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SOLVING A CLASS OF GENERALIZED LYAPUNOV OPERATOR DIFFERENTIAL EQUATIONS WITHOUT THE EXPONENTIAL OPERATOR FUNCTION

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Abstract

In this paper a method for solving operator differential equations of the type $X' = A + BX + XD$; $X(0) = C_0$, avoiding the operator exponential function is given. Results are applied to solve initial value problems related to Riccati type operator differential equations whose associated algebraic equation is solvable.

1. Introduction

It is well-known that the solution of the matrix differential equation

$$X^{(1)}(t) = A + BX(t) + X(t)D; \quad X(0) = C_0 \tag{1.1}$$

where A, B, C_0, D and $X(t)$ are non complex matrices, and D^* denotes the adjoint matrix of D, is given by the expression

$$X(t) = \exp(tB)C_0\exp(tD^*) + \int_0^t \exp(B(t-s))A\exp(D^*(t-s))ds \tag{1.2}$$

see [1, p. 28] for details. It is easy to show that the expression (1.2) defines the solution of problem (1.1) when A, B, C_0 and D are bounded linear operators defined on a Hilbert space H. Although the exponential matrix function has been widely studied ([**13**], [**17**], [**18**]), its computation presents some inconvenients ([**13**]) so, thinking of applications, an expression of the solution of (1.1) avoiding the use of the exponential matrix function is interesting.

The aim of this paper is to present an alternative method for solving (1.1) avoiding the exponential matrix function and the computation of integrals involving exponentials of matrices. Let us denote by $L(H)$ the algebra of all bounded linear operators defined on the Hilbert space H, and for T in $L(H)$ let us denote its spectrum by $\sigma(T)$. We recall that an operator T in $L(H)$ is said to be *algebraic* if there exists a polynomial $p(z)$ such that $p(T) = 0$. It is clear that a finite-dimensional operator is algebraic and from [**4**, p. 569], an algebraic operator in $L(H)$ has a finite spectrum, but there are operators with a finite spectrum that are not algebraic operators in $L(H)$, [**14**]. In [**5**], P.R. Halmos observed that an operator in $L(H)$ that is annihilated by an entire analytic function, is algebraic. An account of the properties of algebraic operators may be found in [**7**], [**14**].

Let T be an operator in $L(H)$ and let $z_0 \varepsilon \sigma(T)$ an isolated point of $\sigma(T)$, then z_0 is said to be a pole of T if the resolvent function $R(z,T) = (zI - T)^{-1}$ has a pole at z_0. By the order $w(z_0)$ of a pole z_0 is meant the order of z_0 as a pole of $R(z,T)$.

In this paper we consider the problem (1.1) where A and C_0 are arbitrary operators in $L(H)$, $\sigma(B) \cap \sigma(-D) = \emptyset$ and

(i) D is an algebraic operator in $L(H)$ and its minimal monic polynomial $p(z)$ only has linear factors, $p(z) = (z - z_1)(z - z_2)\dots(z - z_n)$, $z_i \neq z_j$, for $1 \leq i, j \leq n$, $i \neq j$.

(ii) $B \in L(H)$ has a finite spectrum and each $z_{\varepsilon\sigma}(B)$ is a pole of B.

For the finite-dimensional case the condition (ii) is always satisfied, and the condition (i) means that D is similar to a normal operator, [**7**, p. 14]. Section 2 concerns with the resolution problem (1.1) and section 3 provides an explicit solution for a class of generalized Riccati operator differential equations in terms of a solution of certain generalized Lyapunov equation associated to the problem.

2. Solving generalized Lypunov differential operator differential equations without the exponential operator function

We begin this section with an algebraic result that provides a finite algebraic expression of the solution of generalized algebraic Lyapunov operator equation, under certain uniqueness hypothesis. For the finite-dimensional case, an analogous result is given in [**9**].

Lemma 1. *Let A_1, B_1 and D_1 be operators in $L(H)$ such that D_1 is algebraic and satisfies the condition*

$$\sigma(B_1) \cap \sigma(D_1) = \emptyset \tag{2.1}$$

and let $p(z) = \sum_{k=0}^{n} a_k z^k$, such that $p(D_1) = 0$. Then the only solution of the equation

$$A_1 + B_1 X - X D_1 = 0 \tag{2.2}$$

is given by the expression

$$X = -\Big(\sum_{k=0}^{n} a_k B_1^k\Big)^{-1} \Big(\sum_{j=1}^{n} \sum_{k=1}^{j} a_j B_1^{k-1} A_1 D_1^{j-k}\Big) \tag{2.3}$$

Proof: Under the hypothesis (2.1), the equation (2.2) has only one solution, [**16**], [**3**], and from [**3**], corollary 2, if X is the only solution of such equation one gets

$$V = \begin{bmatrix} B_1 & A_1 \\ 0 & D_1 \end{bmatrix} = W \begin{bmatrix} B_1 & 0 \\ 0 & D_1 \end{bmatrix} W^{-1}; \tag{2.4}$$

$$W = \begin{bmatrix} I & X \\ 0 & I \end{bmatrix}; \quad W^{-1} = \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix}$$

From (2.4), it follows that

$$\begin{aligned} p(V) &= Wp(\begin{bmatrix} B_1 & 0 \\ 0 & D_1 \end{bmatrix})W^{-1} = W \begin{bmatrix} p(B_1) & 0 \\ 0 & p(D_1) \end{bmatrix} W^{-1} \\ &= \begin{bmatrix} p(B_1) & -p(B_1)X \\ 0 & 0 \end{bmatrix} \end{aligned} \tag{2.5}$$

Also, considering the powers V^j, for $j = 0, 1, \ldots, n$, it follows that the $(i, 2)$ block entry of the operator V^j, denoted by $V^j_{i,2}$, for $j = 1, 2, \ldots, n$ and $i = 1, 2$, satisfy

$$V^j_{1,2} = B_1 V^{j-1}_{1,2} + A_1 V^{j-1}_{2,2}; \qquad V^j_{2,2} = D_1^j; \tag{2.6}$$

and $V^0_{1,2} = 0$, $V^0_{2,2} = I$.

Considering the polynomial calculus and computing it follows that for certain operator M one has

$$p(V) = p(\begin{bmatrix} B_1 & A_1 \\ 0 & D_1 \end{bmatrix}) = \begin{bmatrix} p(B_1) & M \\ 0 & p(D_1) \end{bmatrix} = \begin{bmatrix} p(B_1) & M \\ 0 & 0 \end{bmatrix} \tag{2.7}$$

From (2.5) and (2.7), one gets $M = -p(B_1)X$, and from the spectral mapping theorem, [**4**, p. 569], and (2.1), the operator $p(B_1)$ is invertible in $L(H)$. Thus, we have

$$X = -(p(B_1))^{-1} M \tag{2.8}$$

By multiplying the operator $V^j_{1,2}$ by the coefficient a_j, for $j = 0, 1, \ldots, n$, and by addition it follows that the block entry (1,2) of the operator matrix $p(V)$, is given by the expression

$$M = \sum_{j=1}^{n} \sum_{k=1}^{j} a_j B_1^{k-1} A_1 D_1^{j-k}. \tag{2.9}$$

From (2.8) and (2.9) one gets (2.3). ∎

For the sake of clarity in the presentation of the next results we recall some concepts and properties concerned with the Riesz-Dunford functional calculus, [4], and the Laplace transform of operator valued functions, [8].

Let z_0 be an isolated point in the spectrum $\sigma(T)$ of an operator $T \varepsilon L(H)$, then the Laurent expansion of $R(z,T) = (zI-T)^{-1}$ in a neighborhood $0 < |z - z_0| < \delta$, of z_0, is given by

$$R(z,T) = \sum_{n=-\infty}^{\infty} A_n (z_0 - z)^n \tag{2.10}$$

$$A_{-(m+1)} = (z_0 I - T)^m E(z_0;T)$$

where $E(z_0;T)$ denotes the spectral projection corresponding to the spectral set $\{z_0\}$ see [4, p. 573], for details. If z_0 an isolated point in $\sigma(T)$, it follows that z_0 is a pole of order p, if and only if,

$$(z_0 I - T)^p E(z_0;T) = 0 \text{ and } (z_0 I - T)^{p-1} E(z_0;T) \neq 0 \tag{2.11}$$

We recall that a $L(H)$ valued operator function $t \to V(t)$, is said to be an original function if, $V(t) = 0$, for $t < 0$, V is locally integrable and there exist a real number s_0 and a positive number M, such that $\|V(t)\| \leq M \exp(s_0 t)$, for $t \geq 0$. Under these hypotheses the Laplace transform of V, represented by $\hat{V}$, is defined in the usual way, see [8] for details and related properties. In particular, if $V^{(1)}$ is an original function, it follows that $\hat{V}^{(1)}(s) = s\hat{V}(s) - V(0)$. Finally, if $z \to f(z)$, is a $L(H)$ valued meromorphic function and z_0 is a pole of f, we represent by $\operatorname{Res}(f;z_0)$ the residue of f in the pole z_0.

Let us consider the problem (1.1) where A and C_0 are arbitrary operators in $L(H)$ and B, D are operators in $L(H)$ satisfying the properties (i) and (ii) given in page 1. Let X be the function defined by the expression (1.2) for $t \geq 0$, and $X(t) = 0$, for $t < 0$. From (1.2) it follows that $X(t)$ and $X^{(1)}(t)$ are original functions, where $X^{(1)}(0)$ means the right lateral derivative of X at $t = 0$. Let $\hat{X}(s)$ be Laplace transform of X. Taking into account the properties of the Laplace transform, as X satisfies the problem (1.1), by application of the Laplace transform to the differential equation arising in (1.1), it follows that there exists a positive number a such that if $Re(s) > a$ one gets

$$s\hat{X}(s) - C_0 = A/s + B\hat{X}(s) + \hat{X}(s)D \tag{2.12}$$

$$(sI - B)\hat{X}(s) - \hat{X}(s)D = C_0 + A/s.$$

Let $p(z) = \sum_{k=0}^{n} a_k z^k = (z-z_1)(z-z_2)\dots(z-z_n)$, $z_i \neq z_j$, if $i \neq j$, $1 \leq i, j \leq n$, the minimal monic polynomial of D, where $\sigma(D) = \{z_1, \dots, z_n\}$. It is clear that for values of s enough advanced in module, one has $\sigma(sI - B) \cap \sigma(D) = \emptyset$. So, from lemma 1, it follows that for values of s enough advanced in module,

$\hat{X}(s)$ is given by the expression

(2.13)

$$\begin{aligned}\hat{X}(s) &= -(p(sI-B))^{-1}(\sum_{k=1}^{n}\sum_{j=1}^{k} a_j(sI-B)^{j-1}(C_0+A/s)D^{k-j}) \\ &= -(p(sI-B)^{-1}(\sum_{k=1}^{n}\sum_{j=1}^{k} a_j(sI-B)^{j-1}C_0D^{k-j}) \\ &- (sp(sI-B))^{-1}(\sum_{k=1}^{n}\sum_{j=1}^{k} a_j(sI-B)^{j-1}AD^{k-j})\end{aligned}$$

From the spectral mapping theorem, [4], it follows that $p(sI-B)$ is invertible in $L(H)$ for values of s enough advanced in module, and

$$(p(sI-B))^{-1} = \Pi_{j=1}^{n}(sI-B-z_j)^{-1} = \Pi_{j=1}^{n}(sI-R_j)^{-1} \tag{2.14}$$

where $R_j = B + z_j$, for $j = 1,2,\ldots,n$, $z_j \varepsilon \sigma(D)$.

Let $q_1(s)$ and $q_2(s)$ be the holomorphic $L(H)$ valued operator functions defined by the expressions

$$\begin{aligned} q_1(s) &= \sum_{k=1}^{n}\sum_{j=1}^{k} a_j(sI-B)^{j-1}C_0D^{k-j}\,; \\ q_2(s) &= \sum_{k=1}^{n}\sum_{j=1}^{k} a_j(sI-B)^{j-1}AD^{k-j}. \end{aligned} \tag{2.15}$$

From (2.13), (2.14) and (2.15), it follows that

$$\hat{X}(s) = \hat{X}_1(s) + \hat{X}_2(s) \tag{2.16}$$

where

$$\begin{aligned} \hat{X}_1(s) &= -(\Pi_{i=1}^{n}(sI-R_i)^{-1})q_1(s)\,; \\ \hat{X}_2(s) &= -(\Pi_{i=1}^{n}(sI-R_i)^{-1})q_2(s)/s. \end{aligned} \tag{2.17}$$

Let us suppose that $\sigma(B) = \{b_1,\ldots,b_m\}$, then taking into account (2.17), the set of poles of $(sI-R_i)^{-1}$ is the set of points $s_{ij} = b_i + z_j$, where $1 \le j \le m$, and as $q_1(s)$ and $q_2(s)$ are holomorphic functions, by aplication of the Laplace inversion formula, for $t > 0$ we have

(2.18)

$$\begin{aligned} X(t) = \sum_{i=1}^{n}\sum_{j=1}^{m}&(\mathrm{Res}(\hat{X}_1(s)\exp(st);s_{ij}) + \mathrm{Res}(\hat{X}_2(s)\exp(st);s_{ij}) \\ &+ \mathrm{Res}(\hat{X}_2(s)\exp(st);0) \quad \text{if } s_{ij} \neq 0\,,\ 1 \le i \le n\,,\ 1 \le j \le m \end{aligned}$$

and if there exist some $s_{ij}=0$, then

$$(2.19)\qquad X(t)=\sum_{i=1}^{n}\sum_{j=1}^{m}(\mathrm{Res}(\hat{X}_1(s)\exp(st);s_{ij})+\mathrm{Res}(\hat{X}_2(s)\exp(st);s_{ij})$$

In order to compute the residues of $\hat{X}_i(s)\exp(st)$, for $i=1,2$, we need the order of each singularity s_{ij} and 0 for such functions. Note that the spectral projection $E(s_{ij};R_j)=E(b_i+z_j;B+z_j)=E(b_i;B)$, and the order of s_{ij} as a singular point of $(zI-R_j)^{-1}$ coincides with the order of b_i as a singular point of $(zI-B)^{-1}$. Also, considering the decomposition

(2.20)

$$\hat{X}_1(s)\exp(st)=(sI-R_j)^{-1}(\Pi_{\substack{i=1\\ i\neq j}}^{n}(sI-R_i)^{-1})q_1(s)\exp(st)=(sI-R_j)^{-1}Q_j(s).$$

The Taylor expansion of $Q_j(s)$ at the point s_{ij} takes the form

$$(2.21)\qquad Q_j(s)=\sum_{n\geq 0}Q_j^{(n)}(s_{ij})(s-s_{ij})^n/n!$$

and the Laurent expansion of $(sI-R_j)^{-1}$ at the point s_{ij} is given by the expression

$$(2.22)\qquad R(s,R_j)=(sI-R_j)^{-1}=\sum_{n=-w_i}^{\infty}A_n(s_{ij}-s)^n;$$

$$A_{-(m+1)}=-(s_{ij}I-R_j)^mE(s_{ij};R_j)$$

or $A_{-(m+1)}=-(b_i-B)^mE(b_i;B)$, where w_i is the order of b_i as a pole of B. From (2.20)–(2.22) it follows that

(2.23)

$$\mathrm{Res}(\hat{X}_1(s)\exp(st);s_{ij})=(s_{ij}I-R_j)^{w_i}E(b_i;B)Q^{(w_i-1)}(s_{ij})/(w_i-1)!+\dots$$
$$+(s_{ij}I-R_j)E(b_i;R_j)Q_j(s_{ij})=E(b_i;B)((b_iI-B)^{w_i}Q_j^{(w_i-1)}(s_{ij})/(w_i-1)!+\dots$$
$$+(b_iI-B)Q_j(s_{ij})).$$

Let us denote by $Q_{ij}(t)$ the expression

$$(2.24)\quad (b_iI-B)^{w_i}Q_j^{(w_i-1)}(s_{ij})/(w_i-1)!+\dots+(b_iI-B)Q_j(s_{ij})=Q_{ij}(t),$$

then from (2.20) and (2.24), if $t>0$ one gets

$$(2.25)\qquad \sum_{i=1}^{n}\sum_{j=1}^{m}\mathrm{Res}(\hat{X}_1(s)\exp(st);s_{ij})=\sum_{i=1}^{n}\sum_{j=1}^{m}E(b_i;B)Q_{ij}(t).$$

Let $T_j(s) = (\Pi_{\substack{i=1 \\ i\neq j}}^{n}(sI - R_i)^{-1}s^{-1}q_2(s)\exp(st)$, for $j = 1,2,\ldots,n$. Then it follows that $\hat{X}_2(s)\exp(st) = (sI - R_j)^{-1}T_j(s)$. Under the hypothesis $s_{ij} = b_i + z_j \neq 0$, $1 \leq i \leq n$; $1 \leq j \leq m$, or equivalently $\sigma(B) \cap \sigma(-D) = \emptyset$, the set of singularities of the meromorphic opetator function $(sI - R_j)^{-1}T_j(s)$ is $\{0\} \cup \{s_{ij}; 1 \leq i \leq n, 1 \leq j \leq m\}$. Considering the Laurent expansion of $(sI - R_j)^{-1}T_j(s)$ at the point s_{ij}, and taking into account (2.22), it follows that

$$\begin{aligned} &\mathrm{Res}\hat{X}_2(s)\exp(st);\, s_{ij}) = E(b_i;B)((b_iI - B)^{w_i}T_j^{(w_i-1)}(s_{ij})/(w_i-1)! + \ldots \\ (2.26)\qquad &\qquad + \cdots + (b_iI - B)T_j(s_{ij})). \end{aligned}$$

Let us denote by $S_{ij}(t)$ the expression

$$(2.27)\quad (b_iI - B)^{w_i}T_j^{(w_i-1)}(s_{ij})/(w_i-1)! + \cdots + (b_iI - B)T_j(s_{ij}) = S_{ij}(t)$$

then from (2.26) and (2.27) it follows that

$$(2.28)\qquad \sum_{i=1}^{n}\sum_{j=1}^{m}\mathrm{Res}(\hat{X}_2(s)\exp(st);\, s_{ij}) = \sum_{i=1}^{n}\sum_{j=1}^{m}E(b_i;B)S_{ij}(t).$$

In order to compute the residue of $\hat{X}(s)\exp(st)$ at $s = 0$, note that $\mathrm{Res}(X_1(s)\exp(st);\, 0)$ is the operator 0 because $\hat{X}_1(s)\exp(st)$ is holomorphic at $s = 0$ and

$$(2.29)\qquad \hat{X}_2(s)\exp(st) = -s^{-1}(p(sI - B))^{-1}q_2(s)\exp(st).$$

Under the hypothesis $\sigma(B) \cap \sigma(-D) = \emptyset$, the operator $p(-B)$ is invertible, thus the factor $(p(sI - B))^{-1}q_2(s)$ is holomorphic at $s = 0$, and from (2.29) it follows that

$$\begin{aligned} (2.30)\qquad \mathrm{Res}(\hat{X}(s)\exp(st);\, 0) &= -(p(-B))^{-1}q_2(0) \\ &= -(p(-B))^{-1}\sum_{k=1}^{n}\sum_{j=1}^{k}a_j(-B)^{j-1}AD^{k-j} \end{aligned}$$

and from lemma 1, it follows that $\mathrm{Res}\hat{X}(s)\exp(st);0) = X_*, X_*$ being the only solution of the algebraic equation $A + BX + XD = 0$.

Summarizing we have that under the hypothesis $\sigma(B) \cap \sigma(-D) = \emptyset$, the solution $X(t)$ of problem (1.1), for $t > 0$ is given by the expression

$$(2.31)\qquad X(t) = X_* + \sum_{i=1}^{n}\sum_{j=1}^{m}E(b_i;B)(Q_{ij}(t) + S_{ij}(t))$$

where X_* is the only solution of the algebraic equation $A + BX + XD = 0$, given by (2.30), and $Q_{ij}(t)$ and $S_{ij}(t)$ are given by (2.24) and (2.27) respectively. Thus the following result has been proved:

Theorem 1. *Let us consider the problem (1.1) where A and C_0 are operators in $L(H)$ and the operators B and D satisfy the following properties*

(i) $\sigma(B) = \{b_i;\ 1 \le i \le n\}$, $\sigma(D) = \{z_j;\ 1 \le j \le m\}$, and $\sigma(B) \cap \sigma(-D) = \emptyset$.

(ii) Each $b_i \epsilon \sigma(B)$ is a pole of B.

(iii) D is algebraic and its minimal monic polynomial $p(z) = \sum_{k=0}^{n} a_k z^k$, only has linear factors, $p(z) = \Pi_{i=1}^{n}(z - z_i)$; $z_i \ne z_j$, if $i \ne j$.

Then the only solution of problem (1.1) is given by the expression (2.31), where X_* is given by (2.30), $E(b_i; B)$ are the spectral projections of B, and $Q_{ij}(t)$ and $S_{ij}(t)$ are given by (2.24) and (2.27) respectively.

Proof: The result is a consequence of the above comments. In fact for $t > 0$, the expression of the solution coincides with (2.31). On the other hand, the solution of problem (1.1), given by (1.2) is an analytic function of the variable t, and coincides with the expression appearing in the right hand side of (2.31), that is also analytic, in consequence they coincide on all the real line.

Under the hypothesis of theorem 1, note that with the exception of $S_{ij}(t)$ and $Q_{ij}(t)$, all coefficients $E(b_i; B)$ and X_* given by (2.30) do not involve the variable t, thus, in order to study the behaviour of the solution when $t \to \infty$, we have to consider the functions $Q_{ij}(t)$ and $S_{ij}(t)$. Note that $S_{ij}(t)$ and $Q_{ij}(t)$ are defined by (2.27) and (2.24) in terms of the derivatives (with respect to s) of the functions

$$Q_j(s) = \big(\Pi_{\substack{i=1 \\ i \ne j}}^{n}(sI - R_i)^{-1}\big) q_1(s) \exp(st)$$

$$T_j(s) = \big(\Pi_{\substack{i=1 \\ i \ne j}}^{n}(sI - R_i)^{-1}\big) s^{-1} q_2(s) \exp(st),$$

where $q_1(s)$ and $q_2(s)$ are given by (2.15). ∎

Corollary 1. *Let us consider the problem (1.1) under the hypotheses of theorem 1. If $s_{ij} = b_i + z_j$, $1 \le i \le n$, $1 \le j \le m$, and all s_{ij} are contained in the half plane $Re(z) < 0$, then all solution of the differential equation arising in (1.1) converges to X_* when $t \to +\infty$.*

Proof: The result is a consequence of the expression (2.31), (2.27) and (2.24) and theorem 1. ∎

3. An Application to Riccati Operator Differential Equations

The resolution of a Cauchy problem for Riccati operator differenctial equations of the type

$$d/dtX(t) = A + FX(t) + X(t)G + X(t)EX(t);\ X(0) = C_0 \tag{3.1}$$

where A, F, G, E and C_0 are operators in $L(H)$, is important in control theory, [**12**], transport theory, [**15**], and filtering problems, [**2**]. In a recent paper [**10**], an explicit expression of the solution of problem (3.1) is given in terms of the block entries of the operator function

$$S(t)\exp(\begin{bmatrix} -G & -E \\ A & F \end{bmatrix} t) \tag{3.2}$$

but an explicit expression of such entries in terms of data is not known. The aim of this section is to obtain an explicit expression of a class of problems of the type (3.1) in terms of a solution of the corresponding algebraic Riccati operator equation

$$A + FX + XG + XEX = 0 \tag{3.3}$$

and the solution of certain associated generalized Lyapunov operator differential equation.

A resolution method for solving non-symmetric algebraic Riccati operator equation is given in [**6**].

Let us consider the problem (3.1) and let us suppose that there exists a solution X_* of the algebraic equation (3.3) such that $C_0 - X_*$ is invertible in $L(H)$. From [**11**], the problem (3.1) is locally solvable, so, there exists a solution $X(t)$ defined in a neighborhood J of the origin $t = 0$. As $X(0) = C_0$ satisfies $C_0 - X_*$ invertible in $L(H)$, from continuity, it follows that $X(t) - X_*$ is invertible in $L(H)$ when t belongs to some neighborhood of $t = 0$, let us denote this neighborhood by J.

Let F_* and G_* be the operators in $L(H)$ defined by the expressions

$$F_* = F + X_*E, \quad G_* = G + EX_* \tag{3.4}$$

and let $Y(t) = (X(t) - X_*)^{-1}$, $t \epsilon J$. Then $(Y(t))^{-1} = X(t) - X_*$, and by differentiation it follows that $d/dt((Y(t))^{-1}) = d/dt(X(t) - X_*) = X(t)EX(t) + FX(t) + X(t)G + A - (X_*EX_* + FX_* + X_*G + A)$ and from (3.4) one gets

(3.5)

$$d/dt((Y(t))^{-1}) = (X(t) - X_*)E(X(t) - X_*) + (F + X_*E)(X(t) - X_*)$$
$$+ (X(t) - X_*)(G + EX_*) = (Y(t))^{-1}E(Y(t))^{-1} + F_*(Y(t))^{-1}) + (Y(t))^{1}G_*.$$

Thus, $U(t) = (Y(t))^{-1}$ satisfies

$$d/dt\, U(t) = U(t)EU(t) + F_*U(t) + U(t)G_*. \tag{3.6}$$

Premultiplying and postmultiplying by $Y(t)$ both members of equation (3.6), and taking into account that

$$-d/dt\, Y(t) = Y(t)(d/dt(Y(t))^{-1}))Y(t)$$

it follows that

$$d/dt\, Y(t) = -E - Y(t)F_* - G_*Y(t);\ Y(0) = C_0 - X_*)^{-1}. \tag{3.7}$$

So, the solution $X(t)$ of problem (3.1) is given by $X(t) = X_* + (Y(t))^{-1}$, where $Y(t)$ is the solution of (3.7). From the above comments and theorem 1, the following theorem has been established.

Theorem 2. *Let us suppose that there exists a solution X_* of equation (3.3) such that the operators F_* and G_* given by (3.4) satisfy the properties*

(i) $\sigma(-F_*) \cap (G_*) = \emptyset$; $\sigma(-G_*) = \{b_i; 1 \leq i \leq n\}$;$\sigma(-F_*) = \{z_j; 1 \leq j \leq m\}$

(ii) Each $b_i \epsilon \sigma(-G_*)$ is a pole of $-G_*$

(iii) $-F_*$ is algebraic and its minimal monic polynomial $p(z)$ only has linear factors with $p(z) = \sum_{k=0}^{n} a_k z^k = \Pi_{i=1}^{n}(z - z_i)$, with $z_i \neq z_j$, if $i \neq j$.

Let $s_{ij} = b_i + z_j$, for $1 \leq i \leq n$, $1 \leq j \leq m$, and let w_i be the order of b_i as a pole of $-G_*$, and let $E(b_{ij} - G_*)$ be the spectral projection associated to b_i as a pole of $-G_*$, then if $C_0 - X_*$ is invertible in $L(H)$, in a neighborhood of $t = 0$, the solution of problem (3.1) is given by the expression

$$X(t) = X_* + (Y_* + \sum_{i=1}^{n} \sum_{j=1}^{m} E(b_i; -G_*)(Q_{ij}(t) + S_{ij}(t)))^{-1}$$

where $Y_* = -(p(G_*))^{-1}(\sum_{k=1}^{n} \sum_{j=1}^{k} a_j (G_*)^{j-1} E(-F_*)^{k-j})$, and $Q_{ij}(t)$, $S_{ij}(t)$ are given by the expressions analogous to (2.24) and (2.27) respectively, by replacing the operators B, D and A, by $-G_*$, $-F_*$ and $-E$, respectively.

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