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LOCALIZING GROUPS WITH ACTION

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Abstract

When localizing the semidirect product of two groups, the effect on the factors is made explicit. As an application in Topology, we show that the loop space of a based connected CW–complex is a P–local group, up to homotopy, if and only if $\pi_1 X$ and the free homotopy groups $[S^{k-1}, \Omega X]$, $k \geq 2$, are P–local.

Introduction

The study of groups G in which the functions $\rho_p : G \to G$, $\rho_p(g) = g^p$, are, for certain primes p, bijective, has a long history; see Malcev [9], Baumslag [1] and the references there. After Sullivan [15], Bousfield–Kan [3], Hilton [5] and Hilton–Mislin–Roitberg [8] this study appears now in the guise of localizing a group with respect to a given set of primes P. In a P–local group the functions ρ_n are bijective if n belongs to the multiplicative closure of the set of primes P', which is complementary to P.

According to Ribenboim [11], there is a P–localizing functor from the category of groups to the category of P–local groups, $\mathcal{G} \to \mathcal{G}p$. While the properties of this functor, when restricted to the category of nilpotent groups, are well understood (see [5] and [7]), its properties in general are not clear at all.

For example, on nilpotent groups the P–localizing functor is exact, but not in general. E.g., the exact sequence $Z/3 \rightarrowtail S_3 \to Z/2$ for the symmetric group of 3 elements gets sent to $Z/3 \to 0 \to 0$, when localizing at 3. S_3 is a semidirect product $Z/3 \rtimes Z/2$ and the purpose of this paper is to investigate the effect of localization on semidirect products $G = H \rtimes R$:

Since localization is functorial, Gp is again a semidirect product $Gp \cong K \rtimes Rp$. Therefore, it is desirable to understand the relation between H and K. We will discover that K is the P–localization of H with respect to the change of operator groups from R to Rp.

To explain this, we use the category ${}_R\mathcal{G}$ of R–groups (i.e. groups on which the group R acts on the left) and R–homomorphisms (i.e. group homomorphisms $f : H \to H'$ with $f(r \cdot h) = r \cdot f(h)$ for all $h \in H$ and $r \in R$). Further a group

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homomorphism $\gamma : R \to S$ induces the change–of–operator–groups functor $\gamma^* : {}_S\mathcal{G} \to {}_R\mathcal{G}$. For $H \in {}_R\mathcal{G}$, $K \in {}_S\mathcal{G}$, a group homomorphism $f : H \to K$ is a γ–homomorphism, if $f : H \to \gamma^* K$ is an R–homomorphism. We then construct a left adjoint ${}_\gamma Ad$ for γ^*; see 1.5.

Now, ${}_S\mathcal{G}$ contains a subcategory ${}_S\mathcal{G}p$ consisting of such groups on which S acts P–locally ; see 1.2. Accordingly, we construct a left adjoint ${}_SLp : {}_S\mathcal{G} \to {}_S\mathcal{G}p$; see 1.6. The composite ${}_\gamma Lp := S^Lp\,{}_\gamma Ad : {}_R\mathcal{G} \to {}_S\mathcal{G}p$ is left adjoint to the restriction of γ^* to ${}_S\mathcal{G}p$. It then follows that $(H \rtimes R)p \cong ({}_eLpH) \rtimes Rp$, where $e : R \to Rp$ P–localizes.

Remarks.

(1) The functor ${}_\gamma Ad$ is of independent interest. For example, let ${}_SAd$ correspond to the unique homomorphism $\{1\} \to S$. Then ${}_SAd$ provides the foundation for a theory of S–groups by generators and relations.

(2) The problem of localizing semidirect products has also been studied by Casacuberta [4] in the case where the normal subgroup H is abelian, and by A. Reynol when H is finite abelian [**12**].

(3) Our study is also of interest in Topology; see 1.7 and 1.8.

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1. We now take up the announced investigation. So let R be a group acting on another group H via a homomorphism $\phi : R \to AutH$. The corresponding semi–direct product is denoted by $H \rtimes_\phi R$ or $H \rtimes R$ if there is no risk of confusion.

Lemma 1.1. *$G = H \rtimes R$ is P-local $\Leftrightarrow$ the following two conditions hold:*
(i) R is P-local;
(ii) For all $r \in R$ and $n \in P'$, the function

$$\rho_{r,n} : H \to H,\ h \mapsto h\phi_r(h)\phi_{r^2}(h)\ldots\phi_{r^{n-1}}(h)$$

is a bijection, where ϕ_r denotes the automorphism $\phi(r)$ of H.

Proof: This follows from $(h,r)^n = (\rho_{r,n}(h), r^n)$. ■

The functions $\rho_{r,n}$ have been used already by Baumslag in a setting involving wreath products; see [**2**].

Definition 1.2. *R acts P–locally* on H : $\Leftrightarrow$ for all $r \in R$ and $n \in P'$, the function $\rho_{r,n}$ of (1.1) is a bijection.

The notion of a P–local action has independently been introduced by Rodicio, [**13**]. Since $\rho_{1,n}(h) = h^n$, if R acts P–locally on H, then H is P–local. We write ${}_R\mathcal{G}p$ for the category of R–groups on which R acts P–locally.

It is straightforward to prove

Lemma 1.3. *Let*

$$\begin{array}{ccccc} H & \rightarrowtail & H \rtimes_\phi R & \twoheadrightarrow & R \\ \alpha \downarrow & & \beta \downarrow & & \downarrow \gamma \\ K & \rightarrowtail & K \rtimes_\psi S & \twoheadrightarrow & S \end{array}$$

be a commuting diagram of split exact sequences of groups. Then β P-localizes in $\mathcal{G}$ if and only if the following three conditions hold:

(i) γ P-localizes in $\mathcal{G}$;

(ii) S acts P-locally on K;

(iii) For all $L \in {}_S\mathcal{G}$ on which S acts P-locally and every γ-homomorphism $\nu : H \to L$, there is a unique S-homomorphism $\nu' : K \to L$, with $\nu = \nu'\alpha$.

This suggests

Definition 1.4. Let $H \in {}_R\mathcal{G}$, $K \in {}_S\mathcal{G}$ and let $\gamma : R \to S$ be a homomorphism. Then $\alpha : H \to K$ *P-localizes with respect to* γ if and only if the following three conditions hold:

(i) S acts P-locally on K;

(ii) α is a γ-homomorphism;

(iii) α satisfies the universal property 1.3 (iii) above

Thus, Lemma (1.3) can be restated as

Lemma 1.3'. *β P-localizes in $\mathcal{G}$ if and only if γ P-localizes in $\mathcal{G}$ and α P-localizes with respect to γ.*

Now let $\gamma : R \to S$ be given. The construction of a left adjoint functor ${}_\gamma Lp : {}_R\mathcal{G} \to {}_S\mathcal{G}p$ to the composite functor ${}_S\mathcal{G}p \xrightarrow{\text{inclusion}} {}_S\mathcal{G} \xrightarrow{\gamma^*} {}_R\mathcal{G}$ is done in two steps.

Theorem 1.5. *$\gamma^* : {}_S\mathcal{G} \to {}_R\mathcal{G}$ has a left adjoint ${}_\gamma Ad : {}_R\mathcal{G} \to {}_S\mathcal{G}$.*

Theorem 1.6. *The inclusion functor ${}_S\mathcal{G}p \to {}_S\mathcal{G}$ has a left adjoint left inverse ${}_S Lp$: ${}_S\mathcal{G} \to {}_S\mathcal{G}p$.*

It then follows from 1.3' that $(H \rtimes R)p \cong ({}_e LpH) \rtimes Rp$, where $e : R \to Rp$ P-localizes.

Here is an interesting application of P-local actions in Topology.

Theorem 1.7. *Let X be a based connected CW-complex. Then, $\pi_1 X$ and the free homotopy groups $[S^{k-1}, \Omega X]$, $k \geq 2$, are P-local $\Leftrightarrow$ ΩX is a P-local group up to homotopy; i.e. for each $n \in P'$, the map $\bar\rho_n$: $\Omega X \to \Omega X$, $\bar\rho_n(x) = x^n$, is a homotopy equivalence.*

Proof: " $\Leftarrow$ " Recall that $\bar\rho_n$ induces ρ_n on $\pi_0\Omega X$ and on all free homotopy groups $[S^{k-1}, \Omega X]$, $k \geq 2$. If $\bar\rho_n$ is a homotopy equivalence, then ρ_n is a bijection. Thus $\pi_0\Omega X \cong \pi_1 X$ and $[S^{k-1}, \Omega X]$ are P-local.

" $\Rightarrow$ " Recall from [10] that ΩX is an H–semidirect product: $\Omega X \simeq (\Omega X)_0 \rtimes \pi_1 X$ and, as a consequence, that $[S^{k-1}, \Omega X] \cong \pi_k X \rtimes \pi_1 X$, for all $k \geq 2$. Since $\pi_1 X$ is P–local, $\bar{\rho}_n$ determines a bijection of the connected components of ΩX. Since $(\Omega X)_0$ is a simple space, the restriction of ρ_n to $(\Omega X)_0 \times \{r\}$, $r \in \pi_1 X$, induces the homomorphism

$$\pi_{k-1}(\Omega X)_0 \times \{r\} \cong [S^{k-1}, (\Omega X)_0 \times \{r\}] \to [S^{k-1}, (\Omega X)_0 \times \{r^n\}] \cong \\ \cong \pi_{k-1}(\Omega X)_0 \times \{r^n\}.$$

By hypothesis, this is a bijection. Thus, $\bar{\rho}_n$ is a homotopy equivalence. ■

Corollary 1.8. *Loop spaces of P–local nilpotent CW–complexes are P–local groups up to homotopy.*

Proof: If X is a P–local nilpotent space, then $\pi_1 X$ is P–local. Furthermore, the groups $[S^{k-1}, \Omega X]$, $k \geq 2$, are semidirect products of the P–local groups $\pi_k X$ and $\pi_1 X$ with respect to a nilpotent action of $\pi_1 X$ on $\pi_k X$. By a result of Hilton [6], the groups $[S^{k-1}, X]$ are P–local, for $k \geq 2$; compare also Roitberg [14]. Now apply 1.7. ■

2. Proof of Theorem 1.5

We need the following lemma whose proof is a little tedious but straightforward.

Lemma 2.1. *Let R act on H via $\phi : R \to AutH$. Let $D := \{rhr^{-1}\phi_r(h^{-1}) : r \in R, h \in H\} \subset H * R$. Let $\bar{H}$, $\bar{D}$ denote the normal closure of H, D in $H * R$. Then $\bar{D}$ is normal in $\bar{H}$ and $\bar{H}/\bar{D}$ is isomorphic to H.*

Step 1 for the proof of (1.5): Construction of ${}_\gamma Ad$

Let R act on H via $\phi : R \to Aut\ H$ and consider the diagram

$$\begin{array}{ccccc}
\bar{H} & \xrightarrow{i} & H * R & \xrightarrow{\pi} & R \\
\eta \downarrow & 1 & \downarrow Id*\gamma & 2 & \downarrow \gamma \\
\hat{H} & \xrightarrow[i']{} & H * S & \xrightarrow[\pi']{} & S
\end{array}$$

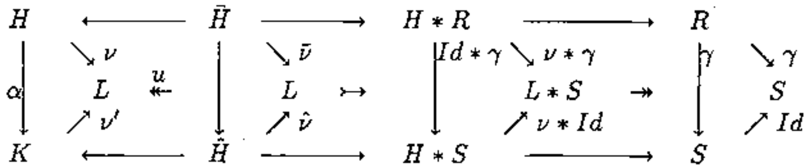
where π, π' are the canonical epimorphisms making square 2 commute. $\bar{H} :=$ ker π and $\hat{H} :=$ ker π'. Note that $(Id * \gamma)(\bar{H}) \subset \hat{H}$ and let η be the restriction of $Id * \gamma$ to $\bar{H}$. Then square 1 also commutes.

By design, R acts on $\bar{H}$ by conjugation and S acts on $\hat{H}$ by conjugation and η is a γ–homomorphism. Using (2.1), we relate these actions to the given action of R on H. We have refined the method of HNN–extensions.

Let $D := \{rhr^{-1}\phi_r(h^{-1}) : r \in R, h \in H\} \subset H * R$. By (2.1), $\bar{D} \subset \bar{H}$. Let $\hat{D}$ be the normal closure of $\eta(D)$ in $H * S$. Since $\eta(D) \subset \hat{H} \lhd (H * S)$, $\hat{D} \subset \hat{H}$. Take $K :=_\gamma Ad(H) := \hat{H}/\hat{D}$. Then η defines $\alpha : H \cong \hat{H}/\hat{D} \to \hat{H}/\hat{D} = K$.

The action of S on $\hat{H}$ by conjugation passes down to an action $\psi : S \to Aut\, K$. Explicitly, $\psi_S(\hat{h}\hat{D}) = s\hat{h}s^{-1}\hat{D}$. The action of R on $\bar{H}$ by conjugation passes down to the original action ϕ, by (2.1). It is clear that α is a γ–homomorphism.

Step 2. Vertification of the universal property of $\alpha : H \to K$. Let S act on L via $\theta : S \to AutL$. Let $\nu : H \to L$ be a γ–homomorphism. Consider the diagram



The right hand prism commutes and induces homomorphisms $\bar{\nu}$, $\hat{\nu}$ by restriction. Further, $\bar{\nu}(D) \subset \ker u$. Consequently, $\hat{\nu}(\hat{D}) \subset \ker u$, showing that $\hat{\nu}$ factors through K with $\nu' : K \to L$. Since $\hat{\nu}$ is an S–homomorphism, so is ν'.

It is straightforward to check uniqueness of ν' on the generators xhx^{-1} of K, where $x \in H * S$, $h \in H$. That ${}_\gamma Ad$ is a functor is immediate. This completes the proof of (1.5). ■

It follows directly from the construction of ${}_\gamma Ad$ that
(2.2): *${}_\gamma Ad$ preserves epimorphisms.*

3. Proof of Theorem 1.6

Let

$$_S\mathcal{U}p := \{K \in {}_S\mathcal{G} : \rho_{s,n} \text{ is } 1-1 \text{ for all } s \in S, n \in P'\}$$
$$_S\mathcal{E}p := \{K \in {}_S\mathcal{G} : \rho_{s,n} \text{ is onto for all } s \in S, n \in P'\}.$$

Then ${}_S\mathcal{G}p := {}_S\mathcal{U}p \cap {}_S\mathcal{E}p$ is the category of S–groups on which S acts P–locally.

We construct functors ${}_S\sqrt{}_p : {}_S\mathcal{G} \to {}_S\mathcal{G}$, which create preimages for the functions $\rho_{s,n}$ as well as ${}_SUp : {}_S\mathcal{G} \to {}_S\mathcal{U}p$, which make preimages of the functions $\rho_{s,n}$ unique.

Let S act on K via $\psi : S \to AutK$. Let FK denote the free group with basis $\{k_{s,n} : k \in K, s \in S, n \in P'\}$ and let $\xi K :=_S Ad\, FK$ denote the free S–group with that basis. If $\theta : S \to Aut\, \xi K$ denotes the corresponding S–action, then S acts on $K * \xi K$ by $S \ni s \mapsto \psi_s * \theta_s \in Aut(K * \xi K)$. Let N denote the

S–invariant normal closure of the set $\{\rho_{s,n}(k_{s,n})k^{-1} : k \in K, s \in S, n \in P'\}$ in $K * \xi K$.

Definition 3.1. $_S\sqrt{}_P K := K * \xi K / N$.

There is a canonical homomorphism $t : K \to {}_S\sqrt{}_P K$. By design, im $t \subset$ im $\rho_{s,n}$, for all $s \in S$ and $n \in P'$. Further, an S–homomorphism $f : K \to K'$ induces the S–homomorphism $\xi f : \xi K \to \xi K'$ via the function $k_{s,n} \to [f(k)]_{s,n}$ on bases. Hence, the S–homomorphism $(f * \xi f) : K * \xi K \to K' * \xi K'$ is defined. Passing to quotients, it yields the S–homomorphism $_S\sqrt{}_P f : {}_S\sqrt{}_P K \to {}_S\sqrt{}_P K'$.

Lemma 3.2.

(i) *$_S\sqrt{}_P : {}_S\mathcal{G} \to {}_S\mathcal{G}$ is a covariant functor.*

(ii) *$_S\sqrt{}_P$ preserves epimorphisms.*

(iii) *The homomorphism $t : K \to {}_S\sqrt{}_P K$ defines a natural transformation of the identity functor on $_S\mathcal{G}$ to $_S\sqrt{}_P$.*

(iv) *If $f : K \to L$ is an S-homomorphism such that $\rho_{\ell,n}$ is $(1-1)$ and onto im f for all $\ell \in L$ and $n \in P'$, then there is a unique S-homomorphism $f' : {}_S\sqrt{}_P K \to L$ with $f = f't$.*

Proof: (i), (ii), (iii) are straightforward from the construction.

(iv) The universal property of $_S Ad$ yields a unique S–homomorphism $d : \xi K \to L$ corresponding to the homomorphism $FK \to L$, $k_{s,n} \mapsto \rho_{s,n}^{-1} f(k)$. Observe that $\ker(K * \xi K \twoheadrightarrow {}_S\sqrt{}_P K) \subset \ker(f * d)$. Hence, f' exists. Uniqueness of f' follows from $f''\rho_{s,n} = \rho_{s,n} f''$, for any $f'' : {}_S\sqrt{}_P K \to L$ with $f = f''t$. ■

Definition 3.3. Let K be any S–group.

$$_S EpK := \varinjlim K \to {}_S\sqrt{}_P K \to \left({}_S\sqrt{}_P\right)^2 K \to \dots$$

By induction, using Lemma 3.2, we get

Proposition 3.4.

(1) *(i) $_S Ep : {}_S\mathcal{G} \to {}_S\mathcal{E}p$ is a covariant functor.*

(2) *(ii) $_S Ep$ preserves epimorphisms.*

(3) *(iii) The canonical homomorphism $\tau : K \to {}_S EpK$ defines a natural transformation of the identity functor on $_S\mathcal{G}$ to $_S Ep$.*

(4) *(iv) If $f : K \to L$ is an S-homomorphism and S acts P-locally on L, then there is a unique S-homomorphism $f' : {}_S EpK \to L$ with $f = f'\tau$.*

To make the functions $\rho_{s,n}$ of an S–group K $(1-1)$, we factor out a suitable subgroup. Let

$$_S apK := \bigcap\{\ker(f : K \to U) : U \in {}_S\mathcal{U}p,\ f \text{ any } S\text{–homomorphism}\}.$$

Definition 3.5. ${}_S UpK := K/{}_S apK$.

It follows that ${}_S UpK \in {}_S\mathcal{U}p$. Further, if $f : K \to K'$ is an S–homomorphism, then $f({}_S apK) \subset {}_S apK'$. So f induces ${}_S Upf :_S UpK \to_S UpK'$.

The lemma below is a direct consequence of this definition.

Lemma 3.6.

(i) *${}_S Up : {}_S\mathcal{G} \to {}_S\mathcal{U}p$ is a covariant functor.*

(ii) *The canonical epimorphism $\sigma : K \twoheadrightarrow {}_S UpK$ defines a natural tranformation of the identity functor on ${}_S\mathcal{G}$ to ${}_S Up$.*

(iii) *${}_S Up$ preserves epimorphisms.*

(iv) *If $f : K \to L$ is an S-homomorphism and $L \in {}_S\mathcal{U}p$, then there is a unique homomorphism $f' : {}_S UpK \to L$ with $f = f'\sigma$. ${}_S Up$ is left adjoint left inverse to the inclusion functor ${}_S\mathcal{U}p \to {}_S\mathcal{G}$.*

Definition 3.7: Let $\gamma : R \to S$ be a group homomorphism. Let ${}_\gamma Lp := {}_S Up\, {}_S Ep\, {}_\gamma Ad$: $\mathcal{G}_R \to_S \mathcal{G}p$ be the composite of the three functors.

Note that the natural transformations associated with ${}_S Up$, ${}_S Ep$, ${}_\gamma Ad$ define a natural transformation $e(=_\gamma ep)$ of the identity functor on ${}_R\mathcal{G}$ to ${}_S Lp$.

Proposition 3.8. *Let $\gamma : R \to S$ be a group homomorphism.*

(i) *${}_S Lp : {}_R\mathcal{G} \to {}_S\mathcal{G}p$ is a covariant functor which is left adjoint to the change-of-operator-groups functor $\gamma^* : {}_S\mathcal{G}p \to {}_R\mathcal{G}$.*

(ii) *${}_S Lp$ preserves epimorphisms.*

Proof: Combine 1.5, 2.2, 3.4, 3.6.
This completes the proof of 1.6. ■

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