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QUALITATIVE PROPERTIES OF THE FREE–BOUNDARY OF THE REYNOLDS EQUATION IN LUBRICATION

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Abstract

The hidrodynamic lubrication of a cylindrical bearing is governed by the Reynolds equation that must be satisfied by the preassure of lubricating oil. When cavitation occurs we are carried to an elliptic free-boundary problem where the free-boundary separates the lubricated region from the cavited region.

Some qualitative properties are obtained about the shape of the free-boundary as well as the localization of the cavited region.

1. Introduction. Existence and uniqueness

Let Ω be the rectangle $(0,2\pi)\times(0,1)\subset \mathbf{R}^2$; let $\Gamma_0=(0,2\pi)\times\{0\}$, $\Gamma_1=(0,2\Pi)\times\{1\}$ and let us introduce the following sets of functions:

$$V=\{\phi\in H^1(\Omega),\phi\,|_{\Gamma_0\cup\Gamma_1}=0,\ \phi \text{ is } 2\pi x-\text{periodic}\}$$
$$V_a=\{\phi\in H^1(\Omega),\phi\,|_{\Gamma_0}=0,\phi\,|_{\Gamma_1}=p_a,\ \phi \text{ is } 2\pi x-\text{periodic}\}$$

where $H^1(\Omega)$ is the Sobolev space of functions such that they and their first derivatives are square summable.

We consider the following:

Problem (P).

Find a pair of functions (p,γ) such that

(1.1) $(p,\gamma)\in V_a\times L^\infty(\Omega)$

(1.2) $p\geq 0$ and $H(p)\leq\gamma\leq 1$ a.e. in Ω

(1.3) $\int_\Omega h^3\nabla p\nabla\xi=\int_\Omega h\gamma\frac{\partial\xi}{\partial x}\qquad\forall\xi\in V,$

where $h=h(x)=1+\alpha\cos x$, with $0<\alpha<1$, and H is the Heaviside function.

This problem is related to the lubrication with cavitation arising in bearings. The first unknow is the pressure distribution $-p-$ in a thin film of lubricant contained in the narrow gap between two circular cylinders of parallel axes (the shaft and the bearing); another unknow is the percentage $-\gamma-$ of oil contained in an elementary volume.

shaft and the bearing); another unknow is the percentage $-\gamma-$ of oil contained in an elementary volume.

Introducing cylindrical coordinates, the gap h depends only on the angular coordinate, being α the eccentricity ratio of the bearing.

The equation (1.3) derives from the Reynolds equation, $\operatorname{div}(h^3 \nabla p) = h'$, which must be satisfied for p on the region $[p > 0]$, and from conservation laws of flow across the free boundary separing the regions $[p > 0]$ and $[p = 0]$ in Ω. In the lubricated region (completely occuped for oil) γ is equal to one, while over the cavited region ($[p = 0]$) γ must satisfy $0 \leq \gamma \leq 1$.

The main goal of this paper is to give some qualitative properties of the free–boundary,

$$\Gamma = \overline{[p > 0]} \cap \overline{[p = 0]} \cap \Omega.$$

The existence of solutions for Problem (P) was proved by Bayada and Chambat in **[B-Ch]**; they prove also uniqueness of solutions under the assumption that the free–boundary is a Lipschitz–continuous function of x. A comparison result and uniqueness was proved by Carrillo and the author in **[A-C]**, without any of the previous assumption related to the free–boundary.

For a more general treatment on physical aspects and the formulation of Problem (P), see **[A]**, **[B-Ch]**, **[D-T]**, **[F]**.

About existence and uniqueness, we recall the following results:

Theorem 1.1. *(Existence and Regularity)*

There exist at least one solution for Problem (P); moreover, if (p, γ) satisfies (1.1), (1.2) and (1.3), then

$$p \in C^0(\bar{\Omega}) \cap C^{0,r}(\Omega \cup (\{0\} \times (0,1)) \cup (\{2\pi\} \times (0,1))).$$

Proof: See **[B-Ch]** and **[A-C]**, as well as the proof of existence for the Dam Problem in **[B-K-S]**. ■

Theorem 1.2. *(Comparison)* (**[A-C]**))

Let (p_1, γ_1) and (p_2, γ_2) be two pairs in $H^1(\Omega) \times L^\infty(\Omega)$, with p_1 and p_2 being 2Π x–periodic functions and satisfying (1.2) and (1.3), as well as the condition,

(1.4) $p_i \,|_{\Gamma_j} = \phi_i^j$ for $i = 1, 2$ and $j = 0, 1$
where for ϕ_i^j we assume
(1.5) $\phi_i^j \in C(\Gamma_j)$ and $\phi_1^j \leq \phi_2^j$.
Then $p_1 \leq p_2$ in Ω.

Like a corollary of this theorem, we have:

Theorem 1.3. *(Uniqueness)* (**[A-C]**)

There exist an unique solution (p, γ) for Problem (P).

Remark. Theorem 1.2 gives a global comparison result in Ω for p_1 and p_2, when we can compare their values on Γ_0 and Γ_1: this remain true to compare solutions of Problem (P) with solutions of a *swiftly modified problem*, as we will precise later in Section 3.

2. Uniforme bounds for solutions in the x-variable

In this section we shall give an upper bound and a lower bound, both independents of x, for solutions of Problem (P).

Let $M = \underset{x\in[0,2\pi]}{\text{maximum}} \frac{h'(x)}{h^3(x)}$, and, for $0 \le y \le 1$, let us define,

$$\overline{v}(y) = -\frac{M}{2}y^2 + (p_\alpha - \frac{M}{2})y \tag{2.1}$$

$$\underline{v}(y) = \left[\frac{M}{2}y^2 + (p_\alpha - \frac{M}{2})y\right]^+ \tag{2.2}$$

Such functions satisfy:

$$\begin{aligned} &\underline{v}(0) = \overline{v}(0) = 0 \\ &\underline{v}(1) = \overline{v}(1) = p_a \\ &\overline{v}'' = -M \\ &\underline{v}'' = \begin{cases} 0 \text{ if } y < 1 - 2p_a/M \\ M \text{ if } y > 1 - 2p_a/M \end{cases} \end{aligned}$$

We have:

Theorem 2.1.
If (p, γ) is the solution of Problem (P), then

$$p(x, y) \le \overline{v}(y) \qquad \text{in } \ddot{\Omega}.$$

Proof: Taking $\xi = (p - \overline{v})^+$, and as $\gamma = 1$ on the support of ξ, we have

$$\int_\Omega h^3 \nabla p \nabla \xi = \int_\Omega h\gamma\xi_x = -\int_\Omega h'\xi$$

Moreover,

$$\int_\Omega h^3 \nabla\overline{v}\nabla\xi = \int_\Omega h^3\overline{v}'\xi_y = -\int_\Omega h^3\overline{v}''\xi = \int_\Omega h^3 M\xi \ge \int_\Omega h'\xi.$$

and, substracting from the above equality:

$$\int_\Omega h^3 \mid \nabla(p-\overline{v})^+ \mid^2 = \int_\Omega h^3 \nabla(p-\overline{v})\nabla\xi \leq 0.$$

So, we obtain

$$(p-\overline{v})^+ = \text{ constant} \qquad \text{in } \Omega$$

and, hence

$$(p-\overline{v})^+ = 0 \qquad \text{i.e. } \; p \leq \overline{v}.$$

In order to complet the boundedness of p, we have:

Theorem 2.2.

If (p,γ) is the solution of Problem (P), then

$$p(x,y) \geq \underline{v}(y) \qquad \textit{in } \bar{\Omega}.$$

Proof: Let $\xi = (\underline{v}-p)^+$; we have:

$$\int_\Omega h^3 \nabla \underline{v} \nabla \xi = \int_\Omega h^3 \underline{v}' \xi_y = -\int_\Omega h^3 \underline{v}'' \xi = -\int_\Omega h^3 M \xi \leq -\int_\Omega h' \xi,$$

since $\underline{v}''(y) = M$ if $\underline{v} \neq 0$, and hence on the support of ξ.

Now, since $\xi_x = [(\underline{v}-p)^+]_x = 0$, on the region $[p=0]$, we have:

$$\int_\Omega h^3 \nabla p \nabla \xi = \int_\Omega h\gamma\xi_x = \int_\Omega h\xi_x + \int_\Omega h(\gamma-1)\xi_x = -\int_\Omega h'\xi$$

and so:

$$\int_\Omega h^3 \mid \nabla(\underline{v}-p)^+ \mid^2 = \int_\Omega h^3 \nabla(\underline{v}-p)\nabla\xi \leq 0.$$

Similarly to Theorem 2.1, we obtain the conclusion. ■

Corollary 2.3.

If (p,γ) is the solution of Problem (P), with $p_a \geq M/2$, then $p>0$ in Ω and so there is not free-boundary.

Proof: If $p_a \geq M/2$ then $\underline{v}(y) > 0$ and $p > 0$ for all $y \in (0,1)$. ■

Remark.

The figures one and two illustrate functions $\underline{v}$ and $\overline{v}$ in the two differents cases: $p_a < M/2$ and $p_a \geq M/2$.

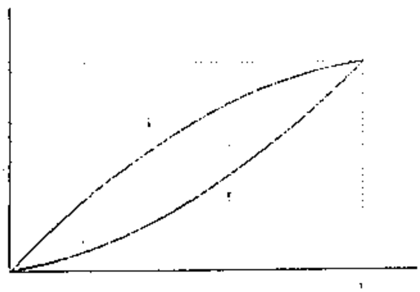

Fig. 1 ($p_a < M/2$)

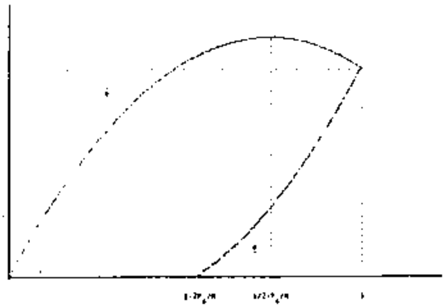

Fig. 2 ($p_a \geq M/2$)

Figure 1 illustrate the region $(0, 2\pi) \times (0, 1 - 2p_a/M)$ where the free–boundary (when it exist) lies. The function $\overline{v}$ attain a maximum in $y = \frac{1}{2} + p_a/M \in (0,1)$; we shall prove later that, fixed $x, p(x, \cdot)$ is a non–decreasing monotone function up to this point.

Figure 2 corresponds to the case where there is not free–boundary; when $p_a >> M/2$ the solution is very close to the function $w(y) = p_a y$, which satisfies that div $(h^3 \nabla w) = 0$, corresponding to the limit case when the eccentricity ratio α is equal to zero, and evidencing that this eccentricity is negligible when the pression on the supply line is very great.

3. Behaviour of the free–boundary in the y–variable

We consider in this section the case $p_a < M/2$, denoting by y_m the value $y_m = \frac{1}{2} + p_a/M$, where the function $\overline{v}$, defined by (2.1), attain a maximum. Let $y_0 = 2p_a/M$, and take y_1 any value in $(y_m, 1)$. Finally, let $\Omega_1 = (0, 2\pi) \times (y_0, y_1)$, denoting by Γ_0^1 and Γ_1^1 the lower and upper boundarys of Ω_1 respectively. (see Fig. 3).

Fig. 3

The equation $z = y_1 - \frac{1}{\beta}(y - y_1)$ with $\beta = \frac{y_1 - y_0}{1 - y_1}$, transform the interval $[y_0, y_1]$ into $[y_1, 1]$. Making use of this transformation we can define a new function on Ω_1, from the solution p of Problem (P), by means of

$$\text{for } (x,y) \in \bar{\Omega}_1 \qquad \bar{p}(x,y) = \beta^2 p(x,z).$$

We have:

Theorem 3.1.

$$p(x,y) \leq \bar{p}(x,y) \qquad \textit{for any } (x,y) \in \Omega_1.$$

*Before to give the proof of Theorem 3.1 we shall first prove some previous results about $\bar{p}(x,y)$. We remark that the technics to prove this theorem are the same that the ones used to prove uniqueness. They are based on the construction of a class of test functions defined in a multidimensional domain. Sucht test functions appear in [**A-C**], [**C-1**] and [**C-2**].*

Proposition 3.2.

If (p,γ) is the solution of Problem (P) and we define $\bar{\gamma}(x,y) = \gamma(x,z)$ for $(x,y) \in \Omega_1$, then the pair $(\bar{p},\bar{\gamma})$ satisfies,

$$\int_{\Omega_1} h^3 \nabla \bar{p} \nabla \xi + \frac{1-\beta^2}{\beta^2} \int_{\Omega_1} h^3 \bar{p}_x \xi_x = \int_{\Omega_1} h \bar{\gamma} \xi_x \tag{3.1}$$

for any $\xi \in H^1(\Omega_1)$, 2π x-periodic and $\xi \mid_{\Gamma_0^1 \cup \Gamma_1^1} = 0$

$$H(\bar{p}) \leq \bar{\gamma} \leq 1 \qquad a.e. \text{ in } \Omega \tag{3.2}$$

Moreover

$$p \mid_{\Gamma_0^1 \cup \Gamma_1^1} \leq \bar{p} \mid_{\Gamma_0^1 \cup \Gamma_1^1} \tag{3.3}$$

Proof: Let $\bar{\xi}(x,z) = \xi(x,y)$, $J = \begin{pmatrix} 1 & 0 \\ 0 & -1/\beta \end{pmatrix}$ the matrix for derivatives of (x,z) with rapport to (x,y), and $\Omega_2 = (0,2\pi) \times (y_1, 1)$ with lower and upper boundarys Γ_0^2 and Γ_1^2 respectively ($\Gamma_0^2 = \Gamma_1^1$, and $\Gamma_1^2 = \Gamma_1$). We get:

$$\begin{aligned}
&\int_{\Omega_1} h^3 \nabla \bar{p} \nabla \xi = \int_{\Omega_1} h^3 \nabla_{x,y} \beta^2 p(x,z) \nabla_{x,y} \xi(x,y) dx\, dy = \\
&= \int_{\Omega_2} h^3 (\nabla_{x,z} \beta^2 p(x,z) J) \cdot (\nabla_{x,z} \bar{\xi}(x,z) J) \beta dx\, dz = \\
&= \beta \int_{\Omega_2} h^3 \{\beta^2 p_x \bar{\xi}_x + p_z \bar{\xi}_z\} dx\, dz = \\
&= \beta \int_{\Omega_2} h^3 \nabla_{x,z} p \nabla_{x,z} \bar{\xi} dx\, dz + \beta(\beta^2 - 1) \int_{\Omega_2} h^3 p_x \bar{\xi}_x dx\, dz = \\
&= \beta \int_{\Omega_2} h \gamma \bar{\xi}_z dx\, dz + \beta(\beta^2 - 1) \int_{\Omega_2} h^3 p_x \bar{\xi}_x dx\, dz
\end{aligned}$$

since $\bar{\xi} \in H^1(\Omega_2)$, is 2π x-periodic and $\bar{\xi} \mid_{\Gamma_0^2 \cup \Gamma_1^2} = 0$.

Now, coming back to the y-variable in Ω_1, we conclude

$$\int_{\Omega_1} h^3 \nabla \bar{p} \nabla \xi + \frac{1-\beta^2}{\beta^2} \int_{\Omega_1} h^3 \bar{p}_x \xi_x = \int_{\Omega_1} h \bar{\gamma} \xi_x$$

and $H(\bar{p}) = H(\beta^2 p(x,z)) \leq \gamma(x,z) = \bar{\gamma}(x,y) \leq 1$ a.e. in Ω_1. ∎

Moreover $p \leq \bar{p}$ on Γ_1^1, because $y = z$ and $\beta^2 > 1$, and $p \leq p_a \leq \beta^2 p_a = \bar{p}$ on Γ_0^1.

We shall distinguish the x-variable for p and $\bar{p}$, using the variables $(x_1, y) \in \Omega_1$ for (p, γ) and $(x_2, y) \in \Omega_1$ for $(\bar{p}, \bar{\gamma})$; we set $Q = (0, 2\pi) \times (0, 2\pi) \times (y_0, y_1)$, and let us consider $\xi(r)$ and $\rho(r)$, real functions such that:

$$\xi(r) \in C_0^\infty(y_0, y_1), \quad \xi \geq 0.$$

$$\rho(r) \in C_0^\infty(\mathbb{R}), \quad \rho \geq 0, \text{ supp } \rho = [-1, 1]$$

$$\rho \text{ is a pair function.}$$

For small $\varepsilon > 0$ we define $\rho_\varepsilon(r) = (1/\varepsilon)\rho(r/\varepsilon)$, and finally for $(x_1, x_2, y) \in \bar{Q}$ let $F(x_1, x_2, y)$ be defined by

$$F(x_1, x_2, y) = \xi(y) \rho_\varepsilon\left(\frac{x_1 - x_2}{2}\right).$$

This function, is identically zero when $\mid x_1 - x_2 \mid \geq 2\varepsilon$ and, since ρ_ε is a pair function, it can be redefined when $(x_1, x_2) \in T_1 \cup T_2 = \{(x_1, x_2) \in [0, 2\pi] \times [0, 2\pi] : \mid x_1 - x_2 \mid \geq 2\pi - 2\varepsilon\}$, by making

$$\rho_\varepsilon\left(\frac{x_1 - x_2}{2}\right) = \rho_\varepsilon\left(\frac{\mid x_1 - x_2 \mid - 2\pi}{2}\right)$$

So we obtain a 2Π-periodic function in the independents variables x_1 and x_2 (see Fig. 4). Moreover $F(\cdot, x_2, \cdot)$, $F(x_1, \cdot, \cdot) \in H^1(\Omega_1)$ and $F(x_1, x_2, y_0) = F(x_1, x_2, y_1) = 0$.

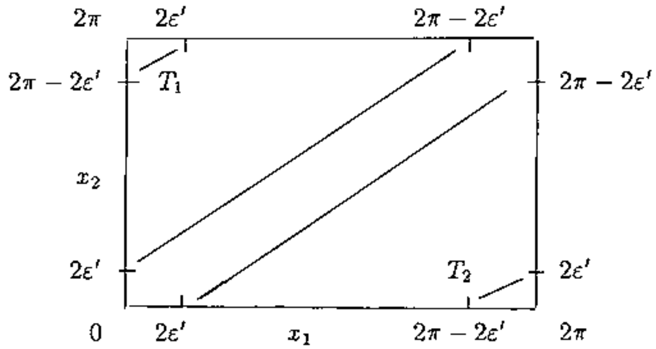

Fig. 4

Now, let us consider a new parameter $\delta > 0$, and define

$$\eta(x_1, x_2, y) = \text{ Min } \left[\frac{(p(x_1,y) - \bar{p}(x_2,y))^+}{\delta}, F\right]$$

Using this function and denoting by ∇_1 and ∇_2 the gradient operator for (x_1, y) and (x_2, y) respectively, we have:

Proposition 3.3.

$$\begin{aligned}&\int_Q \{(h^3(x_1)p_{x_1} - h^3(x_2)\bar{p}_{x_2})(\eta_{x_1} + \eta_{x_2}) + (h^3(x_1)p_y - h^3(x_2)\bar{p}_y)\eta_y\} + \\ &+ \frac{\beta^2 - 1}{\beta^2}\int_Q h^3(x_2)\bar{p}_{x_2}(\eta_{x_1} + \eta_{x_2}) = \int_Q (h(x_1) - h(x_2)\bar{\gamma})(\eta_{x_1} + \eta_{x_2})\end{aligned} \tag{3.4}$$

Proof: For each $x_2 \in (0, 2\pi)$, we have

$$\begin{aligned}&\int_{\Omega_1} h^3(x_1)\nabla_1 p \nabla_1 \eta \, dx_1 \, dy = \int_{\Omega_1} h(x_1)\eta_{x_1} \, dx_1 dy = \\ &= \int_{\Omega_1} (h(x_1) - h(x_2)\bar{\gamma})\eta_{x_1} dx_1 \, dy\end{aligned}$$

since $\gamma_1 \equiv 1$ on $\text{supp}\, \eta(\cdot, x_2, \cdot)$, and $\int_{\Omega_1} h(x_2)\bar{\gamma}\eta_{x_1} = 0$, from the periodicity of η.

By integrating the above equality in the x_2-variable, we get:

$$\int_Q h^3(x_1)\nabla_1 p \nabla_1 \eta = \int_Q (h(x_1) - h(x_2)\bar{\gamma})\eta_{x_1}$$

and, analogously for $\bar{p}$:

$$\begin{aligned}&\int_Q h^3(x_2)\nabla_2 \bar{p} \nabla_2 \eta + \frac{1 - \beta^2}{\beta^2}\int_Q h^3(x_2)\bar{p}_{x_2}\eta_{x_2} = \\ &= \int_Q (h(x_2)\bar{\gamma} - h(x_1))\eta_{x_2}\end{aligned}$$

Substracting the above equalities, we get:

$$\begin{aligned}&\int_Q (h^3(x_1)\nabla_1 p \nabla_1 \eta - h^3(x_2)\nabla_2 \bar{p} \nabla_2 \eta) + \frac{\beta^2 - 1}{\beta^2}\int_Q h^3(x_2)\bar{p}_{x_2}\eta_{x_2} = \\ &= \int_Q (h(x_1) - h(x_2)\bar{\gamma})(\eta_{x_1} + \eta_{x_2})\end{aligned} \tag{3.5}$$

Moreover

$$\int_Q h^3(x_2)\bar{p}_{x_2}\eta_{x_1} = \int_Q h^3(x_1)p_{x_1}\eta_{x_2} = 0$$

and introducing this terms in (3.5), we conclude (3.4). ■

Now, we go to consider the new variables (see [**A-C**], [**C-1**]):

$$t = \frac{x_1 + x_2}{2}, \quad z = \frac{x_1 - x_2}{2},$$

getting for the function η:

$$\eta(t+z, t-z, y) = \text{ Min } \left[\frac{(p(t+z,y) - \bar{p}(t-z,y))^+}{\delta}, \xi(y)\rho_\varepsilon(z)\right]$$

and, for derivatives:

$$\begin{aligned} \phi_{x_1} &= \frac{1}{2}(\phi_t + \phi_z), \\ \phi_{x_2} &= \frac{1}{2}(\phi_t - \phi_z), \qquad \text{for any } \phi = \phi(x_1, x_2, y) \\ \phi_{x_1} &+ \phi_{x_2} = \phi_t \end{aligned}$$

what, in the particular case of $p = p(x_1, y)$ and $\bar{p} = \bar{p}(x_2, y)$, being $p_t(t+z,y) = p_z(t+z,y)$ and $\bar{p}_t(t-z,y) = -\bar{p}_z(t-z,y)$, gives:

$$\begin{aligned} p_{x_1}(x_1, y) &= p_t(t+z, y) \\ \bar{p}_{x_2}(x_2, y) &= \bar{p}_t(t-z, y). \end{aligned}$$

In the new variables, the equation (3.4) becomes:

$$\begin{aligned} &\int_{Q_{tz}} (h^3(t+z)\nabla_{ty}p(t+z,y) - h^3(t-z)\nabla_{ty}\bar{p}(t-z,y))\nabla_{ty}\eta + \\ (3.6) \qquad & + \frac{\beta^2 - 1}{\beta^2}\int_{Q_{tz}} h^3(t-z)\bar{p}_t\eta_t = \int_{Q_{tz}} (h(t+z) - h(t-z)\bar{\gamma})\eta_t, \end{aligned}$$

where we omite the constant due to the coordinates transformation, and denote by Q_{tz} the new domain.

If we consider the sets,

$$A^\delta_\varepsilon = [(p_1 - p_2)^+ > \delta\xi\rho_\varepsilon] \quad B^\delta_\varepsilon = [0 < p_1 - p_2 \le \delta\xi\rho_\varepsilon]$$

(in Q or Q_{tz}) and denote:

$$I_1 = \int_{A_\varepsilon^\delta} (h^3(t+z)\nabla_{ty} p(t+z,y) - h^3(t-z)\nabla_{ty}\bar{p}(t-z,y))\nabla_{ty}(\xi(y)\rho_\varepsilon(z)) =$$
$$= \int_{A_\varepsilon^\delta} (h^3(t+z)p_y(t+z,y) - h^3(t-z)\bar{p}_y(t-z,y))\xi'(y)\rho_\varepsilon(z)$$
$$I_2 = \int_{B_\varepsilon^\delta} (h^3(t+z)\nabla_{ty} p(t+z,y) - h^3(t-z)\nabla_{ty}\bar{p}(t-z,y))\nabla_{ty}\frac{p-\bar{p}}{\delta}$$
$$I_3 = \frac{\beta^2-1}{\beta^2}\int_{B_\varepsilon^\delta} h^3(t-z)\bar{p}_t\frac{(p-\bar{p})_t}{\delta}$$
$$I_4 = \int_{B_\varepsilon^\delta} (h(t+z) - h(t-z)\bar{\gamma})\frac{(p-\bar{p})_t}{\delta},$$

we can write (3.6) in the form:

$$I_1 + I_2 + I_3 = I_4. \tag{3.7}$$

For I_4 we have:

Lemma 3.4. ([A])

$$\lim_{\varepsilon\to 0}\left[\lim_{\delta\to 0} I_4\right] = 0.$$

Let us prove now, the following:

Lemma 3.5.

$$\lim_{\varepsilon\to 0}\left[\lim_{\delta\to 0} I_1\right] \leq 0 \tag{3.8}$$

Proof:

$$I_2 + I_3 = \int_{B_\varepsilon^\delta} (h^3(t+z)\mid \nabla_{ty}\frac{p}{\delta}\mid^2 + h^3(t-z)\mid \nabla_{ty}\frac{\bar{p}}{\delta}\mid^2) -$$
$$-\int_{B_\varepsilon^\delta} (h^3(t+z)\nabla_{ty} p\nabla_{ty}\frac{\bar{p}}{\delta} + h^3(t-z)\nabla_{ty}\bar{p}\nabla_{ty}\frac{p}{\delta}) +$$
$$+\frac{\beta^2-1}{\beta^2}\int_{B_\varepsilon^\delta} h^3(t-z)\bar{p}_t(\frac{p}{\delta})_t - \frac{\beta^2-1}{\beta^2}\int_{B_\varepsilon^\delta} h^3(t-z)(\frac{\bar{p}_t}{\delta})^2$$

denoted by $J_1 - J_2 + J_3 - J_4$, with the following balance:

$J_1 - J_4 \geq 0$ because $0 < \frac{\beta^2-1}{\beta^2} < 1$.

$\mid J_3 \mid \leq \mid J_2 \mid$ and J_2 can be decomposed in two integrals having both of them limit equal to zero, when we pass to the limit first as $\delta \to 0$ and later as $\varepsilon \to 0$. (see [A], [A-C]).

From Lemma 3.4 and (3.7) we conclude (3.8). ∎

Proof of Theorem 3.1: By Lebesgue Theorem,

$$\lim_{\delta\to 0} I_1 = \int_{Q_{tz}} (h^3(t+z)\frac{\partial}{\partial y}p - h^3(t-z)\frac{\partial}{\partial y}\bar{p})\chi([p>\bar{p}])\xi'(y)\rho_\varepsilon(z) =$$

$$= \int_{Q_{tz}} h^3(t+z)\frac{\partial}{\partial y}(p-\bar{p})\chi([p>\bar{p}])\xi'(y)\rho_\varepsilon(z) +$$

$$+ \int_{Q_{tz}} (h^3(t+z) - h^3(t-z))\frac{\partial}{\partial y}\bar{p}\chi([p>\bar{p}])\xi'(y)\rho_\varepsilon(z)$$

denoted by I_1^1 and I_1^2 respectily.

I_1^2 satisfies

$$| I_1^2 | \leq C\int_{Q_{tz}} | h^3(t+z) - h^3(t-z) || \frac{\partial}{\partial y}\bar{p} | \rho_\varepsilon(z) \leq$$

$$\leq C\|\frac{\partial}{\partial y}\bar{p}\|_{L^2(Q_{tz})}\{\int_{Q_{tz}} | h^3(t+z) - h^3(t-z) |^2 | \rho_\varepsilon(z) |^2\}^{1/2} \leq C'\sqrt{\varepsilon}$$

because h^3 is Lipschitz continuous and the measure of supp $\rho_\varepsilon(z)$ is $4\pi\varepsilon$, and then

$$\int_{Q_{tz}} | h^3(t+z) - h^3(t-z) |^2 | \rho_\varepsilon(z) |^2 \leq \text{ cte } \int_{Q_{tz}} | z |^2 \frac{1}{\varepsilon^2}(\rho_\varepsilon(z/\varepsilon))^2 \leq \text{ cte } \varepsilon.$$

From (3.8) we have:

$$0 \geq \lim_{\varepsilon\to 0}\left[\lim_{\delta\to 0} I_1\right] = \lim_{\varepsilon\to 0} I_1^1 + \lim_{\varepsilon\to 0} I_1^2 =$$

$$= \lim_{\varepsilon\to 0}\int_{Q_{tz}} h^3(t+z)\frac{\partial}{\partial y}[(p-\bar{p})^+]\xi'(y)\rho_\varepsilon(z) =$$

$$= -\lim_{\varepsilon\to 0}\int_{Q_{tz}} h^3(t+z)(p-\bar{p})^+\xi''(y)\rho_\varepsilon(z)$$

but, by a classical argument (see **[A]**) we can elimine ε and the z–variable, concluding:

$$\int_{\Omega_1} h^3(t)(p(t,y) - \bar{p}(t,y))^+\xi''(y)dt\,dy \geq 0 \tag{3.9}$$

Now, setting

$$T(y) = \int_0^{2\Pi} h^3(t)(p-\bar{p})^+ dt$$

(3.9) is equivalent to:

$$\left\langle \frac{d^2T}{dy^2}, \xi \right\rangle_{\mathcal{D}'(y_0,y_1)\times\mathcal{D}(y_0,y_1)} \geq 0$$

and we have that the distribution T satisfies:

$$\frac{d^2T}{dt_2^2} \geq 0.$$
$$T(0) = T(1) = 0 \qquad \text{due to (3.3)}\ .$$

Hence, by the maximum principle, we conclude

$$\int_0^{2\Pi} h^3(t)(p-\bar{p})^+ dt \leq 0$$

and then

$$p \leq \bar{p} \quad \text{in } \Omega_1$$

That is,

$$p(x,y) \leq \beta^2 p(x,\bar{y}) = \beta^2 p(x, y_1 - \frac{1}{\beta}(y-y_1))$$

and the proof ends. ■

When $y_1 \leq y_m$ (the point of a maximum for $\overline{v}(y)$), we can obtain the same result with $\beta = 1$. We introduce two cases:

If $1/2 < y_1 \leq y_m$, we make $y_0 = 2y_1 - 1, \Omega_1 = (0,2\pi)\times(y_0,y_1)$,
$\Omega_2 = (0,2\pi)\times(y_1,1)$ and $z = 2y_1 - y$.
If $y_1 \leq 1/2$, we make $y_0 = 0, \quad \Omega_1 = (0,2\pi)\times(0,y_1)$,
$\Omega_2 = (0,2\pi)\times(y_1,2y_1)$ and $z = 2y_1 - y$ (see Fig. 5 and 6).

Fig. 5

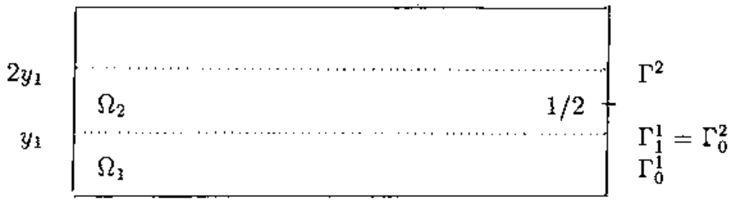

Fig. 6

For both cases $\beta = 1, p\,|_{\Gamma_0^1} \leq p\,|_{\Gamma_1^2}$ and we can conclude as in Theorem 3.1:

Corollary 3.6.
If (p, γ) is the solution of Problem (P), then $p\,(x, \cdot)$ is a monotone incresing function on $[0, y_m]$.

Proof: Let $y^1, y^2 \in [0, y_m]$ and such that $y^1 < y^2$; taking $y_1 = \frac{y^1+y^2}{2}$, we have $y^2 = 2y_1 - y^1$, and applying Theorem 3.1, we conclude

$$p(x, y^1) \leq p(x, y^2).$$

Corollary 3.7.
Let (x, z) be such that $p(x, z) = 0$; then $p(x, y) = 0$ for any $y \in [0, z]$.

Proof: By the above Corollary we must only to prove that $p(x, y) = 0$ in $[y_m, z]$ when $z > y_m$.

For $y \in [y_m, z)$, we take $y_1 \in (y, z)$ such that $y - y_1 = -\frac{y_1 - y_0}{1 - y_1}(z - y_1)$ (see Fig. 7), which is equivalent to $z = y_1 - \frac{1}{\beta}(y - y_1)$ with $\beta = \frac{y_1 - y_0}{1 - y_1} > 1$.

Applying Theorem 3.1, we conclude:

$p(x, y) \leq \beta^2 p(x, z) = 0$ for any $y \in [y_m, z)$, and hence $p(x, y) = 0$ for any $y \in [0, z]$. ■

Fig. 7

Remark. Corollary 3.7 states that the free–boundary does not have horizontal oscillations.

4. Behaviour of γ in the x–variable

We go to study some properties of γ with geometrical consequences on the free–boundary, when $x \in (0, \pi)$.

Theorem 4.1.

Let (p, γ) be the solution of Problem (P), and let χ be the characteristic function of the set $[p > 0]$; then,

$$(h\gamma)_x - h'\chi \geq 0 \qquad \text{in } \mathcal{D}'(\Omega). \tag{4.1}$$

Proof: Let $\phi \in \mathcal{D}(\Omega)$ with $\phi \geq 0$, and for $\varepsilon > 0$ let us consider the test function $\xi = \min(\varepsilon\phi, p)$; we have:

$$\int_\Omega h^3 \nabla p \nabla \xi = \int_{[p<\varepsilon\phi]} h^3 \mid \nabla p \mid^2 + \varepsilon \int_{[p\geq\varepsilon\phi]} h^3 \nabla p \nabla \phi = \int_\Omega h\xi_x = -\int_\Omega h'\xi$$

since $\gamma = 1$ on the support of ξ. Then

$$\int_{[p\geq\varepsilon\phi]} h^3 \nabla p \nabla \phi + \int_\Omega h' \min(\phi, p/\varepsilon) = -1/\varepsilon \int_{[p<\varepsilon\phi]} h^3 \mid \nabla p \mid^2 \leq 0;$$

letting $\varepsilon \to 0$ and using the Lebesgue Theorem, we obtain:

$$\int_\Omega h^3 \nabla p \nabla \phi + \int_\Omega h'\chi\phi \leq 0$$

but

$$\int_\Omega h^3 \nabla p \nabla \phi = \int_\Omega h\gamma\phi_x$$

concluding that

$$\int_\Omega h\gamma\phi_x + \int_\Omega h'\chi\phi \leq 0 \qquad \forall\, \phi \in \mathcal{D}^+(\Omega),$$

which equivales to

$$\langle h'\chi - (h\gamma)_x, \phi \rangle_{\mathcal{D}'(\Omega)\times\mathcal{D}(\Omega)} \leq 0 \qquad \forall \phi \in \mathcal{D}^+(\Omega).$$

and, hence

$$h'\chi - (h\gamma)_x \leq 0 \qquad \text{in } \mathcal{D}'(\Omega).$$

Corollary 4.2.

(4.2) $$\gamma_x \geq 0 \qquad in\ \mathcal{D}'((0,\pi)\times(0,1)).$$

(4.3) $$(h\gamma)_x \geq 0 \qquad in\ \mathcal{D}'((\pi,2\pi)\times(0,1)).$$

Proof: As $h' > 0$ in $(\pi, 2\pi)$ and from (4.1) we deduce that

$$(h\gamma)_x \geq h'\chi \geq 0 \quad \text{in } \mathcal{D}'((\pi,2\pi)\times(0,1)).$$

In $(0,\pi)$:

$$\begin{aligned} &h'\chi - (h\gamma)_x = h'\chi - h'\gamma - h\gamma_x = h'(\chi-\gamma) - h\gamma_x \leq 0 \\ &h' < 0 \\ &\chi - \gamma \leq 0 \end{aligned}$$

so that,

$$\gamma_x \geq \frac{h'(\chi-\gamma)}{h} \geq 0 \qquad \text{in } \mathcal{D}'((0,\pi)\times(0,1))$$

Corollary 4.3.
If $p(x_0,y_0) > 0$ for some $x_0 < \pi$, then there exists $\varepsilon > 0$ such that $p > 0$ on the set $C_\varepsilon = (x_0-\varepsilon,\pi)\times(y_0-\varepsilon,y_0+\varepsilon)$.

Proof: From the continuity of p, there exist $Q_\varepsilon = (x_0-\varepsilon, x_0+\varepsilon)\times(y_0-\varepsilon, y_0+\varepsilon)$ such that $p > 0$ in Q_ε (see Fig. 8) and $\gamma = 1$ a.e. in Q_ε. Like $\gamma_x \geq 0$ we get $\gamma = 1$ a.e. in C_ε.

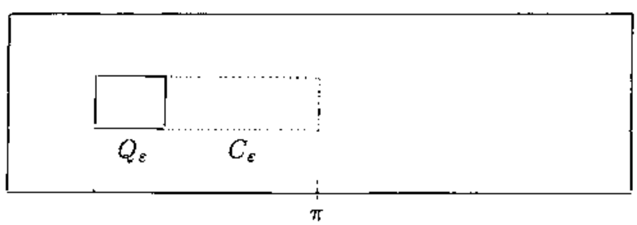

Fig. 8

Now, for $\phi \in C_0^\infty(C_\varepsilon)$ we have

$$\int_{C_\varepsilon} h^3 \nabla p \nabla \phi = \int_{C_\varepsilon} h\phi_x$$

and, hence

$$\text{div } h^3\nabla p = h' < 0 \qquad \text{in } \mathcal{D}'(C_\varepsilon).$$

Using the strong minimum principle, p can not attain the minimum value zero in C_ε and hence

$$p > 0 \qquad \text{in } C_\varepsilon.$$

Remark. As a consequence of this Corollary the free–boundary can not have vertical oscillations in the interval $(0, \pi)$.

Taking account the Corollary 3.7, we conclude that the free–boundary is a monotone decreasing graph $-y = \Gamma(x)$–in the interval $(0, \pi)$ (see Fig. 9).

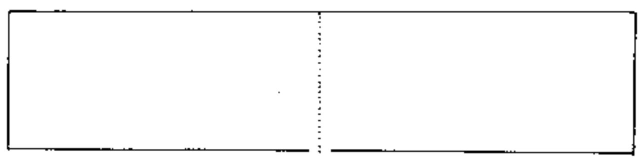

Fig. 9

Theorem 4.4.

If (p, γ) is the solution of Problem (P), then p satisfies:

$$\int_0^{2\pi} h^3(x) p(x,y)\,dx = p_a y \int_0^{2\pi} h^3(x)\,dx$$

Proof: For $\phi(y) \in C_0^\infty(0,1)$ we have

$$\int_\Omega h^3 \nabla p \nabla \phi = \int_\Omega h^3 p_y \phi' = 0$$

Integrating by parts and introducing the function

$$F(y) = \int_0^{2\pi} h^3(x) p(x,y)\,dx$$

we have

$$\left\langle \frac{d^2 F}{dy^2}, \phi \right\rangle_{\mathcal{D}'(0,1)\times\mathcal{D}(0,1)} = 0$$

and hence

$$\frac{d^2 F}{dy^2} = 0 \qquad \text{in } \mathcal{D}'(0,1)$$

but, $F(0) = 0$ and $F(1) = p_a \int_0^{2\pi} h^3$, so we conclude

$$F(y) = p_a y \int_0^{2\pi} h^3(x)\,dx.$$

Corollary 4.5.

Given $y \in (0,1)$ there exist a region of positive measure in $(0, 2\pi)$ where $p > 0$.

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