Publicacions Matemàtiques, Vol 33 (1989), 123–132.

ON SEMIFIR MONOID RINGS

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Abstract

We give a new condition on a monoid M for the monoid ring $F[M]$ to be a 2-fir. Futhermore, we construct a monoid $\mathcal{M}$ that satisfies all the currently known necessary conditions for $F[\mathcal{M}]$ to be a semifir and that the group of units of $\mathcal{M}$ is trivial, but $\mathcal{M}$ is not a directed union of free monoids.

Introduction

Let R be a ring. Recall that R is a fir if every right ideal of R is free of unique rank. For a positive integer m, the ring R is said to be an m-fir if all its m-generator right ideals are free of unique rank. R is a semifir if it is an m-fir for all positive integer m.

There is a conjecture of Dicks which states that if R is a ring and M is a non-trivial monoid then the monoid ring $R[M]$ is a semifir if and only if R is a skew field and M is a directed union of free products of free groups and free monoids (see [4], [5], [6]). The latter condition is known to be sufficient.

Recall that a monoid M is rigid if it is a cancellation monoid such that for $x, y \in M$, $xM \cap yM \neq \emptyset$ implies $xM \subset yM$ or $yM \subset xM$. A monoid M is conical provided that for $a, b \in M$, $ab = 1$ implies $a = b = 1$.

Menal in [6] shows that if R is a ring, M is a non-trivial monoid and $R[M]$ is a 2-fir, then

(a) R is a skew field,
(b) M is a rigid monoid, and
(c) for $a, b, c \in M$, if a is a nonunit and $a = bac$, then $b = c = 1$.

In Proposition 2, we obtain a new condition and its left-right dual:

(d) For $a, b, c, d \in M$, if $ab = cad$, then either $c = 1$ or there exist $n \in \mathbf{Z}$ and $b' \in M$ such that $ab' = c^n$.
(d') For $a, b, c, d \in M$, if $ba = cad$, then either $d = 1$ or there exist $m \in \mathbf{Z}$ and $b'' \in M$ such that $b''a = d^m$.

This work was partially supported by the grant PB86-0353-C02-01 of the CICYT. 1980 Mathematics Subject Clasification. Primary 20M25, 16A06.

Dicks and Schofield in [**4**] show that if F is a skew field and M is a monoid such that $F[M]$ is a semifir then the universal group of M is locally free.

In [**1**], we ask the following question: *Let M be a rigid monoid such that for $x, y, z \in M$, $x = yxz$ implies $y = z = 1$. Let K be a skew field. Is $K[M]$ a 2-fir?*

We construct a monoid (Example 4) satisfying (b), (c), with free universal group of rank two, which does not satisfy (d). Furthermore, this monoid is conical, so we answer in the negative the above question.

We also construct a conical rigid monoid $\mathcal{M}$ satifying (c), (d), (d'), with free universal group of rank two, such that it is not a directed union of free monoids. I do not know whether $\mathcal{M}$ is a counterexample to Dicks' conjecture.

I am indebted to Warren Dicks for the present format of the paper.

Main results

The following example answers in the negative the above question of [**1**].

Example 1. There exists a conical rigid monoid M with universal group $\mathbf{Z} \times \mathbf{Z}$, such that the monoid ring $R[M]$ is not a 2-fir for any ring R.

Proof: Let $\{a_n\}$ be the Fibonacci sequence, i.e. $a_0 = a_1 = 1$ and $a_{n+2} = a_{n+1} + a_n$ for all $n \geq 0$. Consider the following elements of $\mathbf{Z} \times \mathbf{Z}$:

$$\begin{aligned} x_0 &= (1,0), \;\; y_0 = (0,1), \\ x_i &= (a_{2i-2}, -a_{2i-1}), \\ y_i &= (-a_{2i-1}, a_{2i}), \end{aligned}$$

for all $i > 0$. Let M be the submonoid of $\mathbf{Z} \times \mathbf{Z}$ generated by $\{x_i, y_i \mid i \geq 0\}$.

Given $i \geq 0$, it is easy to show, by induction on k, that

$$\begin{aligned} x_i &= a_{2k} x_{i+k} + a_{2k-1} y_{i+k}, \\ y_i &= a_{2k-1} x_{i+k} + a_{2k-2} y_{i+k}, \end{aligned} \tag{1}$$

for all $k > 0$.

In order to prove that M is conical, it is sufficient to see that, for all $i > 0$, the unique solution in $\mathbf{Z}$ of the equation

$$nx_i + my_i = 0 \tag{2}$$

is $n = m = 0$. The equation (2) is equivalent to the system

$$\left.\begin{aligned} na_{2i-2} &= ma_{2i-1} \\ na_{2i-1} &= ma_{2i} \end{aligned}\right\}$$

Now $ma_{2i} = ma_{2i-1} + ma_{2i-2} = (m+n)a_{2i-2}$, so we have

$$\left.\begin{aligned} ma_{2i-1} &= na_{2i-2} \\ na_{2i-1} &= (m+n)a_{2i-2} \end{aligned}\right\}$$

and so we obtain $n^2 - nm - m^2 = 0$. Thus $n = m = 0$ is the unique solution in $\mathbf{Z}$ of (2).

In order to prove that M is rigid, it is sufficient to see that, for each $(z_1, z_2) \in \mathbf{Z} \times \mathbf{Z}$, either $(z_1, z_2) \in M$ or $(-z_1, -z_2) \in M$. Since $(1,0), (0,1) \in M$, we may assume $z_1 < 0 < z_2$. By (1), $(z_1, z_2) = z_1(a_{2k}x_k + a_{2k-1}y_k) + z_2(a_{2k-1}x_k + a_{2k-2}y_k)$ for all $k > 0$. Since $\lim_{n\to\infty} a_{n+1}/a_n = (1+5^{1/2})/2$, it is easy to see that, if $z_2 > -z_1(1+5^{1/2})/2$, then $(z_1, z_2) \in M$, and if $z_2 < -z_1(1+5^{1/2})/2$, then (b) $(-z_1, -z_2) \in M$.

Therefore M is a conical rigid monoid with universal group $\mathbf{Z} \times \mathbf{Z}$. Let R be a non-zero ring. It is clear that $(1-x_0)R[M] \cap (1-y_0)R[M] \neq 0$. Suppose that $R[M]$ is a 2-fir. Then $(1-x_0)R[M] + (1-y_0)R[M]$ is principal and hence $\omega(R[\mathbf{Z} \times \mathbf{Z}])$ is right principal. By [**6**, Lemma 1], R is a skew field. By [**7**, Lemma 13.1.6 and pag. 591], $R[\mathbf{Z} \times \mathbf{Z}]$ is a domain. Thus $\omega(R[\mathbf{Z} \times \mathbf{Z}])$ is free as right $R[\mathbf{Z} \times \mathbf{Z}]$-module, but $\mathbf{Z} \times \mathbf{Z}$ is not free and this contradicts the Stalligs-Swan Theorem [**2**, Theorem A]. Therefore $R[M]$ is not a 2-fir. ■

Proposition 2. *Let R be a ring and let M be a monoid. Let $a, b, c, d \in M$. Suppose that $R[M]$ is a 2-fir.*

(i) *If $ab = cad$, then either $c = 1$ or there exist $n \in \mathbf{Z}$ and $b' \in M$ such that $ab' = c^n$.*

(ii) *If $ba = cad$, then either $d = 1$ or there exist $m \in \mathbf{Z}$ and $b'' \in M$ such that $b''a = d^m$.*

Proof: By [**3**, Theorem 1.1.1], we need only show (i).

Suppose that $ab = cad$. Thus $a(d-b) = (1-c)ad$. If $c \neq 1$ then $aR[M] \cap (1-c)R[M] \neq 0$. Since $R[M]$ is a 2-fir, $aR[M] + (1-c)R[M]$ is right principal say $\alpha R[M]$. Because $1 - c \in \alpha R[M]$, the support of α contains a unit. On the other hand $a = \alpha\beta$, for some $\beta \in R[M]$, and, by [**6**, Lemma 2], α is a unit. Hence $a\gamma + (1-c)\delta = 1$, for some $\gamma, \delta \in R[M]$. Now let N be the cyclic submonoid (subgroup, if c is a unit) of M generated by c and consider the left $R[N]$-linear map $\pi: R[M] \to R[N]$ defined by $\pi(\sum_{x\in M} r_x x) = \sum_{x \in N} r_x x$. Then we have $1 = \pi(a\gamma) + (1-c)\pi(\delta)$ and since $1 - c$ is not invertible in $R[M]$ we get $\pi(a\gamma) \neq 0$. Hence there exists $b' \in M$ such that $ab' = c^n$, for some integer n. ■

Lemma 3. *Let M be a rigid monoid. Suppose that for $a, b, c, d \in M$,*

(d) *if $ab = cad$, then either $c = 1$ or there exist $n \in \mathbf{Z}$ and $b' \in M$ such that $ab' = c^n$,*

(d') *if $ba = cad$, then either $d = 1$ or there exist $m \in \mathbf{Z}$ and $b'' \in M$ such that $b''a = d^m$.*

If a is a nonunit and $a = cad$ then $c = d = 1$.

Proof: By (d) and (d'), we may assume that c and d are nonunits. Let n be the least positive integer such that there exists $b' \in M$ with $ab' = c^n$. Since M is rigid, there exists $b'' \in M$ such that $a = c^{n-1}b''$ and $b''b' = c$. Hence $b'' = cb''d$. Now $b''b' = cb''db'$, thus $c = cb''db'$ and so $1 = b''db'$, but d is a nonunit, a contradiction, therefore $c = d = 1$. ■

Example 1, suggests the following question: *Let M be a rigid monoid such that for $x, y, z \in M$, $x = yxz$ implies $y = z = 1$ and its universal group is locally free. Let K be a skew field. Is $K[M]$ a 2-fir?* We shall see below that the answer is no.

Let H be a free group on the set S. Recall that every element $h \in H$ can be written uniquely as a finite product of the form $h = x_{i_1}^{n_1} x_{i_2}^{n_2} \dots x_{i_r}^{n_r}$, with $x_{i_j} \in S$, $n_j \neq 0$ and $i_j \neq i_{j+1}$. We denote by $l(h) = |n_1| + |n_2| + \cdots + |n_r|$ the *S-length* of h. We call $x_{i_1}^{n_1} x_{i_2}^{n_2} \dots x_{i_r}^{n_r}$ the *reduced S-form* of h.

Example 4. There exists a rigid monoid M such that for $a, b, c \in M$, $a = bac$ implies $b = c = 1$ and its universal group is free of rank two, but $R[M]$ is not a 2-fir for any ring R.

Proof: Let G be the free group on $S_0 = \{x, y\}$. Let φ denote the automorphism of G with $\varphi(x) = y^{-1}x$ and $\varphi(y) = y$. For each $i \in \mathbf{N}$, write $x_i = \varphi^i(x)$, and let $z = y^{-1}x^{-1}yx$. It is easy to see that $\varphi(z) = z$. Note that G is free on $S_i = \{x_i, y\}$. If $g \in G$ we denote by $l_i(g)$ the S_i-length of g.

Let M be the submonoid of G generated by $\{x_i, y, z \mid i \in \mathbf{N}\}$; it can be shown that $\{x_i yz = yx_i,\ x_i = yx_{i+1} \mid i \in \mathbf{N}\}$ is a complete set of defining relations. It is clear that G is the universal group of M. Let M_i denote the submonoid of M generated by $\{x_i, y, z\}$.

Note that if $a \in M_i \setminus \{1\}$, then its reduced S_{i+1}-form is

$$y^{n_1} x_{i+1}^{\varepsilon_1} y^{n_2} x_{i+1}^{\varepsilon_2} \cdots y^{n_j} x_{i+1}^{\varepsilon_j}, \tag{3}$$

where

(i) $n_1 = 0 \Rightarrow \varepsilon_1 = -1$,
(ii) $n_1, \dots, n_j \geq -1$ and $n_2, \dots, n_j \neq 0$,
(iii) $\varepsilon_j = 0, 1$ and $\varepsilon_1, \dots, \varepsilon_{j-1} = \pm 1$,
(iv) $\varepsilon_k = -1 \Rightarrow n_{k+1}, \varepsilon_{k+1} > 0$,
(v) $n_k = -1 \Rightarrow \varepsilon_k = -1$.

Now it is easy to see that the group of units of M is trivial. Note also that if an element a of G has the reduced S_{i+1}-form (3) with the properties (i)-(v), then $a \in M_{i+1}$.

Lemma 4.1. *Let $a, b \in M_i$ and $c = b^{-1}a$. If $l_{i+1}(bc) = l_{i+1}(b) + l_{i+1}(c)$ then $c \in M_{i+1}$.*

Proof: Examine the reduced S_{i+1}-form of c. ■

Lemma 4.2. *M is rigid.*

Proof: Let $a, b \in M$. Suppose that $aM \cap bM \neq \emptyset$. Let $c, d \in M$ such that $ac = bd$. We shall see that either $aM \subset bM$ or $bM \subset aM$. Clearly, we may assume that a, b, c, d are nonunits. Suppose that $a, b, c, d \in M_i$. By Lemma 4.1, we may assume that $l_{i+1}(b) \geq l_{i+1}(a) - 2$. Hence, either there exist $a_1, a', b' \in G$ such that $a = a_1 a'$ and $b = a_1 b'$, with $l_{i+1}(a) = l_{i+1}(a_1) + l_{i+1}(a')$, $l_{i+1}(b) = l_{i+1}(a_1) + l_{i+1}(b')$ and $l_{i+1}(a') = 2$, or $a = y$.

Case 1. $a = y$.

It is easy to see that one of the following cases occurs:

(i) $b = 1$ and $a = by$,
(ii) $b = yb''$, with $b'' \in M_{i+1}$, $l_{i+1}(b) = l_{i+1}(b'') + 1$, and $b = ab''$,
(iii) $b = x_{i+1}^{-1} y^n x_{i+1} b''$, with $b'' \in M_{i+1}$, $n > 0$, $l_{i+1}(b) = l_{i+1}(b'') + 2 + n$, and $b = az(yz)^{n-1} b''$.

Case 2. $a' = yx_{i+1}$.

Then one of the following cases occurs:

(i) $b' = 1$ and $a = byx_{i+1}$,
(ii) $b' = y$ and $a = bx_{i+1}$,
(iii) $b' = yx_{i+1} b''$, with $b'' \in M_{i+1}$, $l_{i+1}(b') = l_{i+1}(b'') + 2$, and $ab'' = b$,
(iv) $b' = x_{i+1} y$ and $a = bz$.

Case 3. $a' = y^2$.

Then one of the following cases occurs:

(i) $b' = 1$ and $a = by^2$,
(ii) $b' = y$ and $a = by$,
(iii) $b' = y^2 b''$, with $b'' \in M_{i+1}$, $l_{i+1}(b') = l_{i+1}(b'') + 2$, and $ab'' = b$,
(iv) $b' = yx_{i+1}^{-1} y^n x_{i+1} b''$, with $b'' \in M_{i+1}$, $l_{i+1}(b') = l_{i+1}(b'') + n + 3$, and $a(zy)^{n-1} zb'' = b$,
(v) $b' = yx_{i+1} b''$, with $b'' \in M_{i+1}$, $l_{i+1}(b') = l_{i+1}(b'') + 2$, and $ax_{i+2} b'' = b$,
(vi) $b' = x_{i+1} b''$, with $b'' \in M_{i+1}$, $l_{i+1}(b') = l_{i+1}(b'') + 1$, and $ax_{i+3} b'' = b$.

Case 4. $a' = x_{i+1} y$.

Then one of the following cases occurs:

(i) $b' = 1$ and $a = bx_{i+1} y$,
(ii) $b' = x_{i+1}$ and $a = by$,
(iii) $b' = x_{i+1} y b''$, with $b'' \in M_{i+1}$, $l_{i+1}(b') = l_{i+1}(b'') + 2$, and $ab'' = b$,
(iv) $b' = y^n x_{i+1} b''$, with $b'' \in M_{i+1}$, $l_{i+1}(b') = l_{i+1}(b'') + n + 1$, and $a(zy)^{n-1} zb'' = b$,
(v) $b' = y^n$ and $a = bx_{i+n+1} y$.

Thus M is rigid. ■

Lemma 4.3. *For $a, b, c \in M$, $a = bac$ implies $b = c = 1$.*

Proof: Suppose that $a, b, c \in M_i$ and $b \neq 1 \neq c$. Clearly $a \neq 1$. If $a \neq y^n$, then $l_{i+3}(a) \geq 4$. The same is true for b and c. Now $l_{i+3}(a) = l_{i+3}(bac) \geq$

$l_{i+3}(b)+l_{i+3}(a)+l_{i+3}(c)-8$, so the only possibilities for b and c are: y, y^2, y^3, y^4 and z. It is easy to see that all these cases give a contradiction, and the lemma follows. ■

Lemma 4.4. *$R[M]$ is not a 2-fir for any ring R.*

Proof: We have $x_0yz = yx_0$. By Proposition 2, if $R[M]$ is a 2-fir, then there exist $n \in \mathbf{Z}$ and $b' \in M$ such that $x_0b' = y^n$, but it is easy to see that this is not possible, thus $R[M]$ is not a 2-fir. ■

Now we shall construct a conical rigid monoid $\mathcal{M}$ satifying the conditions (c), (d), (d') of the introduction, with free universal group of rank two, such that it is not a directed union of free monoids.

Let G be the free group on $S_0 = \{r, t\}$. Let φ denote the automorphism of G with $\varphi(r) = t^{-1}r^2$, $\varphi(t) = r^{-1}t$. For each $i \in \mathbf{N}$, write $r_i = \varphi^i(r)$, $t_i = \varphi^i(t)$, and let $x = r^{-1}t^{-1}rt$. It is easy to see that $\varphi(x) = x$. Note that G is free on $S_i = \{r_i, t_i\}$. If $g \in G$ we denote by $l_i(g)$ the S_i-length of g.

Let $\mathcal{M}$ be the submonoid of G generated by $\{r_i, t_i, x \mid i \in \mathbf{N}\}$; it is easy to see that $\{t_ir_ix = r_it_i,\ r_i = t_{i+1}r_{i+1},\ t_i = t_{i+1}r_{i+1}t_{i+1} \mid i \in \mathbf{N}\}$ is a complete set of defining relations. It is clear that G is the universal group of $\mathcal{M}$. Let $\mathcal{M}_i$ denote the submonoid of $\mathcal{M}$ generated by $\{r_i, t_i, x\}$. By the relation $t_ir_ix = r_it_i$, it is easy to see that $\mathcal{M}_i$ is not a submonoid of a free monoid. So $\mathcal{M}$ is not a directed union of free monoids.

Remark. Let $a \in \mathcal{M} \setminus \{1\}$. There exists $i \geq 0$ such that $a \in \mathcal{M}_i$. Since $r_ix = t_{i+1}r_{i+1}x = r_{i+1}t_{i+1}$, $r_i^n = (t_{i+i}r_{i+1})^n$ and $t_i^n = (t_{i+1}r_{i+1}t_{i+1})^n$, we have that if $n \geq 3$ then t_{i+1}^n, r_{i+1}^n and $r_{i+1}x$ not can appear in the reduced S_{i+1}-form of a. Furthermore, if $n \geq 3$, then t_{i+2}^n, r_{i+2}^{n-1} and $r_{i+2}x$ can not appear in the reduced S_{i+2}-form of a. It is clear that the reduced S_{i+2}-form of a finishes with

$$\ldots t_{i+2}r_{i+2} \quad \text{or}$$
$$\ldots r_{i+2}t_{i+2}\,,$$

and it begins with

$$t_{i+2}r_{i+2}t_{i+2}\ldots \quad \text{or}$$
$$x\ldots$$

Now it is clear that a is a nonunit in $\mathcal{M}$. So the group of units of $\mathcal{M}$ is trivial.

The next lemma is an easy exercise.

Lemma 5. *Let $a, b \in \mathcal{M}_i$ and $c = b^{-1}a$. If $l_i(bc) = l_i(b) + l_i(c)$ then $c \in \mathcal{M}_i$.* ■

Lemma 6. *$\mathcal{M}$ is a rigid monoid.*

Proof: Let $a, b \in \mathcal{M}$. Suppose that $a\mathcal{M} \cap b\mathcal{M} \neq \emptyset$. Let $c, d \in \mathcal{M}$ such that $ac = bd$. We shall see that either $a\mathcal{M} \subset b\mathcal{M}$ or $b\mathcal{M} \subset a\mathcal{M}$. Clearly we may assume that a, b, c, d are nonunits in $\mathcal{M}$. By the Remark, there exists $i \geq 0$ such that $a, b, c, d \in \mathcal{M}_i$ and the reduced S_i-forms of a, b, c, d finish with

$$...t_i r_i \quad \text{or}$$
$$...r_i t_i\,,$$

and they begin with

$$t_i... \quad \text{or}$$
$$x...$$

By Lemma 5, we may assume that $l_i(b) \geq l_i(a) - 2$. Hence there exists $a_1, a', b' \in G_i$ such that $a = a_1 a'$ and $b = a_1 b'$, with $l_i(a) = l_i(a_1) + l_i(a')$, $l_i(b) = l_i(a_1) + l_i(b')$ and $l_i(a') = 2$.

Case 1. $a' = r_i t_i$.

Now $l_i(ac) = l_i(a) + l_i(c)$ and it is easy to see that one of the following cases occurs:

(i) $b' = 1$ and $a = br_i t_i$,
(ii) $b' = r_i$ and $a = bt_i$,
(iii) $b' = r_i t_i b''$, with $b'' \in \mathcal{M}_i$, and $ab'' = b$,
(iv) $b' = t_i r_i$ and $a = bx$.

Case 2. $a' = t_i r_i$.

If the reduced S_i-form of c begins with t_i, then one of the following cases occurs:

(i) $b' = 1$ and $a = bt_i r_i$,
(ii) $b' = t_i$ and $a = br_i$,
(iii) $b' = t_i r_i b''$, with $b'' \in \mathcal{M}_i$, and $ab'' = b$.

If the reduced S_i-form of c begins with x, then the reduced S_i-form of $a'c$ begins with $r_i t_i$. Now one of the following cases occurs:

(i) $b' = 1$ and $a = bt_i r_i$,
(ii) $b' = r_i$ and $a = bt_{i+1} r_i$,
(iii) $b' = r_i t_i b''$, with $b'' \in \mathcal{M}_i$, and $axb'' = b$,
(iv) $b' = t_i r_i$ and $a = b$.

Therefore $\mathcal{M}$ is rigid. ∎

Let $r_i^{n_1} t_i^{m_1} \ldots r_i^{n_k} t_i^{m_k}$ be the reduced S_i-form of $c \in \mathcal{M}_i$. We define $LR_i(c) = \sum_{n_j > 0} n_j$ and $LT_i(c) = \sum_{m_j > 0} m_j$. It is clear that $LR_i(c) + LT_i(c) \leq l_i(c)$.

Lemma 7. *Let $a, b, c, d \in \mathcal{M}$. Suppose that $ab = cad$. Then either $c = 1$ or there exists $n \in \mathbf{Z}$ and $b' \in \mathcal{M}$ such that $ab' = c^n$.*

Proof: Suppose that $a, b, c, d \in \mathcal{M}_i$ and $c \neq 1$. Since $\mathcal{M}$ is rigid and $ab = cad$, we may assume that there exists $a_1 \in \mathcal{M}$ such that $a = ca_1$. Note that if there exists $a_n \in \mathcal{M}$ such that $a = c^n a_n$, then $c^n a_n b = c^{n+1} a_n d$, hence $a_n b = ca_n d$. Since $\mathcal{M}$ is rigid, either there exists $b' \in \mathcal{M}$ such that $a_n b' = c$ or there exists

$a_{n+1} \in \mathcal{M}$ such that $a_n = ca_{n+1}$. If $a_n b' = c$, then $ab' = c^{n+1}$. If $a_n = ca_{n+1}$, then $a = c^{n+1}a_{n+1}$. So we may assume that, for each positive integer m, there exists $a_m \in \mathcal{M}$ such that $a = c^m a_m$.

By the Remark, $LR_{i+2}(c^m) \geq m$. Let m be a positive integer such that $m > 2l_{i+2}(a)$. Now $LR_{i+3}(a) \leq l_{i+2}(a)$, $LT_{i+3}(a) \leq LR_{i+2}(a) + 2LT_{i+2}(a) \leq 2l_{i+2}(a)$, $LR_{i+3}(c^m) \geq m$ and $LT_{i+3}(c^m) \geq m$. Thus

$$\begin{aligned} &LR_{i+3}(c^m) \geq m > LR_{i+3}(a) + l_{i+2}(a) \\ &LT_{i+3}(c^m) \geq m > LT_{i+3}(a). \end{aligned}$$

It is easy to see that

$$\begin{aligned} &LR_{i+k}(c^m) > LR_{i+k}(a) + l_{i+2}(a) \\ &LT_{i+k}(c^m) > LT_{i+k}(a), \end{aligned}$$

for all integer $k \geq 3$. But there exists $k \geq 3$ such that $a_m \in \mathcal{M}_{i+k}$ and $a = c^m a_m$, hence $l_{i+k}(a) = l_{i+k}(c^m a_m) \geq l_{i+k}(c^m)$, a contradiction. ■

Lemma 8. *Let $a, b, c, d \in \mathcal{M}$. Suppose that $ba = cad$. Then either $d = 1$ or there exists $n \in \mathbf{Z}$ and $b' \in \mathcal{M}$ such that $b'a = d^n$.*

Proof: Similar to the proof of Lemma 7. ■

Let F be a skew field. We shall see that $\mathcal{M}$ satisfies other known necessary conditions for $F[\mathcal{M}]$ to be a semifir.

Lemma 9. *$F[\mathcal{M}]$ is a domain.*

Proof: $F[\mathcal{M}] \subset F[G]$ and $F[G]$ is a domain by [**7**, Corollary 13.2.8 and pag. 591]. ■

Lemma 10. *$\mathcal{M}$ is locally atomic.*

Proof: See the proof of [**6**, Lemma 5]. ■

Lemma 11. *Let $c \in G$. Suppose that there exists $n \geq 1$ such that $c^n \in \mathcal{M}_i$. Then $c \in \mathcal{M}_i$.*

Proof: Suppose that $c \neq 1$. There exists $i \geq 0$ such that $c^n \in \mathcal{M}_i$. There exists $d, c' \in G_i$ such that $c' \neq 1$,

$$\begin{aligned} &c = d^{-1}c'd, \\ &l_i(c) = l_i(d^{-1}) + l_i(c') + l_i(d) \quad \text{and} \\ &l_i((c')^2) = 2l_i(c'). \end{aligned}$$

Now $c^n = d^{-1}(c')^n d$ and clearly we have $l_i(c^n) = l_i(d^{-1}) + nl_i(c') + l_i(d)$. Since $c^n \in \mathcal{M}_i$, there are only three possible values for d.

Case 1. $d = 1$.

In this case $l_i(c^n) = nl_i(c)$, and it is easy to see that $c \in \mathcal{M}_i$.

Case 2. $d = r_i$.

In this case the reduced S_i-form of c' begins with $t_i^{-1}r_it_i$ and it does not finish in r_i^{-1}, t_i^{-1} or $t_i^{-1}r_i^n\,(n \geq 1)$. Hence it is easy to see that $c' \in \mathcal{M}_i$ and therefore $c \in \mathcal{M}_i$.

Case 3. $d = t_ir_i$.

In this case the reduced S_i-form of c' begins with r_it_i, and it is easy to see that $c \in \mathcal{M}_i$. ■

Lemma 12. *Let $a, b \in \mathcal{M}\setminus\{1\}$. Suppose that $(1-a)F[\mathcal{M}]\cap(1-b)F[\mathcal{M}] \neq 0$. Then there exists $c \in \mathcal{M}$ such that $(1-a)F[\mathcal{M}] + (1-b)F[\mathcal{M}] = (1-c)F[\mathcal{M}]$.*

Proof: We have $(1-a)F[G] \cap (1-b)F[G] \neq 0$. Since G is a free group, by [**7**, Corollary 10.3.7(iv)], it is easy to see that the subgroup H of G generated by $\{a, b\}$ is an infinite cyclic group. Thus there exists $c \in H$ and $n, m \geq 1$ such that $H = \langle c\rangle$, $a = c^n$ and $b = c^m$. By Lemma 11, $c \in \mathcal{M}$. Now we have

$$1 - a = (1-c)(1 + c + \cdots + c^{n-1}),$$
$$1 - b = (1-c)(1 + c + \cdots + c^{m-1}).$$

Since $H = \langle c\rangle$, we have $(n, m) = 1$. Suppose that $m > n$. There exists $m_1, c_1 > 0$ such that $m = c_1n + m_1$ and $m_1 < n$. Now $1 - b - (1-a)(c^{m-n} + c^{m-2n} + \cdots + c^{m-c_1n}) = 1 - c^{m_1}$. Hence we obtain that there exist $\alpha, \beta \in F[\mathcal{M}]$ such that

$$(1-a)\alpha + (1-b)\beta = 1 - c.$$

Therefore $(1-a)F[\mathcal{M}] + (1-b)F[\mathcal{M}] = (1-c)F[\mathcal{M}]$. ■

Lemma 13. *Let N be a submonoid of $\mathcal{M}$ generated by two elements a, b. Then N is a free monoid of rank two or one.*

Proof: Suppose that N is not free of rank two. If K is a field, then, by [**3**, Exercise 0.11.6], $(1-a)K[\mathcal{M}] \cap (1-b)K[\mathcal{M}] \neq 0$. By the proof of Lemma 12, N is an infinite cyclic monoid. ■

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Rebut el 8 de juny de 1988