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# AN ELEMENTARY PROOF OF A LIMA'S THEOREM FOR SURFACES

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*Abstract*

An elementary proof of the following theorem is given:

THEOREM. *Let $M$ be a compact connected surface without boundary. Consider a $C^\infty$ action of $\mathbb{R}^n$ on $M$. Then, if the Euler–Poincaré characteristic of $M$ is not zero there exits a fixed point.*

The proof given here adapts for dimension two the ideas used by P. Molino and the author in [2] and [3]. Moreover we show that the theorem remains true if $\mathbb{R}^n$ is replaced by a connected nilpotent Lie group $G$.

In the slightly more general case, dealt with by E.L. Lima, of a surface with boundary, it is sufficient gluing together two copies of this surface in order to obtain a surface without boundary.

## 1. Actions of $\mathbb{R}^n$

Let $V$ be the Lie algebra of $\mathbb{R}^n$. The action of $\mathbb{R}^n$ induces a Lie algebra homomorphism $v \in V \to X_v \in \mathcal{X}(M)$ called infinitesimal action. We recall that the infinitesimal isotropy of a point $p$ is the set $I(p) = \{v \in V / X_v(p) = 0\}$. As $V$ is abelian $I(p)$ depends only on the orbit.

Denote by $\Sigma_k$ the set of points $p$ of $M$ whose orbit is $k$–dimensional, i.e. $\operatorname{codim} I(p) = k$.

Suppose $\Sigma_0$ empty. We will gradually arrive to a contradiction.

1) Set $C_2 = \{v \in V / X_v(p) = 0 \text{ for some } p \in \Sigma_2\}$.

As there are at most countably many 2–orbits because they are open sets, $C_2$ is at most countable union of $(n-2)$–planes of $V$.

2) The map on the grassmannian of $(n-1)$–planes $h : p \in \Sigma_1 \to I(p) \in g_{n-1}(V)$ is differentiable, i.e. it can be locally extended to a differentiable map.

Indeed, consider $p \in \Sigma_1$ and $u \in V$ such that $X_u(p) \neq 0$. We can find a coordinate system $(A, x)$, $p \in A$, such that $X_u = \frac{\partial}{\partial x_1}$ and that the image of $A$ on $\mathbb{R}^2$ is a rectangle.

Let $\{v_1,\dots v_{n-1}\}$ a basis of $I(p)$. Set $X_{v_j}=f_j\frac{\partial}{\partial x_1}+g_j\frac{\partial}{\partial x_2}$. We define the map

$$\begin{aligned}\tilde{h}:A &\longrightarrow g_{n-1}(V)\\ x &\longrightarrow \mathbb{R}\{v_1-f_1u,\dots v_{n-1}-f_{n-1}u\}\end{aligned}$$

whose differentiability is clear.

Note that $w\in\tilde{h}(x)$ if and only if $X_w(x)$ is proportional to $\frac{\partial}{\partial x_2}$. If $x\in A\cap\Sigma_1$ this means that $X_w(x)=0$ because it is also proportional to $\frac{\partial}{\partial x_1}$. Then $\tilde{h}$ is a local extension of $h$.

3) Let $Fr(\Sigma_1)$ be the boundary on $M$ of $\Sigma_1$. Then $C_1=\{v\in V/X_v(p)=0$ for some $p\in Fr(\Sigma_1)\}=\bigcup_{p\in Fr(\Sigma_1)} I(p)$ is of the first category (i.e. it is contained in the union of a countable family of closed nowhere dense subsets of $M$).

Since $Fr(\Sigma_1)$ can be covered by a finite family of coordinate systems $(A,x)$ as in 2), it will be sufficient to prove that $\bigcup_{p\in A\cap Fr(\Sigma_1)} I(p)$ is of the first category. Let $T$ be a slice of $A$ obtained by doing $x_1$ constant. As the isotropy is constant on the orbits:

$$\bigcup_{p\in A\cap Fr(\Sigma_1)} I(p)=\bigcup_{p\in T\cap Fr(\Sigma_1)} I(p)$$

Consider the vector bundle $\pi:E\to T$, subbundle of $T\times V$, given by the condition $\pi^{-1}(x)=\{x\}\times\tilde{h}(x)$. Set $\varphi:(x,v)\in E\to v\in V$.

The set $\pi^{-1}(T\cap Fr(\Sigma_1))$ is of the first category in $E$ because $T\cap Fr(\Sigma_1)$ is of the first category in $T$. As $\varphi$ is differentiable and $E$ and $V$ are manifolds of the same dimension, it follows that

$$\varphi(\pi^{-1}(T\cap Fr(\Sigma_1)))=\bigcup_{p\in T\cap Fr(\Sigma_1)} I(p)$$

is of the first category in $V$.

4) Take now $v\in(V-C_1\cup C_2)$. The set $Z(X_v)$ of the zeros of $X_v$ is contained in $\overset{0}{\Sigma}_1$. On the other hand the 1–foliations given by:

(a) $X_v$ on $M-Z(X_v)$

(b) the action of $\mathbb{R}^n$ on $\overset{0}{\Sigma}_1$.

agree on $(M-Z(X_v))\cap\overset{0}{\Sigma}_1$. Then $M$ admits an 1–foliation and $\mathcal{X}(M)=0$, contradiction.

## 2. Case of a connected nilpotent Lie group G

It will be sufficient to adapt the proof of the abelian case. Let $V$ be the Lie algebra of $G$. Since $V$ is nilpotent every subalgebra of codimension one is an ideal. Therefore the isotropy is constant over each 1-orbit and $C_1$ will still be of the first category.

Let $B$ be a 2-orbit. Given $p \in B$ there always exists an ideal I of codimension one which contains $I(p)$. As $B$ is an orbit and $I$ an ideal then $I(q) \subset I$ for all $q \in B$. Consequently $C_2$ is contained in a finite or countable union of $(n-1)$-planes of $V$. In particular $C_1 \cup C_2 \neq V$. The rest is similar.

**Example 1.** See $P(2,\mathbb{R})$ as the plane $\mathbb{R}^2$ plus the infinite points. The vector fields on $\mathbb{R}^2$: $\frac{\partial}{\partial x_1}$, $\frac{\partial}{\partial x_2}$ and $x_1 \frac{\partial}{\partial x_2}$ can be extended, in a natural way, to $P(2,\mathbb{R})$ because they are affine. These vector fields generate an action of a 3-dimensional nilpotent group on $P(2,\mathbb{R})$, whose orbits are $\mathbb{R}^2$; the set of all points of infinity except the vertical one (i.e. the point associated to the vertical direction); and the infinite vertical point, which is the only fixed point.

**Example 2.** Tare now $\frac{\partial}{\partial x_1}$, $\frac{\partial}{\partial x_2}$ and $-x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2}$. One obtains an action of a 3-dimensional solvable group with no fixed point. Their orbits are $\mathbb{R}^2$ and the set of the infinite points.

See [1] for a 2-dimensional example with no fixed point.

## References

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