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ON THE BASIC CHARACTER OF RESIDUE CLASSES

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Abstract

Let t, b be mutually prime positive integers. We say that the residue class t mod b is *basic* if there exists n such that $t^n \equiv -1$ mod b; otherwise t is *not basic*. In this paper we relate the *basic* character of t mod b to the *quadratic* character of t modulo the prime factors of b. If all prime factors p of b satisfy $p \equiv 3$ mod 4, then t is basic mod b if t is a quadratic non-residue mod p for all such p; and t is not basic mod b if t is a quadratic residue mod p for all such p. If, for all prime factors p of b, $p \equiv 1$ mod 4 and t is a quadratic non-residue mod p, the situation is more complicated. We define $d(p)$ to be the highest power of 2 dividing $(p-1)$ and postulate that $d(p)$ takes the same value for all prime factors p of b. Then t is basic mod b. We also give an algorithm for enumerating the (prime) numbers p lying in a given residue class mod $4t$ and satisfying $d(p) = d$. In an appendix we briefly discuss the case when b is even.

0. Introduction

In a series of papers [1,through 4], culminating in the monograph [5], Hilton and Pedersen developed an algorithm – in fact, two algorithms, one being the reverse of the other – for calculating the quasi–order of t mod b, where t, b are mutually prime positive integers, and determining whether t is basic mod b. Here the *quasi–order* of t mod b is the smallest positive integer k such that $t^k \equiv \pm 1$ mod b; and t is said to be *basic* if, in fact, $t^k \equiv -1$ mod b. Thus t is basic if and only if the order of t mod b is twice the quasi–order of t mod b (in the contrary case the quasi–order and the order coincide). Froemke and Grossman carried the number–theoretical investigation considerably further in [6] and drew attention to the importance, where b is prime, of the quadratic character of t mod b in their arguments.

Our object in this paper is to relate the basic character of t mod b to the quadratic character of t modulo the prime factors of b. We assume b odd, but add a few remarks in an appendix on the case when b is even.

Given a pair (t, p) where p is an odd prime not dividing t, we distinguish 4 possibilities as follows: t may or may not be a quadratic residue mod p, and we may have $p \equiv 1$ mod 4 or $p \equiv 3$ mod 4. We restrict attention, in our discussion of the basic character of t mod b, to the situation in which all prime factors p of b place (t,p) in the same class. If t is a quadratic residue mod p and $p \equiv 1$ mod 4, we are unable to draw any general conclusion about the basic character of t mod p. Thus, for example, $7^7 \equiv 1$ mod 29, $7^7 \equiv -1$ mod 113, and 7 is a quadratic residue modulo 29 or 113. If all prime factors p of b satisfy $p \equiv 3$ mod 4, it is easy to draw general conclusions about the basic character of t mod b; our results are given in Section 2.

The most interesting case for our purposes is that in which t is not a quadratic residue mod p and $p \equiv 1$ mod 4, for all prime factors p of b. It then becomes important to be able to calculate the function $d(p)$, where $d(p) = d$ if $p = 1 + 2^d c$, with c odd. Thus d is a positive integer and, in fact, $d \geq 2$ in the case we are discussing. Since the quadratic character of t mod p depends only on the residue class of p mod $4t$, we give an algorithm for enumerating those primes p, as functions of s and d, such that

$$d(p) = d, \quad p \equiv s \bmod 4t, \quad 1 \leq s \leq 4t-3. \tag{0.1}$$

If the prime factors p of b are confined to those satisfying (0.1) for fixed s, d, then, as we show in Section 3, t is basic mod b.

In Section 1 we announce some elementary results which are used in proving our main theorems.

Throughout the paper we use the symbol ϵ for a number which is $+1$ or -1.

1. Some preliminary lemmas

The first result extends to the quasi-order a familiar result on order.

Lemma 1.1. *Let the quasi-order of t mod b be n and let $t^m \equiv \epsilon$ mod b. Then $n|m$.*

Proof: Let $m = qn + r$, $0 \leq r < n$, and $t^n \equiv \eta$ mod b, $\eta = \pm 1$. Then $t^r = t^m(t^n)^{-q} \equiv \epsilon\eta^q = \pm 1$ mod b, so that $r = 0$.

We now restrict b by the condition $b \geq 3$, so that the basic character of t mod b comes into question. ■

Lemma 1.2. *The residue t mod b is basic if and only if $t^m \equiv -1$ mod b for some exponent m.*

Proof: The necessity of the condition is obvious. Suppose then that $t^m \equiv -1$ mod b and that the quasi-order of t mod b is n. Then $n \mid m$, by Lemma 1.1. Thus, if $t^n \equiv 1$ mod b, it follows that $t^m \equiv 1$ mod b. This contradiction shows that $t^n \equiv -1$ mod b, so that the residue t mod b is basic. ■

Lemma 1.3. *The residue t mod b is non-basic if $t^m \equiv 1$ mod b for some odd exponent m.*

Proof: Let the quasi-order of t mod b be n with $t^n \equiv \epsilon$ mod b. Then $n \mid m$, so that $m = nq$. Since m is odd, q is odd. Thus $t^m \equiv \epsilon^q = \epsilon$ mod b, so $\epsilon = 1$ and the residue t mod b is non-basic. ∎

Our next result is of a different kind.

Proposition 1.4. *Let $x \equiv y$ mod m. Then $x^{m^{k-1}} \equiv y^{m^{k-1}}$ mod m^k, $k \geq 1$.*

Proof: We argue by induction on k, the case $k = 1$ being trivial. If we assume $x^{m^{k-1}} \equiv y^{m^{k-1}}$ mod m^k for a certain $k \geq 1$, then

$$\begin{aligned} x^{m^{k-1}} &= y^{m^{k-1}} + \lambda m^k, \text{ so that} \\ x^{m^k} &= (y^{m^{k-1}} + \lambda m^k)^m \\ &= y^{m^k} + \lambda m^{k+1} y^{m^{k-1}(m-1)} + \binom{m}{2} \lambda^2 m^{2k} y^{m^{k-1}(m-2)} + \dots \\ &\equiv y^{m^k} \bmod m^{k+1}. \end{aligned}$$

This establishes the inductive step, and hence the proposition. ∎

We have the immediate consequence:

Lemma 1.5. *Let $c \equiv \epsilon$ mod m, with m odd. Then $c^{m^{k-1}} \equiv \epsilon$ mod $m^k, k \geq 1$.*

Proof: We have only to note that $\epsilon^{m^{k-1}} = \epsilon$ if m is odd. ∎

2. The main results

We recall the following key results on quadratic reciprocity.

Theorem 2.1 (Euler). *Let p be an odd prime. Then*
(i) $t^{\frac{p-1}{2}} \equiv 1$ mod p if and only if t is a quadratic residue mod p
(ii) $t^{\frac{p-1}{2}} \equiv -1$ mod p if and only if t is not a quadratic residue mod p.

Theorem 2.2 (Gauss). *Let p be an odd prime. Then the quadratic character of t mod p depends only on the residue class of p mod $4t$ and is the same for two odd primes p and q such that $p \equiv -q$ mod $4t$.*

Thus, given t and p, we distinguish 4 classes into which the pair (t,p) may fall:

I $p \equiv 1 \bmod 4$, $t^{\frac{p-1}{2}} \equiv 1 \bmod p$;
II $p \equiv 1 \bmod 4$, $t^{\frac{p-1}{2}} \equiv -1 \bmod p$;
III $p \equiv 3 \bmod 4$, $t^{\frac{p-1}{2}} \equiv 1 \bmod p$;
IV $p \equiv 3 \bmod 4$, $t^{\frac{p-1}{2}} \equiv -1 \bmod p$;

We will say nothing further about residues $t \bmod b$ if b admits a factor p such that (t,p) is in class I. We will henceforth, until otherwise stated, assume that b is *odd.*

Theorem 2.3. *Suppose that the prime factors p of b are all such that (t,p) is in Class III. Then the residue t is not basic mod b.*

Proof: Let $b = \Pi_{i=1}^N p_i^{k_i}$, $k_i \geq 1$. Then $\frac{p_i-1}{2}$ is odd and $t^{\frac{p_i-1}{2}} \equiv 1 \bmod p_i$. By Lemma 1.5, $t^{(\frac{p_i-1}{2})p_i^{k_i-1}} \equiv 1 \bmod p_i^{k_i}$. Set $m = \Pi_{i=1}^N(\frac{p_i-1}{2})p_i^{k_i-1}$. Then m is odd, and

$$t^m \equiv 1 \bmod p_i^{k_i}.$$

It follows that $t^m \equiv 1 \bmod b$, so that, by Lemma 1.3, the residue t is not basic mod b. ∎

Theorem 2.4. *Suppose that the prime factors p of b are all such that (t,p) is in Class IV. Then the residue t is basic mod b.*

Proof: We argue as for Theorem 2.3, except that now

$$t^{(\frac{p_i-1}{2})p_i^{k_i-1}} \equiv -1 \bmod p_i^{k_i},$$
$$t^m \equiv -1 \bmod p_i^{k_i},$$
$$t^m \equiv -1 \bmod b,$$

with m odd. We apply Lemma 1.2 to obtain the result. ∎

Example 2.1. Let $t = 7$. Then, by Theorem 2.2, we must consider primes p mod 28. We easily find

$$p \equiv 1 \text{ or } 27 \bmod 28 : 7^{\frac{p-1}{2}} \equiv 1 \bmod p$$
$$p \equiv 3 \text{ or } 25 \bmod 28 : 7^{\frac{p-1}{2}} \equiv 1 \bmod p$$
$$p \equiv 5 \text{ or } 23 \bmod 28 : 7^{\frac{p-1}{2}} \equiv -1 \bmod p$$
$$p \equiv 9 \text{ or } 19 \bmod 28 : 7^{\frac{p-1}{2}} \equiv 1 \bmod p$$
$$p \equiv 11 \text{ or } 17 \bmod 28 : 7^{\frac{p-1}{2}} \equiv -1 \bmod p$$
$$p \equiv 13 \text{ or } 15 \bmod 28 : 7^{\frac{p-1}{2}} \equiv -1 \bmod p$$

Thus

$(7, p)$ is in Class I if $p \equiv 1, 9, 25 \bmod 28$;
$(7, p)$ is in Class II if $p \equiv 5, 13, 17 \bmod 28$;
$(7, p)$ is in Class III if $p \equiv 3, 19, 27 \bmod 28$;
$(7, p)$ is in Class IV if $p \equiv 11, 15, 23 \bmod 28$.

We conclude that 7 is not basic mod b if b is a product of primes p such that $p \equiv 3, 19$ or $27 \bmod 28$; and 7 is basic mod b if b is a product of primes p such that $p \equiv 11, 15$ or $23 \bmod 28$.

As we have said, no inference can be drawn if b is a product of primes p such that $p \equiv 1, 9$ or $25 \bmod 28$. Indeed, the fact that $(7, p)$ is then in Class I is a special case of the following phenomenon, which we describe here for the sake of completeness.

Proposition 2.5. *Let p be an odd prime such that $p = k^2 + 4l$. Then any factor of l is a quadratic residue mod p.*

Proof: It suffices to prove this for prime factors q of l. Now if $q = 2$, then $p \equiv 1 \bmod 8$, so 2 is a quadratic residue mod p. If q is odd, then p is a quadratic residue mod q and $\frac{p-1}{2}$ is even, so that, by quadratic reciprocity, q is a quadratic residue mod p. ■

Note that it follows, by Theorem 2.1, that $7^{\frac{p-1}{2}} \equiv 1 \bmod p$ if $p \equiv 1, 9$ or $25 \bmod 28$.

We will devote the next section to a discussion of the Class II. At this point, we are content to remark

Theorem 2.6. *Suppose that $b = p^k$, where (t, p) is in Class II. Then the residue t is basic mod b.*

Proof: $t^{(\frac{p-1}{2})p^{k-1}} \equiv -1 \bmod p^k$. Apply Lemma 1.2. ■

In the next section we generalize this obvious conclusion.

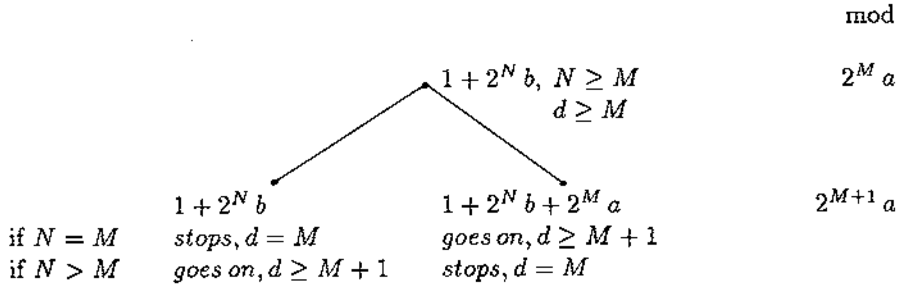
3. The class II situation

We define a function d from positive integers ≥ 2 to non–negative integers by

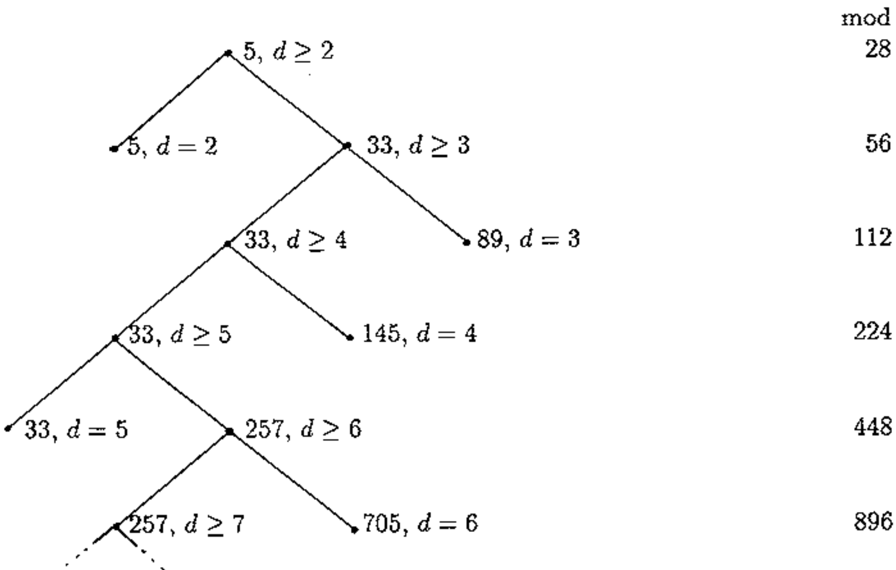
$$d(n) = d \Leftrightarrow 2^d \mid (n-1),\ 2^{d+1} \nmid (n-1). \tag{3.1}$$

Notice that, for an odd prime p, $d(p) \geq 1$ and that $d(p) \geq 2$ if (t, p) is in Class II. Let d be a fixed integer ≥ 2; we then have the following theorem, generalizing Theorem 2.6.

in this way, splitting the inequality $d(p) \geq d$ into the two possibilities $d(p) = d$ or $d(p) \geq d+1$. We demonstrate this tree in Figure 1.



General Case
Figure 1



$p \equiv 5 \bmod 28$
Special Case
Figure 1

The tree provides the conceptual basis for the proof (see Theorem 3.2) that the primes p satisfying the congruence $p \equiv 2^d c + 1 \bmod 2^{d+1}u$ constitute the *totality* of the primes p satisfying $p \equiv s \bmod 4t$ and $d(p) = d$. For we may argue by induction on d that there is only one residue class mod $2^{d+1}u$ containing such primes. We assume $n \geq m+2$ and we rewrite (3.4) as

$$2^d c(d) \equiv 2^n v \bmod u, \tag{3.11}$$

to emphasize the dependence of c on d; recall $d \geq m+2$. We start the induction (and the tree) by rewriting $p \equiv s \bmod 4t$, using (3.11), as

$$p \equiv 1 + 2^{m+2}c(m+2) \bmod 2^{m+2}u. \tag{3.12}$$

Then (3.12) branches into the two congruences

$$\begin{cases} p \equiv 1 + 2^{m+2}c(m+2) \bmod 2^{m+3}u, \\ p \equiv 1 + 2^{m+2}c(m+2) \pm 2^{m+2}u \bmod 2^{m+3}u, \end{cases} \tag{3.13}$$

the sign being chosen so that the right-hand side is in the range $(0, 2^{m+3}u)$. The first possibility in (3.13) yields $d(p) = m+2$ (recall that $c(d)$ is always odd) and the second yields $d(p) \geq m+3$, since u is also odd. Thus our assertion holds in the initial case $d = m+2$.

Now assume inductively that, for $d(p) \geq d$, we require

$$p \equiv 1 + 2^d c(d) \bmod 2^d u, \tag{3.14}$$

for some $d \geq m+2$. Then, as above, we find that, for $d(p) = d$, we require

$$p \equiv 1 + 2^d c(d) \bmod 2^{d+1}u; \tag{3.15}$$

while, for $d(p) \geq d+1$, we require

$$p \equiv 1 + 2^d c(d) \pm 2^d u \bmod 2^{d+1}u. \tag{3.16}$$

This shows the uniqueness of the residue class mod $2^{d+1}u$ of p, given $d(p) = d$. But it also shows that, if $c(d+1)$ is to be odd, to satisfy the inequality $1 \leq c(d+1) \leq 2u - 1$, and to render $p \equiv 1 + 2^{d+1}c(d+1) \bmod 2^{d+1}u$ equivalent to (3.16), then $c(d+1)$ is determined by

$$c(d+1) = \begin{cases} \frac{1}{2}(c(d)+u) \text{ if } & \frac{1}{2}(c(d)+u) \text{ is odd} \\ \frac{1}{2}(c(d)-u) \text{ if } & \frac{1}{2}(c(d)-u) \text{ is odd and positive} \\ \frac{1}{2}(c(d)+3u) \text{ if } & \frac{1}{2}(c(d)-u) \text{ is odd and negative} \end{cases} \tag{3.17}$$

Thus, in any case, $2c(d+1) \equiv c(d) \bmod u$, so that, if $2^d c(d) \equiv 2^n v \bmod u$, then $2^{d+1}c(d+1) \equiv 2^n v \bmod u$. This establishes the inductive step and also gives us a recurrence relation (3.17) for determining $c(d)$. Of course, this recurrence relation is deducible from

$$2c(d+1) \equiv c(d) \bmod u, \tag{3.18}$$

which also shows why the period of $c(d)$ is the order of 2 mod u.

We emphasize that (3.17), together with the initial congruence $c(m+2) \equiv 2^{n-m-2}v \bmod u$, gives a practical algorithm for determining the values $c(d)$

–recall that $c(d)$ is odd with $1 \le c(d) \le 2u - 1$. We then apply Theorem 3.2 to determine the primes p in a given residue class mod $4t$ and satisfying $d(p) = d$.

Example 3.2. Let $t = 11$, $s = 5$, so that $p \equiv 5 \bmod 44$. Thus

$$s = 5,\ u = 11,\ m = 0$$
$$t = 11,\ v = 1,\ n = 2$$

To calculate $c(d)$ we start the induction with $c(2) \equiv 1 \bmod 11$, so $c(2) = 1$. Now l, the order of 2 mod 11, is 10, so the period of $c(d)$ is 10, and (3.17) yields the table

d	2	3	4	5	6	7	8	9	10	11
c	1	17	3	7	9	21	5	19	15	13

The tree diagram is shown in Figure 2.

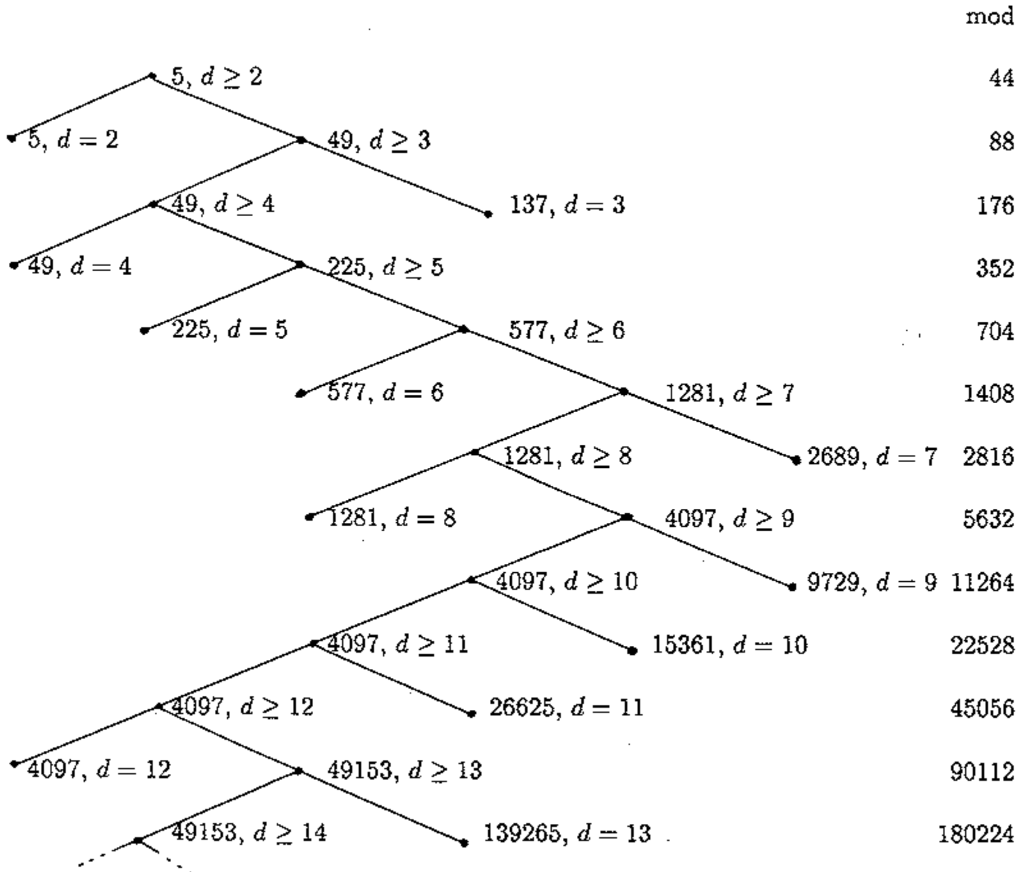


Figure 2

Notice that not only does $c(d)$ repeat from $d = 12$ onwards, but that the whole tree pattern repeats.

4. Appendix

We introduce the equivalence relation $t \sim t'$ in the set of positive integers, defined by

$$t \sim t' \Leftrightarrow t/t' \quad \text{is the square of a rational number.} \tag{4.1}$$

It is then plain that t is a quadratic residue mod p if and only if t' is a quadratic residue mod p (where $t \sim t'$ and t, t' are both prime to p). This shows that the class, in the sense of Section 2, to which (t, p) belongs depends only on the equivalence class of t. Thus our conclusions embodied in Theorems 2.3, 2.4, 2.6 and 3.1 apply to entire equivalence classes of integers t. Moreover, it follows that, insofar as we only exploit these results, we may assume that t is *square-free*. This has the important effect that, in applying the techniques of Section 3 to provide an explicit description of those primes p such that (t, p) is in class II and $d(p) = d$, we may, in practice, confine attention to $m = 0$ or 1. Of course, the formal analysis for $t = 12$, say, is different from that for $t = 3$, but the conclusions are coextensive–and the same!

Indeed, so far as the methods of this paper are concerned, we may really confine ourselves to the case that t is itself a prime. For it is trivial to derive the quadratic character of t from the quadratic characters of its prime factors; and our deductions are exclusively based on the quadratic character of t modulo the prime factors of b. Notice that we are far from saying that the basic character of t mod b can be deduced from that of the prime factors of t–just as we do not claim that, *in general*, the basic character of t mod b depends only on the equivalence class of t under the equivalence relation (4.1). For example, 4 is basic mod 17 but 1 is not. It remains to make a remark if b is even. We do not attempt a careful analysis of this case, but we point out the following

Proposition 4.1. *Let t, b be mutually prime odd numbers. Then the basic character of t mod b coincides with the basic character of t mod $2b$.*

Proof: Let the quasi-order of t mod b be n, and let $t^n \equiv \epsilon$ mod b. Since $t^n - \epsilon$ is even, it follows that $t^n \equiv \epsilon$ mod $2b$. It next follows that n is the quasi–order of t mod $2b$; for the quasi–order of t mod $2b$ is seen to be neither less than nor greater than the quasi–order of t mod b. This proves the proposition. ∎

Finally, we analyse the basic character of t mod 2^n, $n \geq 2$; of course, t is then odd.

Theorem 4.2. *Let $d(t) = q \geq 2$. Then the quasi–order of t mod 2^n is*

$$\begin{array}{ccc} 1 & \textit{if} & n \leq q \\ 2^{n-q} & \textit{if} & n > q. \end{array}$$

Moreover, t is not basic mod 2^n.

Proof: We have $t = 1 + c2^q$, with c odd. The conclusion is obvious if $n \leq q$. Let $n > q$. Now since

$$(t^{2^{r-1}} - 1)(t^{2^{r-1}} + 1) = t^{2^r} - 1, \ r \geq 1$$

it follows by an easy inductive argument on r that

$$d(t^{2^r}) = q + r, \ r \geq 0; \tag{4.2}$$

plainly

$$t^{2^r} + 1 \equiv 2 \bmod 4, \ r \geq 0. \tag{4.3}$$

Thus

$$t^{2^{n-q}} \equiv 1 \bmod 2^n,$$

while

$$t^{2^{n-q-1}} \not\equiv \pm 1 \bmod 2^n,$$

establishing the theorem. ∎

We can also handle the case $q = 1$. Thus let us suppose $d(t) = 1$, so that

$$t = 1 + 2c, \ \text{with } c \text{ odd.}$$

We write

$$t = -1 + 2^{q'}c', \ \text{with } c' \text{ odd}; \tag{4.4}$$

notice that $q' \geq 2$.

Theorem 4.3. *If t is given by (4.4), with $q' \geq 2$, then*
(i) *if $n \leq q'$, the quasi-order of t mod 2^n is 1 and t is basic;*
(ii) *if $n > q'$, the quasi-order of t mod 2^n is $2^{n-q'}$ and t is not basic.*

Proof: (i) is obvious. Thus we suppose $n > q'$. As before, we exploit the identity

$$(t^{2^{r-1}} - 1)(t^{2^{r-1}} + 1) = t^{2^r} - 1,$$

but now only for $r \geq 2$. For we deduce from (4.4) that $t^2 = 1 + 2^{q'+1}c''$, with c'' odd.
Thus

$$d(t^{2^r}) = q' + r, \ r \geq 1,$$

and

$$t^{2^r} + 1 \equiv 2 \bmod 4, \ r \geq 1.$$

This shows that

$$t^{2^{n-q'}} \equiv 1 \bmod 2^n,$$

$$t^{2^{n-q'-1}} \not\equiv \pm 1 \bmod 2^n,$$

establishing the theorem.

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