of the payoff is high the tax auditing agency has to increase the auditing effort in order to discover the true realization of the payoff. However, the expected auditing effort is decreasing with the variance of the errors V_ε. As V_ε increases the taxpayer knows that it is more likely to commit important mistakes. Since the auditing effort is decreasing in reported payoff, the firm knows that low reports will be heavily inspected, while high reports will not be exposed to very severe inspections. This bias in the audit policy will induce the taxpayers to minimize the probability of a rigorous auditing. Consequently, when the variance V_ε increases, the taxpayer will try to report as correct as they can to increase the probability of a sufficient large report, and therefore to decrease the probability that they are rigorously audited. Finally, since the report decreases with the variance V_y, the tax auditing agency will want to audit more rigorously, and that is why the auditing effort is increasing with V_y.

We can inspect next the expected revenue raised by the tax enforcement agency and see how is affected by the different sources of uncertainty. The net revenue per taxpayer is

$$\widetilde{R} = \tau\left(x\left(\widetilde{y}\right) + \widetilde{\varepsilon}\right) + \pi\left(x\left(\widetilde{y}\right) + \widetilde{\varepsilon}\right)f\left(\widetilde{y} - x\left(\widetilde{y}\right) - \widetilde{\varepsilon}\right) - \frac{1}{2}c\pi^2\left(x\left(\widetilde{y}\right) + \widetilde{\varepsilon}\right).$$

Corollary 4.1 *The expected net revenue raised by the tax authority $E\left(R\right)$ is increasing in V_ε and decreasing in V_y.*

A larger variance of the payoff means a larger disadvantage of tax auditors with respect to taxpayers, and, hence, tax auditors end up putting too much effort on low payoff taxpayers, who are the one who pay less fines. When V_ε increases, taxpayers commit more errors so the agency will raise more revenues both from the penalties imposed on the involuntary evaded taxes and from the taxes on the larger amount of voluntarily reported payoff. Therefore, the tax authority benefits from the taxpayer confusion and hence, has no incentives to reduce the complexity of either tax laws or tax forms.

We discuss next the comparative statics concerning the net payoff of the firm. We have that the net payoff of the firm is

$$\widetilde{V} = \widetilde{y} - \tau\left(x\left(\widetilde{y}\right) + \widetilde{\varepsilon}\right) - \pi\left(x\left(\widetilde{y}\right) + \widetilde{\varepsilon}\right) f\left(\widetilde{y} - x\left(\widetilde{y}\right) - \widetilde{\varepsilon}\right).$$

Corollary 4.2 *The expected net payoff is inverted U-shaped with respect to V_y and V_ε.*

The result above suggests that initially an increase in both the variance of the payoff V_y and in the error V_ε will help the firm to increase its value. However, there exists a threshold value such that for higher values the expected net payoff becomes decreasing. As explained above, a high variance of errors V_ε induces the taxpayers to commit more errors and since they pay more penalties this is reflected of course, in the expected net payoff of the firm. We have obtained before that the report is decreasing in the variance of payoff V_y and that the auditing effort is increasing in V_y. Hence, we have here a trade-off between a lower report and a more rigorous auditing, and this explains the inverted U-shape of the expected payoff. Similarly, we have that the report is increasing in the variance of payoff V_ε and that the auditing effort is decreasing in V_ε, and the same trade-off takes place.

4.3 Market Indicators

We study next the implications of endogenizing the public signal and the liquidation value of the firm on the market performance. To do that we compute few market indicators: the market depth, the volatility of prices, the information content of prices, and the expected profit of different market participants and characterize their variation with

respect to the variance of the payoff V_y. The variance of the payoff can be seen here as a measure of the riskiness of the firm's project. This exercise will allow us to emphasize the differences between our model, with endogenous public signal and liquidation value, and the one of Kyle (1985). As we have already mentioned, Kyle (1985) presents a model where a risk neutral informed trader exploits his informational advantage by behaving strategical. In his model, the market indicators behave monophonically with respect to the riskiness of the project (the market depth and the profits of the noise traders are decreasing, while the informativeness of prices, the volatility of prices and the profits of the insider are increasing). We will see, that in the case we endogenize the public signal and the liquidation value, the monotonicity of the market performance indicators does not longer hold.

Let us first write the price in a concise form. Since the effective report is $z = x(y) + \varepsilon$ and the demand of the insider is $d = d(y, z)$ we obtain that (4.1) can be simplified to

$$p(\widetilde{y}, \widetilde{\varepsilon}, \widetilde{\omega}) = \frac{U(1+S)}{2}\widetilde{y} + US\widetilde{\varepsilon} + \frac{U}{2\rho}\widetilde{\omega} + Q + U(1-S)\overline{y} - \alpha^2\gamma f, \qquad (4.2)$$

where, as before, α, γ, ρ are the equilibrium values and U, S and Q are given by (4.8) in the Appendix. From this formula we can see that the unconditional expectation of the equilibrium price is

$$E(\widetilde{p}) = E\left(\widetilde{V}\right) = U\overline{y} + Q - R\alpha,$$

so the price is an unbiased estimator of the liquidation value. It is also interesting to remark that in our model the expected value of the firm is quadratic in the mean of the payoff $\overline{y}$, tax rate τ, penalty rate f and auditing costs c. This is due to the fact that both the tax report and the auditing policy are chosen such that to account for the effects of this variables on the net payoff of the firm. Notice also that since $U < 1$ and $S < 1$ we have that an increase in the payoff $\widetilde{y}$ is reflected less than half in the price, while one unit increase in the report $\widetilde{z}$ is reflected in the price by less than one.

We will use as a measure of liquidity the market depth, as defined by Kyle (1985), which represents the volume of trading needed to move prices by one unit. As can be

easily seen from the formula for the price (4.2), the market depth is

$$\frac{1}{\nu} = \frac{2\rho}{U} = \frac{\left(\dfrac{V_\omega\,(V_y + 4V_\varepsilon)}{V_y V_\varepsilon}\right)^{1/2}}{\dfrac{1}{2}\left(\dfrac{V_y - 4V_\varepsilon}{V_y + 4V_\varepsilon}\right)\dfrac{f^2}{c}\overline{y} + 1 - \dfrac{1}{8}\tau\left(\dfrac{V_y + 4V_\varepsilon}{V_\varepsilon}\right)}.$$

We can immediately see that the market depth is increasing in the variance of the noise trading V_ω, the payoff mean $\overline{y}$, the penalty rate f and the tax rate τ, and decreasing in the cost of auditing c. Notice that an increase in the tax rate increases the market depth. As we have seen already a higher tax rate induces a lower report. Since the market maker uses as a signal the report, if this is not very accurate it will help the insider to preserve his informational advantage. Finally, the behaviour of the market depth with respect to the variance of payoff V_y is very different, depending on the values of parameters.

Corollary 4.3 *The market depth with respect to V_y is*

i) U-shaped if $\overline{y} \in (0, 2c/3f^2)$ and $\tau \in (1/3 + f^2\overline{y}/3c, 1)$,

ii) Inverted U-shaped if $\overline{y} \in [c/f^2, 2c/f^2)$ and $\tau \in (\min\{2 - f^2\overline{y}/c, \tau^*\}, \tau^*)$,

iii) decreasing, otherwise.

The market depth with respect to V_ε is

i) Inverted U-shaped in the following cases: $\overline{y} \in (2c/5f^2, 3c/5f^2)$ and $\tau < 5f^2\overline{y}/c - 2$;
$\overline{y} \in (3c/5f^2, c/f^2)$; $\overline{y} \in [c/f^2, 2c/f^2)$ and $\tau \in (\min\{f^2\overline{y}/c - 1, \tau^*\}, \tau^*)$

ii) decreasing, if $\overline{y} \in [c/f^2, 2c/f^2)$ and $\tau \le \min\{f^2\overline{y}/c - 1, \tau^*\}$ or if $\overline{y} > 2c/f^2$,

iii) increasing, otherwise.

As we can see from the previous Corollary the market depth is not monotonic in the variance of the payoff V_y. This is driven by the fact that we endogenize the public signal. Adding an exogenous public signal in Kyle (1985) model decreases the total variance of the payoff. In our model, the variance of the signal and the net payoff are determined endogenously at the reporting stage and they both depend on V_y. While the variance of the signal is monotonic in V_y, this is not the case of the variance of the net payoff (the liquidation value of the firm). We have obtained in Corollary 4.2 that the expected net

payoff is U-shaped with respect to the variance of the payoff V_y. Since the manager knows the net payoff, this non monotonicity is transmitted to the demand and from here to the order flow. As we know, the market maker reacts to changes in the informativeness of the order flow by changing the elasticity of the price with respect to the order flow. However, when we have a public signal, he reacts to this type of change also by adjusting the weights he puts on the order flow and public signal, respectively. Consequently, the effects of endogenizing the signal and the liquidation value, propagate themselves in the financial markets through multiple channels and induce a non monothonic behaviour of the market indicators.

Notice also that if the tax rate τ is small, the behaviour is similar to the one in Kyle (1985) (i.e. market depth is decreasing in V_y). However, for high values of τ, we might obtain very different behaviour. This suggests that very high taxes have a distortionary effect. It is clear that the higher the tax rate, the lower the tax report made by the firm. Since the tax report affects the liquidation value of the firm and the price set by the market maker, it is intuitive that a report that is far from the real realization of payoff will induce important changes in the price, the demand of the insider and all the market performance indicators. We have actually here a trade-off. There are two ways the variance of the payoff affects the market depth. The first one is the same as in Kyle (1985) where increasing the variance of payoff V_y, increases the informational advantage of the informed agent and thus, decreases the market depth. However, as we already pointed out before, a high tax rate induces a low tax report, and the use of the report as a signal of the payoff becomes not very appropriate when both the variance of payoff V_y and the tax rate increase.

We characterize next the amount of the private information that is revealed through prices. We define the informativeness (or the information content) of prices as the difference between the prior variance of the payoff conditional on the report and the variance conditional on prices and the report. This measure will give us the decrease in variance due to revelation of private information, after conditioning on the private signal. Using the normality assumptions we obtain the expression presented in the following Corollary:

Corollary 4.4 *The informativeness of prices is*

$$Var\left(\widetilde{V}\,\middle|\,\widetilde{z}\right) - Var\left(\widetilde{V}\,\middle|\,\widetilde{p},\widetilde{z}\right)$$

$$= \frac{V_y\left(V_y + 2V_\varepsilon\right)^2}{8V_\varepsilon\left(V_y + 4V_\varepsilon\right)^3}\left(4\frac{f^2}{c}V_\varepsilon\left(4V_\varepsilon - V_y\right)\overline{y} - 8V_\varepsilon\left(V_y + 4V_\varepsilon\right) + \tau\left(4V_\varepsilon + V_y\right)^2\right)^2.$$

Moreover, the effects of changes in V_y on the informativeness are ambiguous.

Similarly to Kyle (1985), we have that the informativeness of prices does not depend on the noise trading. In Kyle (1985), the order flow was the only source of information for the market maker and the changes in the payoff of the firm that affected the price where off-set by the changes in the quantity traded by the insider. Thus, when the variance of the noise trading increase, this allow the informed trader to hide better and therefore, to trade more aggressively, making use of their informational advantage. However, the informativeness of prices depends in our case on the variance of the payoff V_y and the variance of errors V_ε and the main reason for that is that the market maker uses one more signal to set the price and the changes in prices are not completely off-set by the change in the insider trading. We performed the comparative statics with respect to the parameters and we obtained that the informativeness of prices might be increasing, inverted U-shaped, U-shaped or oscillating (decreasing, increasing and decreasing) with respect to the variance of the payoff V_y. As we have already explained in the discussion of the market depth, this behaviour is a consequence of the fact that in our model the public signal and the liquidation value of the firm are endogenous. A deeper analysis suggests us that the tax rate is again an important parameter of the model. As we can see in Lemma A.1 in the Appendix for small τ, the behaviour of informativeness of prices is similar to the one in Kyle (1985). The main conclusions we have reached is that the informativeness of prices is increasing in V_y if the mean of the payoff $\overline{y}$ is small relative to the costs c/f^2 and τ is small (see Figure 4.1) and is inverted U-shaped when τ increases (see Figure 4.2). As the mean payoff increases up to $2c/f^2$ we have either that the informativeness of prices is either increasing for τ small or oscillating for higher τ (see Figure 4.3). Finally, when the mean payoff increases above $2c/f^2$ we have that the informativeness of prices is always

oscillating. We have found also that the informativeness of prices with respect to tax rate τ is either increasing if $\overline{y} > 2\dfrac{c}{f^2}\left(\dfrac{V_y + 4V_\varepsilon}{4V_\varepsilon - V_y}\right)$, or U-shaped, otherwise. However, the second order condition for the insider's problem rules out the first case, so the informativeness of prices is always U-shaped with respect to tax rate. As we have seen already the expected net payoff is quadratic in the tax rate (we have again the trade-off between low report - high auditing effort. This relationship is transmitted in prices (which are an unbiased estimator of the liquidation value) and consequently in the informativeness of prices. Finally, the informativeness of prices is U-shaped as respect to the mean of payoff $\overline{y}$ and the penalty rate f and inverted U-shaped as respect to the cost of auditing c.

Corollary 4.5 *The price volatility, measured as the variance of price, is*

$$Var\left(\widetilde{p}\right) = \frac{V_y\left(V_y + 2V_\varepsilon\right)}{64V_\varepsilon^2\left(V_y + 4V_\varepsilon\right)^3}\left(4\frac{f^2}{c}V_\varepsilon\left(4V_\varepsilon - V_y\right)\overline{y} - 8V_\varepsilon\left(V_y + 4V_\varepsilon\right) + \tau\left(4V_\varepsilon + V_y\right)^2\right)^2.$$

Furthermore, the effects of changes in V_y on price volatility are ambiguous.

We have studied also how changes in the variance of payoff affects the volatility of prices. The behaviour of volatility with respect to the variance of errors V_y depends critically on the values of parameters, and we have 3 possible cases: the volatility being increasing, inverted U-shaped (Figure 4.4), or oscillating (Figure 4.5). We have found also that the volatility of prices is U-shaped with respect to tax rate τ, mean of payoff $\overline{y}$ and the penalty rate f and inverted U-shaped as respect to the cost of auditing c. Moreover, the volatility of prices does not depend on the noise trading, the reasons being the ones explained for the informativeness of prices.

Corollary 4.6 *The expected profit of the insider*

$$E\left(\left(V - p\right)d\right) = \frac{U}{2\rho}V_\omega = \left(\frac{V_\omega V_y V_\varepsilon}{V_y + 4V_\varepsilon}\right)^{1/2}\left(\frac{f^2}{2c}\left(\frac{V_y - 4V_\varepsilon}{V_y + 4V_\varepsilon}\right)\overline{y} + 1 - \frac{1}{8}\tau\left(\frac{V_y + 4V_\varepsilon}{V_\varepsilon}\right)\right).$$

The comparative statics with respect to the variance of payoff V_y is ambiguous.

The non monotonicity of the market depth implies that both the insider's expected profit and noise traders' expected profit are non-monotonic (the noise traders profits is the negative of the insider's profit). We can see here, one of the important implications of endogenizing the public signal. Adding an exogenous public signal in Kyle (1985) model makes the noise traders positively better off because the public signal reduces the variance of the private information (relative to the all information in the market). In our setup, the report still decreases the total variance, but it has an indirect effect on the variance of the liquidation payoff. Following the same reasoning as in the case of the market depth, since the variance of liquidation value is ambiguous with respect to the variance of payoff V_y it is also ambiguous whether they are better off or worse off.

While the welfare of the firm it is always inverted U-shaped with respect to V_y, the welfare of the insider depends critically on the tax level. Notice that when the tax rate is small a very high variance of payoff V_y improves the welfare of the insider (because of the extra noise in the order flow he can make better use of his informational advantage) but it does not improve the welfare of the firm. What happens actually is that initially, the board tries to make use of this informational advantage, and therefore, their expected report is decreasing with V_y. However, their action is overcome by the tax auditing agency, their auditing effort being increasing with V_y. On the other hand, the insider uses the information less restricted. It is true that still the market maker, by including the information from the report in prices, can diminish his informational advantage. However, since he has two sources of uncertainty - about the order flow and about the realization of the payoff - he cannot disentangle the effect of the realization of payoff from the one of noise traders' order flow, and consequently he cannot restrict the insider fully to use his informational advantage.

4.4   Conclusions

In this paper we have presented an insider trading model where we study how interaction between a firm and a tax auditing agency may affect the trading in the financial

markets. We show that uncertainty about the realizations of the payoff of the firm together with the errors produced during the reporting stage, have an important effect on the reporting strategy of the firm and the auditing policy of the tax authority. Allowing for this interaction between the firm and the tax auditing agency permits us to endogenize the public signal used by the market maker in financial markets. As a result, the uncertainty and the decisions of the agents made at the initial stage, affect the value of the firm and the public signal and therefore, the trading in the financial markets. Thus, endogenizing the public signal and the liquidation value of the firm brings about substantial changes in the behaviour of the market depth, profits of the market participants, informativeness of prices when the variance of the payoff changes. Our results suggest also that the market performance becomes in this case very sensitive to the values of the parameters, the most relevant parameter being the variance of the payoff. The tax rate plays also a crucial role in our model, because depending on its level the firm chooses the tax report and this report affects both the liquidation value of the firm and the price set in the financial markets.

Our results are significantly different from the ones in Kyle (1985), despite of the fact that the models are very similar. The difference between the two models lies in the fact that the distribution of the net payoff is affected by the actions taken in the reporting stage and the fact that the market maker uses an additional signal (which is endogenous in our model) when setting the price.

We have studied how the expected net revenue of the tax auditing agency, and the welfare of the firm and insider is affected by the quality of signal and accuracy of filling the tax report. We obtain that a larger variance of income brings about a larger disadvantage for the tax agency, which is forced to put forth additional effort, which decreases its expected revenue. However, the tax auditing agency is benefited by large variance of errors because both the penalties on the involuntarily evaded payoff and taxes on the voluntarily reported payoff are higher. In the case of the welfare of the firm, when we have a change in the variance of the payoff or in the variance of errors we have a trade-off between the effect on the report and the one on the auditing effort. While the tax rate

does not influence the behaviour of firm's welfare when the parameters of the model are changing, in the case of the insider's welfare the tax rate level becomes crucial. Thus, if the tax rate is small the insider is able to use his informational advantage, and therefore, his profit increases with the variance of the payoff. However if the tax rate increases the changes in profit of the insider determined by a change in variance of payoff becomes strongly dependent on the other parameters. As we have already explained there exist different channels through which a change propagates itself. First, a change affects the report and the auditing effort, and therefore, the liquidation value of the firm. This change in the value of the firm affects the demand and consequently, the order flow. The changes in the report and in the order flow affect the pricing strategy of the market maker, affecting the weights he puts on different signals and therefore, it transmits further in the market.

4.5   Appendix

Proof of Proposition 4.1.   Since the firm's choice of report affects the expected payoff after taxes, this means that it will affect also the demand strategy he uses in the second stage. We solve the problem backward considering the auditing effort π as given. The insider conjectures that the market maker is going to use the pricing rule $p\left(d+\widetilde{\omega}\right)=\mu+\nu\left(d+\widetilde{\omega}\right)+\eta\widetilde{z}$.

Thus, the insider is solving the following maximization problem

$$\max_{d} E\left[\left(\widetilde{y}-\tau\widetilde{z}-\pi f(\widetilde{y}-\widetilde{z})-p\left(d+\widetilde{\omega}\right)\right)d\middle|\widetilde{y},\widetilde{z}\right] \tag{4.3}$$

$$\Longleftrightarrow \max_{d} E\left[\left(\widetilde{y}-\tau\widetilde{z}-\pi f(\widetilde{y}-\widetilde{z})-\mu-\nu\left(d+\widetilde{\omega}\right)-\eta\widetilde{z}\right)d\middle|\widetilde{y},\widetilde{z}\right].$$

The first order condition for this problem is

$$E\left[\widetilde{y}-\tau\widetilde{z}-\pi f(\widetilde{y}-\widetilde{z})-\mu-\nu\left(d+\widetilde{\omega}\right)-\eta\widetilde{z}-\nu d\middle|\widetilde{y},\widetilde{z}\right]=0$$

or equivalently,

$$d=\frac{1}{2\nu}E\left[\widetilde{y}-\tau\widetilde{z}-\pi f(\widetilde{y}-\widetilde{z})-\mu-\nu\widetilde{\omega}-\eta\widetilde{z}\middle|\widetilde{y},\widetilde{z}\right].$$

The insider observes $\widetilde{y}$ and $\widetilde{z}$ and conjectures that the tax auditing agency has used the strategy $\pi = \delta + \gamma z$.

Let us define by

$$
\begin{aligned}
U &= \widetilde{y} - \tau\widetilde{z} - \pi f(\widetilde{y} - \widetilde{z}) - \eta\widetilde{z} - \mu = (1 - \delta f - \gamma f(\alpha + \beta y + \varepsilon))\, y - \\
&\quad (\tau - \delta f - \gamma f(\alpha + \beta y + \varepsilon))(\alpha + \beta y + \varepsilon) - \eta(\alpha + \beta y + \varepsilon) - \mu
\end{aligned}
$$

then

$$
E\left[\widetilde{U}\,\middle|\,\widetilde{y}, \widetilde{z}\right] = E\left(\widetilde{U}\right) + \begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} \widetilde{y} - \overline{y} \\ \dfrac{\widetilde{z} - \alpha}{\beta} - \overline{y} \end{pmatrix},
$$

where

$$
\begin{aligned}
E\left(\widetilde{U}\right) &= \overline{y}\left(1 - \delta f - \gamma f\alpha - \tau\beta + \delta f\beta + 2\gamma f\alpha\beta - \eta\beta\right) + \\
&\quad \alpha\left(-\tau + \gamma f\alpha + \delta f - \eta\right) + \beta\gamma f\left(\beta - 1\right)\left(V_y + \overline{y}^2\right) + \gamma f V_\varepsilon - \mu,
\end{aligned}
$$

and

$$
\begin{pmatrix} A & B \end{pmatrix} = \mathrm{cov}\left(\widetilde{U}, \left(\widetilde{y},\ \widetilde{y} + \frac{\widetilde{\varepsilon}}{\beta}\right)\right)\left(\mathrm{var}\left(\widetilde{y},\ \widetilde{y} + \frac{\widetilde{\varepsilon}}{\beta}\right)\right)^{-1}.
$$

After some calculations we obtain that

$$
\mathrm{cov}\left(\widetilde{U}, \left(\widetilde{y},\ \widetilde{y} + \frac{\widetilde{\varepsilon}}{\beta}\right)\right) = \begin{pmatrix} V_y m & V_y m + \dfrac{n}{\beta} V_\varepsilon \end{pmatrix},
$$

where

$$
\begin{aligned}
m &= 1 - \delta f - \gamma f\alpha + \beta\left(-\tau + \delta f + 2\gamma f\alpha - \eta\right), \text{ and} \\
n &= \left(-\tau + \delta f + 2\gamma f\alpha - \eta\right).
\end{aligned}
$$

We also have that

$$
\left(\mathrm{var}\left(\widetilde{y},\ \widetilde{y} + \frac{\widetilde{\varepsilon}}{\beta}\right)\right)^{-1} = \frac{\beta^2}{V_y V_\varepsilon}\begin{pmatrix} V_y + \dfrac{V_\varepsilon}{\beta^2} & -V_y \\ -V_y & V_y \end{pmatrix}.
$$

After some straightforward algebra it results that

$$A = 1 - \delta f - \gamma f \alpha$$

$$B = -\beta \left(\tau - \delta f - 2\gamma f \alpha \right).$$

Since the insider uses the linear strategy $d = \theta + \rho y + \kappa z$, it results by identifying the coefficients that

$$\theta = \frac{1}{2\nu} \left(E\left(\widetilde{U}\right) - A\overline{y} - B\frac{\alpha}{\beta} - B\overline{y} \right)$$

$$\rho = \frac{A}{2\nu}$$

$$\kappa = \frac{1}{2\nu}\frac{B}{\beta}.$$

The second order condition for the insider problem in this second stage is $\nu > 0$.

The market maker observes now the order flow $\widetilde{u}$ and the report received by the auditing agency $\widetilde{z} = \widetilde{x} + \widetilde{\varepsilon}$ and sets the price to be equal to the expected value of the firm conditional on the order flow and report he observes.

$$p = E\left[\widetilde{V}\,\middle|\,\widetilde{u}, \widetilde{z}\right].$$

Observing $\widetilde{u}$ is informationally equivalent to observe

$$\frac{(d + \widetilde{\omega}) - \theta - \kappa\alpha}{\rho + \kappa\beta} = \widetilde{y} + \frac{\widetilde{\omega}}{\rho + \kappa\beta} + \frac{\kappa\widetilde{\varepsilon}}{\rho + \kappa\beta}$$

and observing $\widetilde{z}$ is equivalent to observing $\dfrac{\widetilde{z} - \alpha}{\beta} = \widetilde{y} + \dfrac{\widetilde{\varepsilon}}{\beta}$. Let us denote by

$$\widetilde{\phi} \equiv \frac{\widetilde{\omega}}{\rho + \kappa\beta} + \frac{\kappa\widetilde{\varepsilon}}{\rho + \kappa\beta}.$$

With these notations and equivalencies we can write that the price is

$$p = E\left[\widetilde{V}\,\middle|\,\widetilde{y} + \widetilde{\phi}, \widetilde{y} + \frac{\widetilde{\varepsilon}}{\beta}\right] = E\left(\widetilde{V}\right) + \begin{pmatrix} C & D \end{pmatrix} \begin{pmatrix} \dfrac{(d + \widetilde{\omega}) - \theta - \kappa\alpha}{\rho + \kappa\beta} - \overline{y} \\ \dfrac{\widetilde{z} - \alpha}{\beta} - \overline{y} \end{pmatrix},$$

where

$$\begin{pmatrix} C & D \end{pmatrix} = cov\left(\widetilde{V}, \left(\widetilde{y} + \widetilde{\phi}, \ \widetilde{y} + \frac{\widetilde{\varepsilon}}{\beta}\right)\right) \cdot \left(var\left(\widetilde{y} + \widetilde{\phi}, \ \widetilde{y} + \frac{\widetilde{\varepsilon}}{\beta}\right)\right)^{-1}.$$

Since the market maker observes $\widetilde{z} = \widetilde{x} + \widetilde{\varepsilon}$ and conjectures that the firm uses the strategy $\widetilde{x} = \alpha + \beta\widetilde{y}$ and the tax authority uses the strategy $\pi = \delta + \gamma z$ we can write

$$
\begin{aligned}
\widetilde{V} &= \widetilde{y}(1 - \pi f) - (\tau - \pi f)(\alpha + \beta\widetilde{y} + \widetilde{\varepsilon}) \\
&= \widetilde{y}(1 - \delta f - \gamma f(\alpha + \beta y + \varepsilon)) - (\tau - \delta f - \gamma f(\alpha + \beta y + \varepsilon))(\alpha + \beta y + \varepsilon),
\end{aligned}
$$

and it results that

$$
cov\left(\widetilde{V}, \ \left(\widetilde{y} + \widetilde{\phi}, \quad \widetilde{y} + \frac{\widetilde{\varepsilon}}{\beta}\right)\right) = \left(V_y M - N\frac{\kappa}{\rho + \kappa\beta}V_\varepsilon, \ V_y M - \frac{N}{\beta}V_\varepsilon\right)
$$

where

$$
\begin{aligned}
M &= (1 - \delta f - \gamma f\alpha) - (\tau - \delta f - 2\gamma f\alpha)\beta, \text{ and} \\
N &= \tau - \delta f - 2\gamma f\alpha.
\end{aligned}
$$

On the other hand, we have that

$$
var\left(\widetilde{y} + \widetilde{\phi}, \ \widetilde{y} + \frac{\widetilde{\varepsilon}}{\beta}\right) = \begin{pmatrix} V_y + \dfrac{V_\omega}{(\rho + \kappa\beta)^2} + \dfrac{\kappa^2 V_\varepsilon}{(\rho + \kappa\beta)^2} & V_y + \dfrac{\kappa}{\beta(\rho + \kappa\beta)}V_\varepsilon \\[3mm] V_y + \dfrac{\kappa}{\beta(\rho + \kappa\beta)}V_\varepsilon & V_y + \dfrac{V_\varepsilon}{\beta^2} \end{pmatrix}
$$

and

$$
\begin{aligned}
&\left(var\left(\widetilde{y} + \widetilde{\phi}, \ \widetilde{y} + \frac{\widetilde{\varepsilon}}{\beta}\right)\right)^{-1} \\[2mm]
&= \frac{1}{\Delta}\begin{pmatrix} V_y + \dfrac{V_\varepsilon}{\beta^2} & -\left(V_y + \dfrac{\kappa}{\beta(\rho + \kappa\beta)}V_\varepsilon\right) \\[3mm] -\left(V_y + \dfrac{\kappa}{\beta(\rho + \kappa\beta)}V_\varepsilon\right) & V_y + \dfrac{V_\omega}{(\rho + \kappa\beta)^2} + \dfrac{\kappa^2 V_\varepsilon}{(\rho + \kappa\beta)^2} \end{pmatrix},
\end{aligned}
$$

where

$$
\Delta = \frac{\rho^2 V_y V_\varepsilon + \beta^2 V_\omega V_y + V_\omega V_\varepsilon}{(\rho + \kappa\beta)^2 \beta^2}.
$$

We multiply the covariance and inverse of the variance matrix and we obtain that

$$
C = \frac{\rho V_y V_\varepsilon}{\beta^2(\rho + \kappa\beta)}\frac{(1 - \delta f - \gamma f\alpha)}{\Delta}
$$

$$
D = \frac{\kappa\rho V_y V_\varepsilon(1 - \delta f - \gamma f\alpha) - V_y(M\beta V_\omega - N\rho^2 V_\varepsilon) + N V_\varepsilon V_\omega}{\Delta(\rho + \kappa\beta)^2 \beta}
$$

Finally, we have that the price is

$$p = E\left(\widetilde{V}\right) + C\left(\frac{(d+\omega) - \theta - \kappa\alpha}{\rho + \kappa\beta} - \overline{y}\right) + D\left(\frac{z - \alpha}{\beta} - \overline{y}\right) =$$

$$E\left(\widetilde{V}\right) - \overline{y}\left(C + D\right) - C\frac{\theta}{\rho + \kappa\beta} - C\frac{\kappa\alpha}{\rho + \kappa\beta} - D\frac{\alpha}{\beta} + \frac{C}{\rho + \kappa\beta}(d+\omega) + \frac{D}{\beta}z.$$

We make the following notations which we will use in what it follows. We define, thus

$$
\begin{aligned}
U &\equiv 1 - \delta f - \gamma f \alpha \\[6pt]
R &\equiv (\tau - \gamma f \alpha - \delta f) \\[6pt]
Q &\equiv \beta\gamma f (\beta - 1)\left(V_y + \overline{y}^2\right) + \gamma f V_\varepsilon \\[6pt]
S &\equiv \frac{\beta V_y}{2\left(\beta^2 V_y + V_\varepsilon\right)}.
\end{aligned}
$$

We calculate the expected value of the firm and we obtain that

$$E\left(\widetilde{V}\right) = (U + \alpha\beta\gamma f)\overline{y} - R(\alpha + \beta\overline{y}) + Q$$

Identifying the coefficients in $p(d+\omega) = \mu + \nu(d+\omega) + \eta z$ we obtain

$$
\begin{aligned}
\mu &= E\left(\widetilde{V}\right) - C\frac{\theta}{\rho + \kappa\beta} - C\frac{\kappa\alpha}{\rho + \kappa\beta} - D\frac{\alpha}{\beta} - \overline{y}(C + D) \\[6pt]
\nu &= \frac{C}{\rho + \kappa\beta} \\[6pt]
\eta &= \frac{D}{\beta}.
\end{aligned}
$$

So, we have that

$$
\begin{aligned}
\mu &= E\left(\widetilde{V}\right) - C\frac{\theta}{\rho + \kappa\beta} - C\frac{\kappa\alpha}{\rho + \kappa\beta} - D\frac{\alpha}{\beta} - \overline{y}(C + D) \\[6pt]
\nu &= \frac{C}{\rho + \kappa\beta} \\[6pt]
\eta &= \frac{D}{\beta} \\[6pt]
\theta &= \frac{1}{2\nu}\left(E\left(\widetilde{U}\right) - A\overline{y} - B\frac{\alpha}{\beta} - B\overline{y}\right) \\[6pt]
\rho &= \frac{A}{2\nu} \\[6pt]
\kappa &= \frac{1}{2\nu}\frac{B}{\beta}.
\end{aligned}
$$

From here we conclude first that

$$C = \nu \left(\rho + \kappa \beta \right) = \frac{\left(1 - \delta f - \gamma f \alpha \right) \left(\rho + \kappa \beta \right)}{2\rho} = \frac{U}{2} - \frac{\beta R}{2}.$$

On the other hand from the formula for C we have obtained before it results that

$$\Delta = 2 V_y V_\varepsilon \frac{\rho^2}{\left(\rho + \kappa \beta \right)^2 \beta^2}.$$

However, when we calculated the determinant we obtained that

$$\Delta = \frac{\rho^2 V_y V_\varepsilon + \beta^2 V_\omega V_y + V_\omega V_\varepsilon}{\left(\rho + \kappa \beta \right)^2 \beta^2},$$

so eliminating Δ we obtain an expression for ρ

$$\frac{V_\omega}{\rho^2} = \frac{V_\varepsilon V_y}{V_y \beta^2 + V_\varepsilon}. \tag{4.4}$$

Plugging it in D and after some tedious but straightforward algebra we obtain

$$\begin{aligned} D &= \beta V_y \left(1 - \delta f - \gamma f \alpha \right) \frac{\beta V_\omega - \rho \kappa V_\varepsilon}{2 V_\omega \left(V_y \beta^2 + V_\varepsilon \right)} - \beta \left(\tau - \delta f - 2 \gamma f \alpha \right) \\ &= \beta U S - \frac{\beta R}{2} + \alpha \beta \gamma f. \end{aligned}$$

To conclude, using the formulas we have derived above we obtain that

$$\begin{aligned} \nu &= \frac{U}{2\rho} \\ \eta &= U S - \frac{R}{2} + \alpha \gamma f \\ \rho &= \left(\frac{V_\omega \left(\beta^2 V_y + V_\varepsilon \right)}{V_y V \varepsilon} \right)^{1/2} \\ \kappa &= -\frac{R\rho}{U}. \end{aligned}$$

Finally, we solve for μ and θ and we obtain

$$\begin{aligned} \mu &= E \left(V \right) - \eta \left(\alpha + \beta \overline{y} \right) = U \overline{y} + Q - \alpha^2 \gamma f + \left(\alpha + \beta \overline{y} \right) \left(S U + \frac{R}{2} \right) \\ \theta &= -\rho \overline{y} + \frac{R\rho}{U} \left(\alpha + \beta \overline{y} \right). \end{aligned}$$

In the first stage the firm maximizes the net payoff after taxes choosing the tax report x such that

$$\max_{x} E(V) \Leftrightarrow \max_{x} E\left[(1 - (\delta + \gamma x + \gamma \varepsilon) f) y - (\tau - (\delta + \gamma (x + \varepsilon)) f)(x + \varepsilon)\right]. \quad (4.5)$$

The first order condition for this problem is

$$E\left(-f\gamma y + 2f\gamma x + 2f\gamma \varepsilon - \tau + \delta f\right) = 0,$$

which leads to

$$x = \frac{1}{2}y + \frac{1}{2}\frac{\tau - \delta f}{f\gamma}.$$

We identify the coefficients in $x = \alpha + \beta y$ and we have that

$$\alpha = \frac{1}{2}\frac{\tau - \delta f}{f\gamma}$$
$$\beta = \frac{1}{2}.$$

The second order condition for this problem is $f\gamma < 0$. Since the penalty rate is always positive, it results that we need $\gamma < 0$.

Finally, we consider the maximization problem of the tax auditing agency. The agency maximizes the expected profit conditional on the report received from the firm z. As we said, although the firm reports x, the report received by the agency is $\tilde{z} = x + \tilde{\varepsilon}$. Therefore, the agency's problem is the following:

$$\pi(z) = \arg\max_{\pi} E\left[\tau z + \pi f\left(\tilde{y} - z\right) - \frac{1}{2}c\pi^2 \,\middle|\, z\right].$$

The first order condition for this problem is

$$E\left[f\left(\tilde{y} - z\right) - c\pi \,\middle|\, z\right] = 0,$$

or equivalent the auditing strategy $\pi = \pi(z)$ is

$$\pi = \frac{f}{c}\left[E\left[\tilde{y}\,\middle|\,z\right] - z\right]. \quad (4.6)$$

The tax agency conjectures that the firm uses a linear reporting strategy $x = \alpha + \beta y$. Since the report perceived by the agency $\widetilde{z} = \widetilde{x} + \widetilde{\varepsilon}$, he concludes that $\widetilde{z} = \alpha + \beta \widetilde{y} + \widetilde{\varepsilon}$. Observing $\widetilde{z}$ is informationally equivalent to observing $\dfrac{\widetilde{z} - \alpha}{\beta} = \widetilde{y} + \dfrac{\widetilde{\varepsilon}}{\beta}$. Consequently, we obtain that

$$E\left[\widetilde{y}|\, z\right] = E\left[\widetilde{y}\left|\widetilde{y} + \frac{\widetilde{\varepsilon}}{\beta}\right.\right] = \overline{y} + \frac{V_y}{V_y + V_\varepsilon/\beta^2}\left(\frac{\widetilde{z} - \alpha}{\beta} - \overline{y}\right).$$

Plugging this in (4.6) we obtain that

$$\pi = \frac{f}{c}\left[\overline{y}\frac{V_\varepsilon/\beta^2}{V_y + V_\varepsilon/\beta^2} - \frac{V_y}{V_y + V_\varepsilon/\beta^2}\frac{\alpha}{\beta} + \left(\frac{V_y}{V_y + V_\varepsilon/\beta^2}\frac{1}{\beta} - 1\right)\widetilde{z}\right]$$

The auditor uses the linear strategy $\pi(z) = \delta + \gamma z$, and by identifying the coefficients we have that

$$\begin{aligned}
\delta &= \frac{f}{c}\left[\overline{y}\left(\frac{V_\varepsilon/\beta^2}{V_y + V_\varepsilon/\beta^2}\right) - \left(\frac{V_y}{V_y + V_\varepsilon/\beta^2}\right)\frac{\alpha}{\beta}\right] \\
\gamma &= \frac{f}{c}\left[\left(\frac{V_y}{V_y + V_\varepsilon/\beta^2}\right)\frac{1}{\beta} - 1\right].
\end{aligned} \tag{4.7}$$

We solve now the system of equations with the unknowns, α, δ, γ as a function of β and we obtain that

$$\begin{aligned}
\alpha &= \frac{1}{2}\overline{y} - \frac{\tau c}{8f^2}\left(\frac{V_y + 4V_\varepsilon}{V_\varepsilon}\right) \\
\beta &= \frac{1}{2} \\
\delta &= -\frac{f}{c}\left(\frac{V_y - 4V_\varepsilon}{V_y + 4V_\varepsilon}\right)\overline{y} + \frac{1}{4}\left(\frac{\tau V_y}{fV_\varepsilon}\right) \\
\gamma &= \frac{f}{c}\left(\frac{V_y - 4V_\varepsilon}{V_y + 4V_\varepsilon}\right).
\end{aligned}$$

Finally, using these values for α, β, δ and γ we obtain that the coefficients in the second stage are

$$\mu = U\overline{y} + Q - \alpha^2\gamma f + (\alpha + \beta\overline{y})\left(SU + \frac{R}{2}\right)$$

$$\nu = \frac{U}{2\rho}$$

$$\eta = US - \frac{R}{2} + \alpha\gamma f$$

$$\theta = -\rho\overline{y} - \kappa(\alpha + \beta\overline{y})$$

$$\rho = \left(\frac{V_\omega(V_y + 4V_\varepsilon)}{4V_yV\varepsilon}\right)^{1/2}$$

$$\kappa = -\frac{R\rho}{U}$$

where

$$U = \frac{1}{2}\left(\frac{V_y - 4V_\varepsilon}{V_y + 4V_\varepsilon}\right)\frac{f^2}{c}\overline{y} + 1 - \frac{1}{8}\tau\left(\frac{V_y + 4V_\varepsilon}{V_\varepsilon}\right) \tag{4.8}$$

$$R = \frac{1}{2}\frac{f^2}{c}\left(\frac{V_y - 4V_\varepsilon}{V_y + 4V_\varepsilon}\right)\overline{y} - \frac{1}{8}\tau\frac{V_y}{V_\varepsilon}$$

$$S = \frac{V_y}{V_y + 4V_\varepsilon}$$

$$Q = -\frac{1}{4}\frac{f^2}{c}\left(\frac{(V_y - 4V_\varepsilon)^2}{V_y + 4V_\varepsilon}\right) - \frac{1}{4}\frac{f^2}{c}\left(\frac{V_y - 4V_\varepsilon}{V_y + 4V_\varepsilon}\right)\overline{y}^2.$$

We also replace them in the above formulas for insider's demand, firm's report and auditing effort and we obtain thus that the demand of the insider in the second stage is

$$d(y,z) = \rho\left((y - \overline{y}) - \frac{R}{U}(z - \alpha - \beta\overline{y})\right)$$

$$= \rho(y - \overline{y})\left(1 - \frac{R}{2U}\right) - \frac{R\rho}{U}\varepsilon,$$

the tax report the firm is making in the first stage

$$x(y) = \frac{y + \overline{y}}{2} - \frac{\tau c}{8f^2}\left(\frac{V_y + 4V_\varepsilon}{V_\varepsilon}\right),$$

and the auditing effort of the tax agency

$$\pi(z) = \frac{f}{c}\left(\frac{V_y - 4V_\varepsilon}{V_y + 4V_\varepsilon}\right)(z - \overline{y}) + \frac{1}{4}\left(\frac{\tau V_y}{fV_\varepsilon}\right).$$

As we have seen the equilibrium price is

$$p = \mu + \nu d + \eta z,$$

where μ, ν, η were determined in the previous proposition

$$
\begin{aligned}
\mu &= E\left(V\right) - \left(US - \frac{R}{2} + \alpha\gamma f\right)\left(\alpha + \beta\overline{y}\right) \\
\nu &= \frac{U}{2\rho} \\
\eta &= US - \frac{R}{2} + \alpha\gamma f
\end{aligned}
$$

and the demand of the insider is

$$d(y, z) = \rho\left(\left(y - \overline{y}\right) - \frac{R}{U}\left(z - \alpha - \beta\overline{y}\right)\right)$$

Consequently, the price can be written as

$$
\begin{aligned}
p &= E\left(V\right) + \frac{U}{2}\left(y - \overline{y}\right) - \left(z - \alpha - \beta\overline{y}\right)\left(\eta - \frac{R}{2}\right) + \frac{U}{2\rho}\omega \\
&= \alpha\beta\gamma f\overline{y} - R\left(\alpha + \beta\overline{y}\right) + U\frac{\left(y + \overline{y}\right)}{2} + Q + \left(US - R + \alpha\gamma f\right)\left(z - \alpha - \beta\overline{y}\right) + \frac{U}{2\rho}\omega \\
&= U\frac{\left(y + \overline{y}\right)}{2} + \left(US - R + \alpha\gamma f\right)z + Q - US\left(\alpha + \beta\overline{y}\right) - \alpha^2\gamma f + \frac{U}{2\rho}\omega.
\end{aligned}
$$

Finally, we study the second order conditions for the insider's problem and board's problem. The second order conditions for the insider's problem (4.3) is $\nu > 0$. This is equivalent to $U > 0$ or

$$\frac{1}{2}\left(\frac{V_y - 4V_\varepsilon}{V_y + 4V_\varepsilon}\right)\frac{f^2}{c}\overline{y} + 1 - \frac{1}{8}\tau\left(\frac{V_y + 4V_\varepsilon}{V_\varepsilon}\right) > 0.$$

We solve for τ in this equation and we obtain that this is equivalent to

$$\tau < \tau^* \equiv 4V_\varepsilon\frac{2(V_y + 4V_\varepsilon) + \dfrac{f^2}{c}\overline{y}\left(V_y - 4V_\varepsilon\right)}{\left(V_y + 4V_\varepsilon\right)^2}.$$

However if $\frac{f^2}{c}\overline{y} \leq \frac{V_y + 4V_\varepsilon}{4V_\varepsilon}$, $\tau^* > 1$ always. If $\frac{f^2}{c}\overline{y} \in \left(\frac{V_y + 4V_\varepsilon}{4V_\varepsilon}, 2\left(\frac{V_y + 4V_\varepsilon}{4V_\varepsilon - V_y}\right)\right)$ then $\tau^* \in (0, 1)$. Consequently, the second order conditions are always satisfied for

$\overline{y} \leq \dfrac{c}{f^2} \dfrac{V_y + 4V_\varepsilon}{4V_\varepsilon}$ and they are never satisfied for $\overline{y} \geq 2\dfrac{c}{f^2} \dfrac{V_y + 4V_\varepsilon}{4V_\varepsilon}$. If we have that

$\overline{y} \in \left(\dfrac{c}{f^2} \dfrac{V_y + 4V_\varepsilon}{4V_\varepsilon}, 2\dfrac{c}{f^2} \left(\dfrac{V_y + 4V_\varepsilon}{4V_\varepsilon - V_y} \right) \right)$ then the second order conditions are satisfied only for $\tau < \tau^*$.

Finally, the second order condition for the problem (4.5) is $\gamma < 0$ which is equivalent to $V_y < 4V_\varepsilon$. $\blacksquare$

Proof of Corollary 4.1. The expected net revenue raised by a tax auditor before observing the realization of the report z is

$$E\left(\widetilde{R}\right) = E\left[\tau\left(\alpha + \beta\widetilde{y} + \widetilde{\varepsilon}\right) + f\left(\delta + \gamma\left(\alpha + \beta\widetilde{y} + \widetilde{\varepsilon}\right)\right)\left(\widetilde{y} - \alpha - \beta\widetilde{y} - \widetilde{\varepsilon}\right)\right.$$

$$\left. -\frac{1}{2}\left(\delta + \gamma\left(\alpha + \beta\widetilde{y} + \widetilde{\varepsilon}\right)\right)^2\right] = \left(\alpha + \beta\overline{y}\right)\tau +$$

$$f\left(\left(\delta + \gamma\alpha\right)\left(1 - \beta\right)\overline{y} - \alpha\left(\delta + \gamma\alpha\right) - \alpha\gamma\beta\overline{y} - \gamma\varepsilon + \gamma\beta\left(1 - \beta\right)\left(V_y + \overline{y}^2\right)\right)$$

$$-\frac{1}{2}c\left(\delta^2 + 2\delta\gamma\alpha + 2\delta\gamma\beta\overline{y} + \gamma^2\alpha^2 + 2\gamma^2\alpha\beta\overline{y} + \gamma^2\beta^2\left(V_y + \overline{y}^2\right) + \gamma^2\varepsilon\right).$$

Using the equilibrium values for $\alpha, \beta, \delta, \gamma$ and after some algebra, we obtain that

$$E\left(\widetilde{R}\right) = \frac{1}{128cf^2V_\varepsilon^2\left(V_y + 4V_\varepsilon\right)}\left[256f^4V_\varepsilon^4 - 128f^4V_\varepsilon^3V_y + 16f^4V_\varepsilon^2V_y^2 + 128\overline{y}f^2V_\varepsilon^2\tau cV_y\right.$$

$$\left. +512\overline{y}f^2V_\varepsilon^3\tau c - 4\tau^2c^2V_y^2V_\varepsilon + \tau^2c^2V_y^3 - 192\tau^2c^2V_\varepsilon^3 - 80\tau^2c^2V_yV_\varepsilon^2\right].$$

We compute the derivative of the expected net revenue with respect to V_y and we have that

$$\frac{\partial E\left(\widetilde{R}\right)}{\partial V_y} = -\frac{1}{64}\left(4V_\varepsilon - V_y\right)\frac{96f^4V_\varepsilon^3 + 16\tau^2c^2V_\varepsilon^2 + 8V_\varepsilon^2f^4V_y + 8\tau^2c^2V_yV_\varepsilon + \tau^2c^2V_y^2}{cf^2V_\varepsilon^2\left(V_y + 4V_\varepsilon\right)^2}.$$

The equation

$$96f^4V_\varepsilon^3 + 16\tau^2c^2V_\varepsilon^2 + 8V_\varepsilon^2f^4V_y + 8\tau^2c^2V_yV_\varepsilon + \tau^2c^2V_y^2 = 0$$

has either no real solution when $V_\varepsilon < \dfrac{4\tau^2c^2}{f^4}$, or two real solutions. However, it is always the case that both solutions are negative and this implies that on the interval $(0, 4V_\varepsilon)$ the derivative $\dfrac{\partial E\left(\widetilde{R}\right)}{\partial V_y}$ is negative.

We compute next the following derivative:

$$\frac{\partial E\left(R\right)}{\partial V_\varepsilon} = \frac{1}{64}\left(4V_\varepsilon - V_y\right)\frac{128f^4V_\varepsilon^4 + 96f^4V_yV_\varepsilon^3 + 16\tau^2c^2V_yV_\varepsilon^2 + 8\tau^2c^2V_y^2V_\varepsilon + \tau^2c^2V_y^3}{cf^2V_\varepsilon^3\left(V_y + 4V_\varepsilon\right)^2},$$

and it is obvious that it is always positive for $V_\varepsilon > \dfrac{V_y}{4}$. ∎

Proof of Corollary 4.2.   The expected payoff of the firm net of taxes is

$$
\begin{aligned}
E\left(\widetilde{V}\right) &= E\left[y - \tau\left(x\left(y\right) + \varepsilon\right) - \pi\left(x\left(y\right) + \varepsilon\right)f\left(y - x\left(y\right) - \varepsilon\right)\right]\\
&= \overline{y}\left(1 - \tau\beta + f\left(\delta\left(\beta - 1\right) - \gamma\alpha + 2\gamma\alpha\beta\right)\right) - \alpha\left(\tau - f\left(\delta + \gamma\alpha\right)\right)\\
&\quad - f\gamma\beta\left(1 - \beta\right)\left(V_y + \overline{y}^2\right) + \gamma f V_\varepsilon\\
&= \frac{1}{64f^2V_\varepsilon^2c\left(V_y + 4V_\varepsilon\right)}\left[64f^2\overline{y}V_\varepsilon^2V_yc + 256f^2\overline{y}V_\varepsilon^3c - 64f^2\overline{y}V_\varepsilon^2\tau V_yc\right.\\
&\quad\left. - 256f^2\overline{y}V_\varepsilon^3\tau c - \tau^2c^2V_y^3 + 16\tau^2V_yc^2V_\varepsilon^2 - 4\tau^2c^2V_\varepsilon V_y^2\right.\\
&\quad\left. + 64\tau^2c^2V_\varepsilon^3 - 16f^4V_\varepsilon^2V_y^2 + 128f^4V_\varepsilon^3V_y - 256f^4V_\varepsilon^4\right].
\end{aligned}
$$

We compute the derivative with respect to V_y and we have that

$$\frac{\partial E\left(\widetilde{V}\right)}{\partial V_y} = -\frac{\tau^2c^2V_y^3 + 8\tau^2c^2V_\varepsilon V_y^2 + 16\tau^2V_yc^2V_\varepsilon^2 + 8f^4V_\varepsilon^2V_y^2 - 384f^4V_\varepsilon^4 + 64f^4V_\varepsilon^3V_y}{32f^2V_\varepsilon^2c\left(V_y + 4V_\varepsilon\right)^2}.$$

We define by

$$g\left(V_y\right) = \tau^2c^2V_y^3 + 8\tau^2c^2V_\varepsilon V_y^2 + 16\tau^2V_yc^2V_\varepsilon^2 + 8f^4V_\varepsilon^2V_y^2 - 384f^4V_\varepsilon^4 + 64f^4V_\varepsilon^3V_y,$$

and we have that

$$g'\left(V_y\right) = 3\tau^2c^2V_y^2 + 16\tau^2c^2V_\varepsilon V_y + 16\tau^2c^2V_\varepsilon^2 + 16V_yf^4V_\varepsilon^2 + 64f^4V_\varepsilon^3 > 0$$

Since $g\left(0\right) < 0$, and $g\left(4V_\varepsilon\right) > 0$, it results that it exists V_y^* such that $g\left(V_y\right) < 0$ for $V_y \in \left(0, V_y^*\right)$ and $g\left(V_y\right) > 0$, otherwise. Consequently, $\frac{\partial E\left(\widetilde{V}\right)}{\partial V_y} > 0$, for $V_y \in \left(0, V_y^*\right)$ and $\frac{\partial E\left(\widetilde{V}\right)}{\partial V_y} < 0$, otherwise.

We compute next the derivative with respect to V_ε

$$\frac{\partial E\left(\widetilde{V}\right)}{\partial V_\varepsilon} = -\frac{256f^4V_\varepsilon^4V_y - 96f^4V_\varepsilon^3V_y^2 - \tau^2c^2V_y^4 - 8V_\varepsilon\tau^2c^2V_y^3 - 16\tau^2c^2V_\varepsilon^2V_y^2 + 512f^4V_\varepsilon^5}{32f^2V_\varepsilon^3c\left(V_y + 4V_\varepsilon\right)^2}.$$

To study the sign of this derivative, we define

$$g\left(V_\varepsilon\right) = 256f^4V_\varepsilon^4V_y - 96f^4V_\varepsilon^3V_y^2 - \tau^2c^2V_y^4 - 8V_\varepsilon\tau^2c^2V_y^3 - 16\tau^2c^2V_\varepsilon^2V_y^2 + 512f^4V_\varepsilon^5$$

and we have that $g''\left(V_\varepsilon\right) > 0$ for $V_\varepsilon > \frac{V_y}{4}$. We have that $g'\left(\frac{V_y}{4}\right) = 8V_y^3\left(f^4V_y - 2c^2\tau^2\right)$. If $g'\left(\frac{V_y}{4}\right) > 0$, since $g''\left(V_\varepsilon\right) > 0$, it results that $g'\left(V_\varepsilon\right) > 0$ always. If $g'\left(\frac{V_y}{4}\right) < 0$, we have that it exists $V_\varepsilon^* > \frac{V_y}{4}$ such that $g'\left(V_\varepsilon\right) < 0$, for $V_\varepsilon \in \left(\frac{V_y}{4}, V_\varepsilon^*\right)$, and $g'\left(V_\varepsilon\right) > 0$, otherwise. However, since $g\left(\frac{V_y}{4}\right) = -4\tau^2c^2V_y^4 < 0$, in both cases we obtain that it exists $V_\varepsilon^{**} > \frac{V_y}{4}$ such that $g\left(V_\varepsilon\right) < 0$, for $V_\varepsilon \in \left(\frac{V_y}{4}, V_\varepsilon^{**}\right)$, and $g\left(V_\varepsilon\right) > 0$, otherwise. ∎

Proof of Corollary 4.3. We compute the market depth

$$\frac{1}{\nu} = \frac{2\rho}{U} = \frac{\left(\dfrac{V_\omega\left(V_y + 4V_\varepsilon\right)}{V_yV_\varepsilon}\right)^{1/2}}{\dfrac{1}{2}ft\left(\dfrac{V_y - 4V_\varepsilon}{V_y + 4V_\varepsilon}\right)\overline{y} + 1 - \dfrac{1}{8}\tau\left(\dfrac{V_y + 4V_\varepsilon}{V_\varepsilon}\right)}.$$

We can easily see that the market depth is increasing in the variance of the noise trading V_ω, the cost of auditing c and in the tax rate τ, and decreasing in the payoff mean $\overline{y}$ and in the penalty rate f.

To see how the function varies with respect to V_y we define $a = ft\overline{y}$ and

$$f(V_y) = \frac{\left(\dfrac{V_\omega\left(V_y + 4V_\varepsilon\right)}{V_yV_\varepsilon}\right)^{1/2}}{\dfrac{1}{2}\left(\dfrac{V_y - 4V_\varepsilon}{V_y + 4V_\varepsilon}\right)a + 1 - \dfrac{1}{8}\tau\left(\dfrac{V_y + 4V_\varepsilon}{V_\varepsilon}\right)}.$$

The sign of the derivative of $f(V_y)$ is the same as the one of the function

$$g\left(V_y\right) = 32aV_\varepsilon^3 - 64V_\varepsilon^3 + 32\tau V_\varepsilon^3 + 32V_\varepsilon^2V_y\tau - 16V_\varepsilon^2V_y - 40V_\varepsilon^2V_ya + 10\tau V_y^2V_\varepsilon + \tau V_y^3.$$

The equation $g'\left(V_y\right) = 0$ has two solutions, one of them being always negative. The second equation is negative if $\tau > \frac{1}{2} + \frac{5}{2}a$, is in the interval $\left(0, 4V_\varepsilon\right)$, if $\tau \in \left(\frac{1}{10} + \frac{1}{4}a, \frac{1}{2} + \frac{5}{2}a\right)$, and greater than $4V_\varepsilon$, otherwise. We studied the sign of $g(V_y)$ for different values of parameters and we have obtained the following cases.

In what it follows we will use the second order condition for the insider's problem. This implies that if $a \in (1, 2)$ we need to have that $\tau < \tau^*$.

Case 1 $a \in \left(0, \frac{2}{3}\right)$

If $\tau < \frac{1}{10} + \frac{1}{4}a$, we have that $g(0) < 0$, $g'(0) < 0$, $g(4V_\varepsilon) < 0$ and since the second solution of the equation $g'(V_y) = 0$ is greater than $4V_\varepsilon$, it results that $g(V_y)$ is decreasing. Consequently, we obtain that $g(V_y) < 0$ for any $V_y \in (0, 4V_\varepsilon)$.

If $\tau \in \left(\frac{1}{10} + \frac{1}{4}a, \frac{1}{3} + \frac{1}{3}a\right)$, we have that $g(0) < 0$, $g'(0) < 0$, $g(4V_\varepsilon) < 0$ and the second solution of the equation $g'(V_y) = 0$ is in the interval $(0, 4V_\varepsilon)$, it results that $g(V_y)$ is U-shaped. However, since $g(4V_\varepsilon) < 0$ we have that again $g(V_y) < 0$ for any $V_y \in (0, 4V_\varepsilon)$.

If $\tau \in \left(\frac{1}{3} + \frac{1}{3}a, \frac{1}{2} + \frac{5}{4}a\right)$, we have that $g(0) < 0$, $g'(0) < 0$ but $g(4V_\varepsilon) > 0$ and the second solution of the equation $g'(V_y) = 0$ is in the interval $(0, 4V_\varepsilon)$. It results again that $g(V_y)$ is U-shaped but since $g(4V_\varepsilon) > 0$ we have that it exists $V_y^* \in (0, 4V_\varepsilon)$, such that $g(V_y) < 0$ for any $V_y \in \left(0, V_y^*\right)$, and $g(V_y) > 0$, otherwise.

If $\tau \in \left(\frac{1}{2} + \frac{5}{4}a, 1\right)$, we have that $g(0) < 0$, $g'(0) > 0$ but $g(4V_\varepsilon) > 0$. Moreover, the second solution of the equation $g'(V_y) = 0$ is negative, so $g(V_y)$ is increasing. We obtain again that it exists $V_y^* \in (0, 4V_\varepsilon)$, such that $g(V_y) < 0$ for any $V_y \in \left(0, V_y^*\right)$, and $g(V_y) > 0$, otherwise.

Case 2 $a \in \left[\frac{2}{3}, 1\right)$

If $\tau < \frac{1}{10} + \frac{1}{4}a$, we have that $g(0) < 0$, $g'(0) < 0$, $g(4V_\varepsilon) < 0$ and since the second solution of the equation $g'(V_y) = 0$ is greater than $4V_\varepsilon$, it results that $g(V_y)$ is decreasing. Consequently, we obtain that $g(V_y) < 0$ for any $V_y \in (0, 4V_\varepsilon)$.

If $\tau \in \left(\frac{1}{10} + \frac{1}{4}a, 1\right)$, we have that $g(0) < 0$, $g'(0) < 0$, $g(4V_\varepsilon) < 0$ and the second solution of the equation $g'(V_y) = 0$ is in the interval $(0, 4V_\varepsilon)$. However, since $g(4V_\varepsilon) < 0$ we have that again $g(V_y) < 0$ for any $V_y \in (0, 4V_\varepsilon)$.

Case 3 $a \in \left[1, \frac{38}{25}\right)$

Case 3 $a \in \left[1, \frac{38}{25}\right)$

If $\tau < \frac{1}{10} + \frac{1}{4}a$, we have that $g(0) < 0$, $g'(0) < 0$, $g(4V_\varepsilon) < 0$ and since the second solution of the equation $g'(V_y) = 0$ is greater than $4V_\varepsilon$, it results that $g(V_y)$ is decreasing. Consequently, we obtain that $g(V_y) < 0$ for any $V_y \in (0, 4V_\varepsilon)$.

If $\tau \in \left(\frac{1}{10} + \frac{1}{4}a, \min\{2 - a, \tau^*\}\right)$, we have that $g(0) < 0$, $g'(0) < 0$, $g(4V_\varepsilon) < 0$ and the second solution of the equation $g'(V_y) = 0$ is in the interval $(0, 4V_\varepsilon)$. However, since $g(4V_\varepsilon) < 0$ we have that again $g(V_y) < 0$ for any $V_y \in (0, 4V_\varepsilon)$.

If $\tau \in \left(\min\{2 - a, \tau^*\}, \tau^*\right)$, we have that $g(0) > 0$, $g'(0) < 0$, $g(4V_\varepsilon) < 0$ and the second solution of the equation $g'(V_y) = 0$ is in the interval $(0, 4V_\varepsilon)$. Consequently, it exists $V_y^{**} \in (0, 4V_\varepsilon)$. such that $g(V_y) > 0$ for any $V_y \in \left(0, V_y^{**}\right)$, and $g(V_y) < 0$, otherwise.

Case 4 $a \in \left[\frac{38}{25}, 2\right)$

If $\tau < 2 - a$ then $g(0) < 0$, $g'(0) < 0$, $g(4V_\varepsilon) < 0$, $g'(V_y)$ is U-shaped and it results that $g(V_y) < 0$ for any $V_y \in (0, 4V_\varepsilon)$.

If $\tau \in \left(2 - a, \frac{1}{10} + \frac{1}{4}a\right)$ we have that $g(0) > 0$, $g'(0) < 0$, $g(4V_\varepsilon) < 0$ and the second solution of the equation $g'(V_y) = 0$ is greater than $4V_\varepsilon$. Consequently, it exists $V_y^{**} \in (0, 4V_\varepsilon)$. such that $g(V_y) > 0$ for any $V_y \in \left(0, V_y^{**}\right)$, and $g(V_y) < 0$, otherwise.

If $\tau \in \left(\frac{1}{10} + \frac{1}{4}a, \tau^*\right)$ we have that $g(0) > 0$, $g'(0) < 0$, $g(4V_\varepsilon) < 0$ and the second solution of the equation $g'(V_y) = 0$ is in the interval $(0, 4V_\varepsilon)$. Consequently, it exists $V_y^{**} \in (0, 4V_\varepsilon)$. such that $g(V_y) > 0$ for any $V_y \in \left(0, V_y^{**}\right)$, and $g(V_y) < 0$, otherwise.

Case 5 $a \in \left[2, 2\frac{V_y + 4V_\varepsilon}{4V_\varepsilon - V_y}\right)$

In this case $\tau > 2 - a$ always and

If $\tau \in \left(0, \frac{1}{10} + \frac{1}{4}a\right)$ we have that $g(0) > 0$, $g'(0) < 0$, $g(4V_\varepsilon) < 0$ and the second solution of the equation $g'(V_y) = 0$ is greater than $4V_\varepsilon$. Consequently, it exists $V_y^{**} \in (0, 4V_\varepsilon)$. such that $g(V_y) > 0$ for any $V_y \in \left(0, V_y^{**}\right)$, and $g(V_y) < 0$, otherwise.

If $\tau \in \left(\frac{1}{10} + \frac{1}{4}a, \tau^*\right)$ we have that $g(0) > 0$, $g'(0) < 0$, $g(4V_\varepsilon) < 0$ and the second solution of the equation $g'(V_y) = 0$ is in the interval $(0, 4V_\varepsilon)$. Consequently, it exists $V_y^{**} \in (0, 4V_\varepsilon)$. such that $g(V_y) > 0$ for any $V_y \in \left(0, V_y^{**}\right)$, and $g(V_y) < 0$, otherwise.

We collect the results and we have the following:

If $a \in \left(0, \frac{2}{3}\right)$ and $\tau \in \left(\frac{1}{3} + \frac{1}{3}a, 1\right)$ it exists $V_y^* \in (0, 4V_\varepsilon)$, such that $g(V_y) < 0$ for any $V_y \in \left(0, V_y^*\right)$, and $g(V_y) > 0$, otherwise. In this case we obtain that $f(V_y)$ is U shaped.

If $a \in \left[1, \frac{38}{25}\right)$ and if $\tau \in \left(\min\{2 - a, \tau^*\}, \tau^*\right)$ it exists $V_y^{**} \in (0, 4V_\varepsilon)$. such that $g(V_y) > 0$ for any $V_y \in \left(0, V_y^{**}\right)$, and $g(V_y) < 0$, otherwise. In this case we obtain that

$f(V_y)$ is inverted U shaped.

If $a \geq \frac{38}{25}$ and $\tau \in (2 - a, \tau^*)$ we have that it exists $V_y^{**} \in (0, 4V_\varepsilon)$. such that $g(V_y) > 0$ for any $V_y \in (0, V_y^{**})$, and $g(V_y) < 0$, otherwise. In this case we obtain that $f(V_y)$ is inverted U shaped. ∎

Proof of Corollary 4.4. We compute now $Var\left(\widetilde{V}\middle|\widetilde{z}\right) - Var\left(\widetilde{V}\middle|\widetilde{p}, \widetilde{z}\right)$. Due to the normality assumptions we have that

$$Var\left(\widetilde{V}\middle|\widetilde{z}\right) - Var\left(\widetilde{V}\middle|\widetilde{p}, \widetilde{z}\right) = (Var(\widetilde{p}|\widetilde{z}))^{-1}\left(Cov\left(\widetilde{V}, \widetilde{p}\middle|\widetilde{z}\right)\right)^2.$$

First, we compute the covariance of the payoff with the price conditional on the report z and we have that

$$
\begin{aligned}
Cov\left(\widetilde{V}, \widetilde{p}\middle|\widetilde{z}\right) &= \frac{U^2(1+S)}{2}Var\left(|\widetilde{y}\ \widetilde{z}\right) = \frac{U^2(1+S)}{2}\left(Var(\widetilde{y}) - \frac{(Cov(\widetilde{y}, \widetilde{z}))^2}{Var(\widetilde{z})}\right) \\
&= \frac{U^2(1+S)}{2}\left(V_y - \frac{(\beta V_y)^2}{\beta V_y + V_\varepsilon}\right) = 2U^2(1+S)\left(\frac{V_y V_\varepsilon}{V_y + 4V_\varepsilon}\right) \\
&= \frac{4U^2 V_y V_\varepsilon (V_y + 2V_\varepsilon)}{V_y + 4V_\varepsilon}.
\end{aligned}
$$

Using the formula for the conditional variance of prices we obtained below, we have that

$$Var\left(\widetilde{V}\middle|\widetilde{z}\right) - Var\left(\widetilde{V}\middle|\widetilde{p}, \widetilde{z}\right) = \frac{\left(\dfrac{4U^2 V_y V_\varepsilon (V_y + 2V_\varepsilon)}{V_y + 4V_\varepsilon}\right)^2}{\dfrac{2U^2 V_y V_\varepsilon}{V_y + 4V_\varepsilon}} = \frac{8U^2 V_y V_\varepsilon (V_y + 2V_\varepsilon)^2}{V_y + 4V_\varepsilon}.$$

To characterize the behaviour of the informativeness of prices we establish the following:

Claim *The informativeness of prices with respect to the variance of the payoff V_y may be increasing, inverted U-shaped, oscillating (decreasing, increasing, decreasing) or U-shaped.*

Proof of Claim. We have obtained that the informativeness of prices when we condition on the public signal is

$$Var\left(\widetilde{V}\middle|\widetilde{z}\right) - Var\left(\widetilde{V}\middle|\widetilde{p}, \widetilde{z}\right) = \frac{8U^2 V_y V_\varepsilon (V_y + 2V_\varepsilon)^2}{V_y + 4V_\varepsilon}.$$

We study the derivative of this function with respect to V_y and we have that it has the same sign as

$$f(V_y) = g(V_y) h(V_y),$$

where

$$\begin{aligned}
g(V_y) &\equiv -2aV_\varepsilon V_y^3 + 80V_y\tau V_\varepsilon^3 + 8V_y aV_\varepsilon^3 + 50V_y^2\tau V_\varepsilon^2 + 12\tau V_\varepsilon V_y^3 - 20V_y^2 aV_\varepsilon^2 \\
&\quad -4V_\varepsilon V_y^3 - 40V_\varepsilon^2 V_y^2 - 112V_\varepsilon^3 V_y + 32\tau V_\varepsilon^4 + 32aV_\varepsilon^4 + \tau V_y^4 - 64V_\varepsilon^4,
\end{aligned}$$

and

$$h(V_y) \equiv -4aV_\varepsilon V_y + 16aV_\varepsilon^2 - 8V_y V_\varepsilon - 32V_\varepsilon^2 + 16\tau V_\varepsilon^2 + 8\tau V_y V_\varepsilon + \tau V_y^2.$$

The equation $h(V_y)$ has two solutions. One is always greater than $4V_\varepsilon$. If $\tau < 2 - a$, the second solution V_2 is negative. Otherwise, it is between 0 and $4V_\varepsilon$. It results that if $a \in (0,1)$ or $a \in (1,2)$ and $\tau < 2 - a,$ the solution V_2 is negative, and it results $h(V_y) < 0$, for any $V_y \in (0, 4V_\varepsilon)$. Otherwise, $V_2 \in (0, 4V_\varepsilon)$, and it results that $h(V_y) > 0$, for any $V_y \in (0, V_2)$, and $h(V_y) < 0$, otherwise.

Next, $g(V_y) = 0$ is an equation of forth degree, so it has 4 solutions. We have studied this function depending on the parameters and we obtain the following cases:

Case 1 $a \in [0,1]$

In this case $\tau < 2 - a$ always, and it results that $h(V_y) < 0$ for any $V_y \in (0, 4V_\varepsilon)$. Then, we show that for $a < 4$ (which is the case when we impose the second order condition for insider's problem) we have that $h'(0) < 0$. We have studied the first derivative and we obtain:

if $\tau < \frac{3}{17}a + \frac{11}{17}$ then $g(V_y) < 0$ for any $V_y \in (0, 4V_\varepsilon)$;

if $\tau > \frac{3}{17}a + \frac{11}{17}$ then it exists $V_y^* \in (0, 4V_\varepsilon)$ such that $g(V_y) < 0$ for any $V_y \in (0, V_y^*)$ and $g(V_y) > 0$, otherwise.

This implies that if $\tau < \frac{3}{17}a + \frac{11}{17}$ then $f(V_y) > 0$ for any $V_y \in (0, 4V_\varepsilon)$ (so the informativeness is increasing) and if $\tau > \frac{3}{17}a + \frac{11}{17}$, $f(V_y) > 0$ for any $V_y \in (0, V_y^*)$ and $f(V_y) < 0$, otherwise (so the informativeness is inverted U-shaped).

Case 2 $a \in [1, 2]$

We obtain that if $\tau < 2 - a$, $h(V_y) < 0$, $g(V_y) < 0$, and therefore, $f(V_y) > 0$ for any $V_y \in (0, 4V_\varepsilon)$.

If $\tau \in \left(2 - a, \frac{3}{17}a + \frac{11}{17}\right)$ then $h(V_y)$ changes once the sign, and the same happens with $g(V_y)$ (is negative and then positive). It results that the informativeness is oscillating (decreasing, increasing, decreasing).

If $\tau > \frac{3}{17}a + \frac{11}{17}$ then $h(V_y)$ changes once the sign. In this case $g(V_y)$ might be either all the time positive or changing twice the sign. However the last case is ruled out for this values of parameters and we obtain that $g(V_y) > 0$ and therefore that the informativeness is U-shaped.

Case 3 $a \in \left[2, \frac{86}{31}\right]$

In this case $\tau > 2 - a$ so $h(V_y)$ changes once the sign.

If $\tau < \frac{3}{17}a + \frac{11}{17}$ then $g(V_y)$ changes once the sign (is negative and then positive). It results that the informativeness is oscillating (decreasing, increasing, decreasing).

If $\tau > \frac{3}{17}a + \frac{11}{17}$ then as above, $g(V_y)$ might be either all the time positive or changing twice the sign. However the last case is ruled out for this values of parameters and we obtain that $g(V_y) > 0$ and therefore that the informativeness is U-shaped.

Case 4 $a \in \left[\frac{86}{31}, 2\frac{V_y + 4V_\varepsilon}{4V_\varepsilon - V_y}\right]$, the behaviour of the derivatives of g is different, but the results are the same as in the previous case.

To conclude,

if $a \in [0, 1]$ and $\tau < \frac{3}{17}a + \frac{11}{17}$, the informativeness of prices is increasing in V_y and inverted U-shaped otherwise;

if $a \in [0, 1]$ and $\tau < 2 - a$, then the informativeness of prices is increasing in V_y, if $\tau \in \left(2 - a, \frac{3}{17}a + \frac{11}{17}\right)$ is oscillating (decreasing, increasing, decreasing) and U-shaped otherwise;

if $a \in \left[\frac{86}{31}, 2\frac{V_y + 4V_\varepsilon}{4V_\varepsilon - V_y}\right]$ and $\tau < \frac{3}{17}a + \frac{11}{17}$ then the informativeness of prices is oscillating (decreasing, increasing, decreasing) and U-shaped otherwise. ∎

Proof of Corollary 4.5. Let us now compute the volatility of prices and characterize

it with respect to c, f, τ, V_y and V_ε.

$$
\begin{aligned}
Var\left(\widetilde{p}\right) &= Var\left(U\frac{(y+\overline{y})}{2} + (US - R + \alpha\gamma f)\, z + Q - US\left(\alpha + \beta\overline{y}\right) - \alpha^2\gamma f + \frac{U}{2\rho}\omega\right) \\
&= Var\left(\frac{U}{2}y + (\alpha + \beta y + \varepsilon)\left(US - R + \alpha\gamma f\right) + \frac{U}{2\rho}\omega\right) \\
&= \left(\frac{U}{2} + \beta\left(US - R + \alpha\gamma f\right)\right)^2 V_y + (US - R + \alpha\gamma f)^2\, V_\varepsilon + \left(\frac{U}{2\rho}\right)^2 V_\omega \\
&= \frac{V_y}{4}\left(U^2\left(1 + 3S\right) + \frac{1}{S}\left(R - \alpha\gamma f\right)\left(\left(R - \alpha\gamma f\right) t - 4US\right)\right) + \left(\frac{U}{2\rho}\right)^2 V_\omega = \\
&= \frac{U^2 V_y\left(V_y + 2V_\varepsilon\right)}{\left(V_y + 4V_\varepsilon\right)} \\
&= \frac{V_y\left(V_y + 2V_\varepsilon\right)}{64 V_\varepsilon^2\left(V_y + 4V_\varepsilon\right)^3}\left(4V_\varepsilon f t\overline{y}\left(4V_\varepsilon - V_y\right) - 8V_\varepsilon\left(V_y + 4V_\varepsilon\right) + \tau\left(4V_\varepsilon + V_y\right)^2\right)^2 .
\end{aligned}
$$

We compute also the variance of prices conditional on the report

$$
\begin{aligned}
Var\left(\widetilde{p}|\, \widetilde{z}\right) &= Var\left(\widetilde{p}\right) - \frac{\left(Cov\left(\widetilde{p}, \widetilde{z}\right)\right)^2}{Var\left(\widetilde{z}\right)} \\
&= \frac{U^2 V_y\left(V_y + 2V_\varepsilon\right)}{\left(V_y + 4V_\varepsilon\right)} - \left(\frac{UV_y}{2}\right)^2 \frac{4}{V_y + 4V_\varepsilon} = \frac{2U^2 V_y V_\varepsilon}{V_y + 4V_\varepsilon} .
\end{aligned}
$$

∎

Proof of Corollary 4.6. Since the demand of insider is linear in y which is a normal variable it results that d is also a normal variable with mean $\mu_d = 0$ and variance $Var(d)$ is

$$
\begin{aligned}
V_d &= Var\left(d\right) = Var\left(\rho\left(\left(1 - \frac{R}{2U}\right)(y - \overline{y}) - \frac{R}{U}\varepsilon\right)\right) = \\
&= \rho^2\left(\left(1 - \frac{R}{2U}\right)^2 V_y + \left(\frac{R}{U}\right)^2 V_\varepsilon\right)
\end{aligned}
$$

Then, since d is $\mathcal{N}\left(\mu_d, V_d\right)$ it results that the expected volume of trade

$$
\begin{aligned}
E\left(|d|\right) &= \int_{-\infty}^{\infty} |d|\frac{1}{V_d\sqrt{2\pi}}\exp\left(-\frac{d^2}{2V_d}\right) d\left(d\right) = \mu_d + \sqrt{\frac{2}{\pi}}Var(d) = \\
&\quad \left(\frac{2}{\pi}\right)^{1/2}\rho^2\left(\left(1 - \frac{R}{2U}\right)^2 V_y + \left(\frac{R}{U}\right)^2 V_\varepsilon\right) .
\end{aligned}
$$

Let us compute now the unconditional expected profit of the insider.

$$\Pi = E\left(\left(\widetilde{V} - \widetilde{p}\right)\widetilde{d}\right).$$

Since we have a zero-sum game, the profit of the insider is equal to the loss made by the noise traders

$$E\left((V-p)\,d\right) = -E\left((V-p)\,\omega\right) = -E\left((V-\mu-\nu\,(d+\omega)-\eta z)\,\omega\right) = \nu V_\omega = \frac{U}{2\rho}V_\omega.$$

∎

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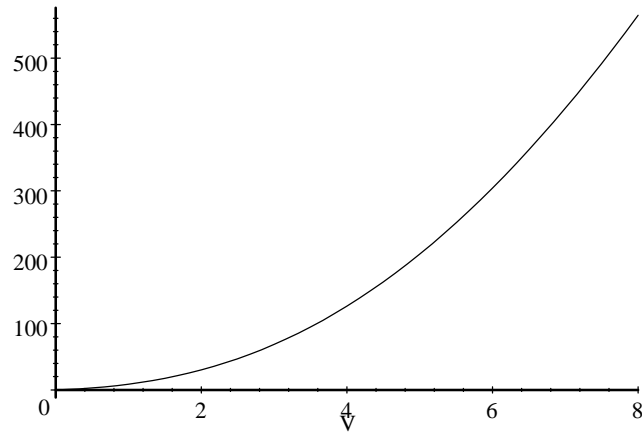[22] Yitzhaki,S., 1974, A Note on Income Tax Evasion: A Theoretical Analysis, Journal of Public Economics, 3, 201-202.

Figure 4.1: Informativeness of prices with respect to V_y. The values of the parameters are $V_\varepsilon = 2, \tau = 0.3, a = 1$.
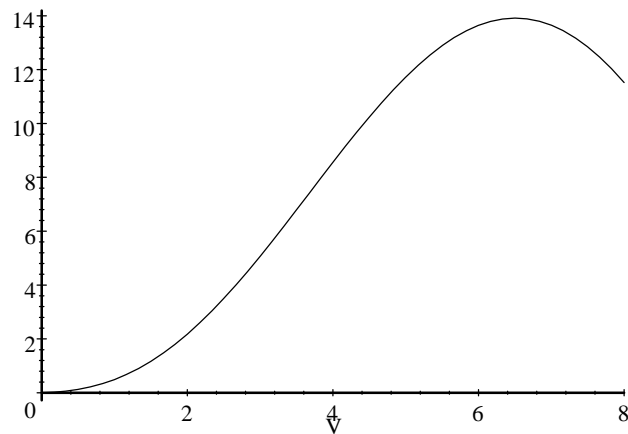

Figure 4.2: The informativeness of prices with respect to V_y. The values of parameters are $V_\varepsilon = 2, \tau = 0.9, a = 1$.
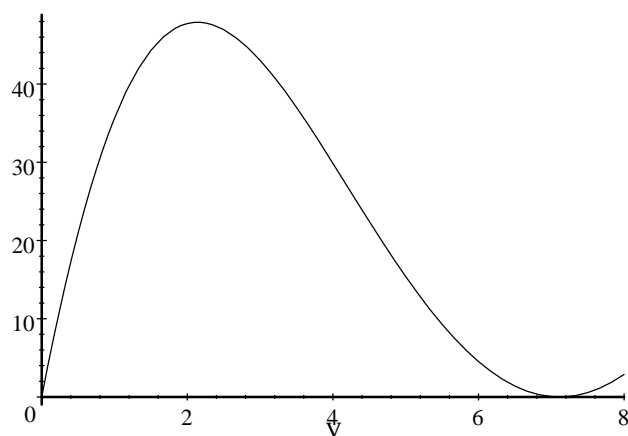
Figure 4.3: The informativeness of prices with respect to V_y. The values of parameters are $V_\varepsilon = 2, \tau = 0.95, a = 2$.
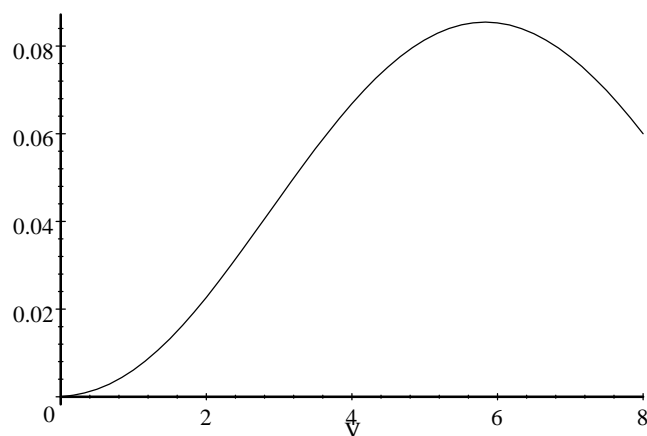

Figure 4.4: The volatility of prices with respect to V_y. The values of parameters are $V_\varepsilon = 2, \tau = 0.9, a = 1$.
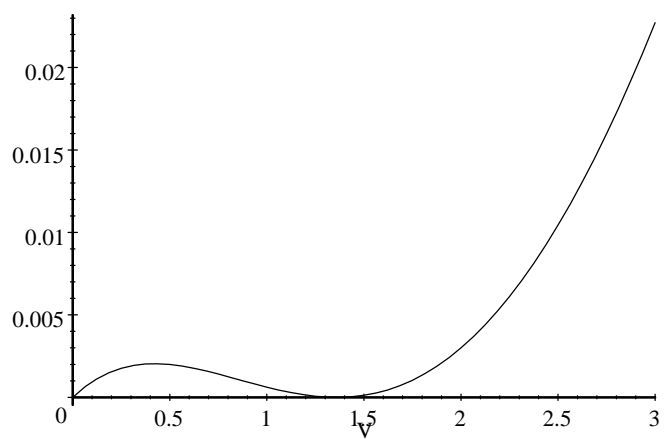
Figure 4.5: The volatility of prices with respect to V_y. The values of parameters are $V_\varepsilon = 2, \tau = 0.8, a = 1.5$.