

## RATIONAL APPROXIMATION NEAR ZERO SETS OF FUNCTIONS

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### Abstract

The paper deals with the relation between global rational approximation and local approximation off the zero set. Also connections with the problem  $f^2 \in R(X) \Rightarrow f \in R(X)$  are studied.

The main result of this paper (Theorem 2 in §2) was obtained in Moskow and was discussed in some seminars and conferences in USSR. It was announced in the proceedings of the 1983 Kiev conference on complex analysis, but no proof was provided. The other results were found while the author was visiting the CRM in Barcelona in the fall of 1987.

1. Let  $X$  be a compact in  $C$ , and  $R(X)$  the closure in  $C(X)$  (with sup-norm) of the space of functions which are holomorphic on  $X$ .

For  $f \in C(X)$  we will write  $f \in R(X, x)$  for some  $x \in X$ , if there exists a neighbourhood  $U$  of  $x$  such that

$$f|_{\overline{X \cap U}} \in R(\overline{X \cap U}).$$

We let  $N(f)$  stand for  $\{x \in X \mid f(x) = 0\}$  - the zero set of  $f$ .

We will consider the following two closely related problems.

**Power problem (P-Pr).** Is it true that for any function  $f \in R(X)$  and  $q > 0$  the condition  $f^q \in C(X)$  implies  $f^q \in R(X)$  (same fixed branch)?

**Zero set problem (Z-Pr).** Let  $f \in C(X)$  and  $f \in R(X, x)$  for all  $x \in X \setminus N(f)$ .

Is it true that  $f \in R(X)$  ?

It is clear that these problems are trivial for  $N(f) = \emptyset$ , and that a positive answer to the Z-Pr implies the same answer to the P-Pr. Also it is not difficult to see, that P-Pr and Z-Pr are equivalent for  $f$  in the class

$$L(X) \equiv \cup_{\rho > 0} \text{Lip}(\rho, X).$$

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For the P-Pr the simpler case  $q > 1$  was considered in [5]. It was proved there that this problem has a positive answer for all  $X$  and all  $f \in L(X)$ , but really the proof presented in [5] works only under the additional assumption  $f^{q-1} \in L(X)$ . One can find a simple example ( $X \subseteq \mathbb{R}$ ), for which  $f \in L(X)$ ,  $q > 1$ ,  $f^q$  and  $f^{q-1} \in C(X)$ , but  $f^q$  and  $f^{q-1}$  are not in  $L(X)$  (the branches are fixed and corresponding).

Given two compact sets  $X$  and  $Y$ ,  $Y \subseteq X$  and  $h(z) \in C(X)$ . We will write

$$h(z) \in L_X(Y)$$

if there exist  $\rho > 0$  and a constant  $c > 0$ , such that for every  $z \in X$  and  $w \in Y$  we have

$$|h(z) - h(w)| \leq c|z - w|^\rho.$$

The following Theorem has a proof absolutely like Theorem 1 in [5], except for a small change, which will be described below.

**Theorem 1.** *Let  $f \in R(X)$ ,  $h \in L_X(N(f))$  and  $h \in R(X, x)$  for all  $x \in X \setminus N(f)$ . Then  $f(z) \cdot h(z) \in R(X)$ .*

**Corollary.** *Let  $f \in R(X) \cap L_X(N(f))$ , then for every  $q > 1$ ,  $f^q \in R(X)$  is equivalent to  $f^q \in C(X)$ .*

The corollary follows applying Theorem 1 to  $h = f^{q-1}$ .

To prove Theorem 1 we proceed as in [5] observing that we just need to worry about the squares (coming from the Vitushkin localization procedure) which intersect  $N(f)$ . To deal with these squares we use Lemma 3 in [5, p. 416] which turns out to be true under our weaker hypothesis. Concretely what we need is the following Lemma.

**Lemma.** *Let  $h$  be as in Theorem 1. There exist  $\rho = \rho(h) > 0$  and  $A = A(h)$ , such that for every square  $T_\delta$  with side length  $\delta$ ,  $T_\delta \cap N(f) \neq \emptyset$ , and every function  $g(z) \in C_0^1(T_\delta)$ , we have*

$$\left| \int_{T_\delta} \int h(z) \frac{\partial g}{\partial z} dz d\bar{z} \right| \leq A\delta \|\nabla g\|_\infty (\delta^\rho \alpha(T_\delta \setminus X) + M^{1+\rho}(T_\delta \cap N(f))),$$

where  $\alpha(\cdot)$  is  $C$ -analytic capacity and  $M^{1+\rho}(\cdot)$  is the Hausdorff content of order  $1 + \rho$  (see [6], p. 145).

We close this section by stating an open problem dealing with  $R^\rho(X)$ , the closure in  $\text{Lip}(\rho, X)$  of the space of holomorphic functions on  $X$ .

**Problem.** Let  $f \in R^\rho(X)$ , and assume that  $f^q \in \text{lip}(\rho, X)$  for some  $q > 1$ . Is it true that  $f^q \in R^\rho(X)$ ?

2. In [3,4] we proved, that the Z-Pr has a positive answer for all  $X$  and all functions  $f \in \text{Lip}(\rho, X)$ ,  $\rho > \frac{1}{2}$ . Now we are going to prove one theorem concerning this problem for all  $\rho > 0$ , but for some special compact sets  $X$ . This result points in the direction that the Z-Pr has a positive answer also for all  $X$  and all  $f \in L(X)$ . We need some notation.

Let  $X$  be compact and  $x \in X$ . We say that  $x$  is a point of stability (of the capacity of  $C \setminus X$ ) if one of the following two conditions holds.

i) There exists  $\lim_{\delta \rightarrow 0} \log_{\delta}(\alpha(T(x, \delta) \setminus X))$ , where  $T(x, \delta)$  (here and below) is the square with center  $x$  and side length  $\delta$ .

ii)  $\liminf_{\delta \rightarrow 0} \log_{\delta}(\alpha(T(x, \delta) \setminus X)) \geq 2$ . If we denote the lim in (i) by  $\lambda(x)$ , then (i) is equivalent to

$$\alpha(T(x, \delta) \setminus X) = \delta^{\lambda(x)+o(1)} \quad \text{as } \delta \rightarrow 0,$$

and (ii) is the same that

$$\alpha(T(x, \delta) \setminus X) = O(\delta^{2-\epsilon}) \quad \text{for any fixed } \epsilon > 0 \text{ as } \delta \rightarrow 0.$$

Let now  $X^* = \{x \in X \mid x \text{ is unstable}\}$ .

**Theorem 2.** *If  $X^* = \emptyset$  then the Z-Pr has a positive answer for all  $f \in L(X)$ .*

**Corollary 1.** *Let  $f \in L(X)$  and assume  $f \in R(X, x)$  for all  $x \in X \setminus N(f)$ . Then  $f \in R(X, x)$  also for all  $x \in X \setminus (N(f) \cap \bar{X}^*)$ .*

The proof of Theorem 2 and the main idea in the proof of Lemma 2 [5] allow to get also the following corollary:

**Corollary 2.** *Let  $f \in \text{Lip}(\rho, X)$ ,  $\rho > 0$ , and assume  $f \in R(X, x)$  for all  $x \in X \setminus N(f)$ . If*

$$M^{1/(1-\rho+\epsilon)}(N(f) \cap \bar{X}^*) = 0 \quad \text{for some } \epsilon > 0, \text{ then } f \in R(X).$$

We remark that for  $\rho > \frac{1}{2}$  the last hypothesis is automatically satisfied and so we obtain the main result of [3,4].

**Some preliminary results and remarks.** For  $T_{\delta} = T(z, \delta)$  we will write  $r \cdot T_{\delta} = T(z, r \cdot \delta)$ , for each  $r > 1$ . As in [4] we will need the following

**Theorem 0.** *Let  $T$  be a square and  $0 < \tau \leq 1$ . If  $f \in C(4T)$ ,  $\|f\|_{4T} \leq 1$  and  $f$  is analytic on  $4T^0 \setminus M_{\epsilon}(f)$ , where  $M_{\epsilon}(f) = \{z \in 4T \mid |f(z)| \leq \epsilon\}$ , then*

$$(1) \quad \left| \int_{\partial T} f(z) dz \right| \leq C(\tau) \cdot \epsilon^{1-\tau} \cdot \alpha(M_{\epsilon}(f) \cap T).$$

**Remark 1.** Assume  $\mathcal{D}$  is Jordan domain with rectifiable boundary  $\partial\mathcal{D}$ , and  $G$  any open neighbourhood of  $\bar{\mathcal{D}}$ . If the estimate of the integral due to Melnikov and Vitushkin ([6], p. 158) is satisfied in  $\mathcal{D}$ , then it is easily seen that we also have

$$(2) \quad \left| \int_{\partial\mathcal{D}} f(z) dz \right| \leq C(\mathcal{D}, G, \tau) \cdot \varepsilon^{1-\tau} \cdot \|f\|_{\bar{G}} \alpha(M_\varepsilon(f) \cap \mathcal{D}),$$

where  $M_\varepsilon(f) = \{z \in G \mid |f(z)| \leq \varepsilon\}$ ,  $f \in C(\bar{G})$  and  $f$  is analytic on  $G \setminus M_\varepsilon(f)$ .

**Problem.** Is estimate (2) true if we take  $G = \mathcal{D}$ ?

In applications to rational approximation we really don't need the condition  $G = \mathcal{D}$ , but it seems useful to have (2) with  $\tau = 0$ .

As it turns out estimates (2) and (1) are not true for  $\tau = 0$ . We present an example here.

Let  $S$  be a compact with  $m(S) > 0$  (where  $m(\cdot)$  is plane Lebesgue measure), with empty interior and with connected complement. Set  $\varphi(z) = \int_S \frac{dm(\zeta)}{z - \zeta}$ , so that  $\varphi \in C(\bar{C})$ ,  $\varphi(\infty) = 0$ , and  $\varphi$  is analytic on  $\bar{C} \setminus S$ . Let now  $T$  be a square, containing  $S$ . Then

$$\int_{\partial T} \varphi(z) dz = 2\pi i \lim_{z \rightarrow \infty} z\varphi(z) = 2\pi i m(S) \neq 0.$$

By the Mergelian's Theorem [2] for any  $\varepsilon > 0$  there exists polynomial  $P(z)$  such that  $\|\varphi(z) - P(z)\|_S \leq \varepsilon$ .

Applying (2) with  $f = \varphi - P$ ,  $\mathcal{D} = T$ ,  $G = 4T$  (and  $\tau = 0$ ) we would get

$$2\pi m(S) = \left| \int_{\partial T} f(z) dz \right| \leq C \cdot \varepsilon,$$

which is a contradiction for  $\varepsilon$  small enough.

We will use also two theorems of Vitushkin ([6], p. 158; for Theorem V1 see [3], p. 104).

**Theorem V1.** Let  $f \in C(C)$ . Then  $f \in R(X)$  provided there exist  $r \geq 1$  and  $a(\delta) \rightarrow 0$  as  $\delta \rightarrow 0$ , such that for any square  $T_\delta$  of side length  $\delta$  we have

$$(3) \quad \left| \int_{\partial T_\delta} f(z) dz \right| \leq a(\delta) [\alpha(rT_\delta \setminus X) + \delta^2].$$

Conversely if  $f \in R(X)$  we obtain (3) with  $a(\delta) = c\omega(f, \delta)$ ,  $r = 1$  and without  $\delta^2$  in the right hand-side.

**Theorem V2.** Let  $E$  be a bounded set with  $\alpha(E) = \alpha > 0$ , and  $\{E_j\}_{j=1}^N$  a finite number of sets  $E_j \subseteq E$  such that any square  $T_\alpha$  with side  $\alpha$  intersects at most  $p$  ( $p \geq 1$  is a fixed integer)  $E_j$ 's. Then for some absolute constant  $c$

$$\sum_{j=1}^N \alpha(E_j) \leq c \cdot p \cdot \alpha(E).$$

*Proof of Theorem 2:* Now we fix a compact  $X$  with  $X^* = \emptyset$ , a function  $f \in \text{Lip}(\rho, X)$ ,  $\rho > 0$ , and let us suppose, that  $f \notin R(X)$ . After several lemmas we will have a contradiction.

We will denote by  $C$  an absolute constant and by  $A$  a constant depending only on  $f$  (on  $\rho$ ).

Both of them may vary from an inequality to another.

Let  $T_\delta$  be any square with side  $\delta$  and put  $\alpha(5T_\delta \setminus X) \equiv 5 \cdot \delta^\beta$ . Then  $\beta > 1$ .

Take  $\delta_1 = \delta^\beta$  and consider a non-overlapping family  $\{T_j\}$  of squares of side  $\delta_1$  and centers  $\{z_j\}$ , covering the plane. Let  $\varphi_j \in C_0^1(2T_j)$ ,  $|\nabla \varphi_j| \leq C\delta_1^{-1}$  and  $\sum_j \varphi_j \equiv 1$  on  $C$ . We denote by  $j'$  the indexes  $j$  for which  $f|_{3T_j} \in R(X \cap 3T_j)$  and  $3T_j \subset 5T_\delta$ .

**Lemma 1.** *Let  $t$  be a point in  $4T_\delta$  such that*

$$\alpha((T(t, \varepsilon) \cap 5T_\delta) \setminus X) \leq C \cdot \varepsilon^{\frac{1-\nu}{\nu}}$$

for some,  $\nu$ ,  $0 \leq \nu < 1$ , and all  $\varepsilon$ ,  $\delta_1 \leq \varepsilon \leq 10\delta_1$ .

Then we have

$$(4) \quad \sum_{j': t \notin 3T_{j'}} \frac{\alpha(2T_{j'} \setminus X)}{|t - z_{j'}|} \leq C\delta_1^\nu$$

*Proof:* Inequality (4) will follow from Theorem V2 and the following elementary result.

**Sublemma.** *Let  $0 < a_1 \leq a_2, \dots$ , and  $a_1 < a$ . If a sequence  $\{\alpha_n, \alpha_n \geq 0\}$  satisfies the conditions  $\sum_{p=1}^m \alpha_p \leq a_m$  for all  $m \geq 1$  and  $\sum_{p=1}^\infty \alpha_p \leq a$ , then*

$$\sum_{p=1}^\infty \frac{\alpha_p}{p} \leq \sum_{m=1}^M \frac{a_m}{m^2} + \frac{a}{M},$$

where  $M$  is defined by  $a_M \leq a < a_{M+1}$ .

In particular, if  $a_m \leq a$  for all  $m \geq 1$ , then

$$\sum_{p=1}^\infty \frac{\alpha_p}{p} \leq \sum_{m=1}^\infty \frac{a_m}{m^2}$$

Proof: The sum  $\sum_{p=1}^{\infty} \frac{\alpha_p}{p}$  is maximized by

$$\alpha_p = \begin{cases} a_p - a_{p-1} & \text{for } p \leq M \text{ and } a_0 = 0, \\ a - a_M & \text{for } p = M + 1, \\ 0 & \text{for } p > M + 1. \end{cases}$$

A computation now gives the sublemma. To prove Lemma 1 let  $\mathcal{D}_p = T(t, (p+4)\delta_1) \setminus T(t, p\delta_1)$ ,  $p \geq 1$ , and  $\alpha_p = \alpha(\mathcal{D}_p \cap 5T_\delta \setminus X)/\delta_1$ . By Theorem V2

$$\sum_{j: 2T_{j'} \subseteq \mathcal{D}_p} \frac{\alpha(2T_{j'} \setminus X)}{|t - z_{j'}|} \leq \frac{C_1 \alpha_p}{p}.$$

Now we use the sublemma with  $a = C_1$  and  $a_m = C_1 \delta_1^{-1} [(m+4)\delta_1]^{1-\frac{1}{\nu}}$ . Lemma 1 is proven.

We will construct by induction squares  $S_n$  such that  $f \notin R(X_n)$ , where  $X_n = X \cap S_n$ , and for any square  $T_\delta$  we will have

$$(5) \quad \left| \int_{\partial T_\delta} f dz \right| \leq a(\delta) \cdot [\alpha(5T_\delta \setminus X_n) + \delta^{1-\frac{1}{\nu n}}],$$

with  $a(\delta) \rightarrow 0$  as  $\delta \rightarrow 0$ , and  $\theta_n \geq \theta_1 + \frac{n-1}{2}\rho$ ,  $\theta_1 < \frac{1}{2}$ .

For some large  $n$ :  $\frac{1}{1-\theta_n} \geq 2$ , and so Theorem V1 will imply  $f \in R(X_n)$ , which is the desired contradiction.

For  $n=1$  we let  $S_1$  be any square such that  $f \notin R(X_1)$ , where  $X_1 = S_1 \cap X$ . For any square  $T_\delta$  we have (since  $f \in \text{Lip}(\rho, X)$ )

$$\left| \int_{\partial T_\delta} f dz \right| \leq A\delta^{1+\rho} = a(\delta)\delta^{1+\rho/2} = a(\delta) \cdot \delta^{1-\frac{1}{\nu_1}},$$

where  $\theta_1 = \frac{\rho/2}{1+\rho/2}$ . So (5) is satisfied for  $n=1$ .

For the inductive step we need two lemmas.

**Lemma 2.** Let  $Q_1$  be a square and

$$f \notin R(X \cap Q_1) \equiv R(Y_1), \quad \text{where } Y_1 = X \cap Q_1.$$

Suppose, that there exist some  $\theta$ ,  $0 < \theta < 1$ , and  $a_1(\delta) \rightarrow 0$  as  $\delta \rightarrow 0$  such that for any square  $T_\delta$  we have (5) with  $X_n = Y_1$  and  $\theta_n = \theta$ . Fix any  $\theta'$ ,  $0 < \theta' < \theta$ .

Then there exists a square  $Q_2 \subseteq Q_1$  such that  $f|_{Y_2} \notin R(Y_2)$ , where  $Y_2 = Y_1 \cap Q_2 = X \cap Q_2$ , and for every square  $T_\varepsilon \subseteq Q_2$ , satisfying the condition

$$(6) \quad \alpha(T_\varepsilon \setminus Y_2) \geq \varepsilon^{1-\frac{1}{\nu}}$$

we have  $f \in R(T_\varepsilon \cap Y_2)$ .

*Proof:* If for some square  $T_\delta$  inequality (3) is not true (for  $Y_1$  instead of  $X$ ), then using (5) with  $X_n = Y_1$  and  $\theta_n = \theta$ , we get

$$(7) \quad \alpha(T_\delta \setminus Y_1) \leq \delta^{\frac{1}{1-\theta}}$$

But if Lemma 2 is not true, then, using (6) and (7), by induction, we will find a point  $t \in Y_1$  such that

$$\liminf_{\delta \rightarrow \infty} (\log_\delta(\alpha(T(t, \delta) \setminus Y_1))) \leq \frac{1}{1-\theta'} < \frac{1}{1-\theta} \leq \limsup_{\delta \rightarrow \infty} (\log_\delta(\alpha(T(t, \delta) \setminus Y_1))).$$

As  $\theta$  in (5) is less than  $\frac{1}{2}$  (otherwise  $f \in R(Y_1)$  by Theorem V1), we have  $t \in Y_1^* \subseteq X^* \neq \emptyset$ , contradicting our assumption  $X^* = \emptyset$ . ■

**Lemma 3.** Let  $Q_2$  be the square in the statement of Lemma 2, e.g.

(a)  $f|_{Y_2} \notin R(Y_2)$ , where  $Y_2 = X \cap Q_2$ .

(b) For some  $0 < \theta < 1$  we have for any square  $T_\delta$ :

$$\left| \int_{\partial T_\delta} f(z) dz \right| \leq \alpha(\delta) [\alpha(5T_\delta \setminus Y_2) + \delta^{\frac{1}{1-\theta}}]$$

(c) Any fixed  $\theta'$ ,  $0 < \theta' < \theta$ , has the following property: for any square  $T_\varepsilon \subseteq Q_2$  and satisfying (b) we have  $f \in R(T_\varepsilon \cap Y_2)$ . Then for any fixed  $\tau$ ,  $0 < \tau < 1$ , there exist  $a_1(\delta) \rightarrow 0$  as  $\delta \rightarrow 0$  such that for any square  $T_\delta$ :

$$(5') \quad \left| \int_{\partial T_\delta} f(z) dz \right| \leq a_1(\delta) [\alpha(5T \setminus Y_2) + \delta^{\frac{1}{1-(\theta+\theta')(1-\tau)}}]$$

**Remark.** The meaning of Lemma 3 is that (5) holds with  $X_n$  replaced by  $Y_2$  and  $\theta_n$  by  $\theta_{(2)} = (\theta' + \rho)(1 - \tau) \geq \theta + \rho/2$  for  $\tau$  and  $(\theta - \theta')$  small enough. Before going on the proof of Lemma 3 we complete the proof of Theorem 2. By induction we find compacts  $X_n$  with  $f \notin R(X_n)$ ,  $X_n = X \cap S_n$  (put  $S_{n-1} = Q_1$  and  $S_n = Q_2$  in Lemmas 2 and 3), for which (5) holds with  $\theta_n = \theta_1 + \frac{n-1}{2}\rho$ . When  $\frac{1}{2} < \theta_n < 1$  we have a contradiction.

*Proof of Lemma 3:* Fix any square  $T_\delta$  with side length  $\delta$ , assuming also that  $T_\delta$  is diadic [1]. The cases  $5T_\delta \not\subseteq Q_2$  or  $\alpha(5T_\delta \setminus Y_2) \geq \delta^{\frac{1}{1-\theta'}}$  are trivial, so we will consider the case

$$5T_\delta \subseteq Q_2 \quad \text{and} \quad \alpha(5T_\delta \setminus Y_2) \equiv 5\delta^\beta < \delta^{\frac{1}{1-\theta'}}$$

So we have  $5\delta^\beta \ll \delta$  if  $\delta$  is small enough.

Now we divide  $5T_\delta$  into equal diadic squares  $\{T(j_1)\}$  with side length  $\delta_1$ ,  $\frac{1}{2}\delta^\beta \leq \delta_1 < \delta^\beta$ . We call a square  $T(j_1)$  "good" and write  $j_1 = j'_1$  if  $T(j_1)$  satisfies the following condition (for  $n = 1$ ,  $J'_n = j'_1$ ):

$$(8) \quad f|_{4T(J'_n)} \in R(4T(J'_n) \cap Y_2).$$

For the remaining "bad" squares we use the notation  $T(j''_n)$ . We will continue our procedure only with "bad" squares, for which (6) gives

$$\alpha(4T(j''_1) \setminus Y_2) \leq (4\delta_1)^{\frac{1}{1-\theta'}} \leq 16\delta^{\frac{1}{1-\theta'}}.$$

Assume that the squares  $\{T(j''_1, \dots, j''_{n-1}, j_n)\}$  have been constructed. For short we put

$$(j''_1, \dots, j''_{n-1}, j_n) = (J_n).$$

For the "good" squares  $T(J_n)$ , which satisfy (8) we use  $T(J'_n)$  and they will not be divided again. Every "bad" square  $T(J''_n)$  will be divided into equal diadic squares  $\{T(J''_n, j_{n+1})\} \equiv \{T(J_{n+1})\}$  with centers  $\{z_{J_{n+1}}\}$  and side length  $\delta_{n+1}$  satisfying

$$\frac{1}{2}\delta_n^{\frac{1}{1-\theta'}} \leq \delta_{n+1} < \delta_n^{\frac{1}{1-\theta'}}$$

Using (6) we find that, for every  $2T(J''_n)$ ,

$$(9) \quad \alpha(2T(J''_n) \setminus X) \leq \alpha(4T(J''_n) \setminus X) \leq (4\delta_n)^{\frac{1}{1-\theta'}} \leq 16\delta_{n+1}.$$

We stop this procedure when  $\delta_N \leq (\delta^\beta)^{\frac{1}{\rho}} < \delta_{N-1}$ .

Evidently, as  $\delta_n < (\delta^\beta)^{\frac{n-1}{1-\theta'}}$ , we have

$$N \leq 1 + \frac{1-\theta'}{\rho} \leq \frac{2}{\rho}.$$

Then

$$5T_\delta = [\cup_{n=1}^N \cup_{(J'_n)} T(J'_n)] \cup [\cup_{(J''_n)} T(J''_n)].$$

By a lemma of Harvey and Polking ([1], p. 43) there exists a partition of unity  $\{\varphi_{J'_n}\}_{n=1}^N \cup \{\varphi_{J''_n}\}$  with  $\text{Supp} \varphi_{J'_n} \subseteq 2T(J'_n)$ ,  $|\nabla \varphi_{J'_n}| \leq \frac{c}{\delta_n}$  and

$$\sum_{n=1}^N \sum_{(J'_n)} \varphi_{J'_n} + \sum_{(J''_n)} \varphi_{J''_n} \equiv 1 \quad \text{on } 4T_\delta.$$

Let  $f_{J'_n}(z) \equiv \frac{1}{2\pi i} \int \frac{f(\zeta) - f(z)}{\zeta - z} \frac{\partial \varphi_{J'_n}(\zeta)}{\partial \bar{\zeta}} d\bar{\zeta} \wedge d\zeta$ . By Vitushkin Theorem ([6], p. 148) we have  $f_{J'_n} \in R(Y_2)$ ,  $\|f_{J'_n}\|_\infty \leq c\delta_n^\rho$ , and for  $t \notin 3T_{(J'_n)}$

$$|f_{J'_n}(t)| \leq \frac{c\delta_n^\rho \alpha(2T(J'_n) \setminus X)}{|t - z_{J'_n}|}.$$

Consider the function

$$G(z) = \sum_{n=1}^N \sum_{(J'_n)} f_{J'_n}(z) \in R(X), \quad \text{and}$$

$$F(z) = f(z) - G(z).$$

Let  $\Omega$  be a set of nonanalyticity of  $F$  in  $4T_\delta$ , then  $\Omega$  belong to  $4\delta_N$ -neighbourhood of  $N(f)$  ([6], p. 148).

For every  $t \in \Omega$  and  $\delta_n < \varepsilon < 10\delta$  we have by (6) and (9):

$$\alpha(T(t, \varepsilon) \cap 5T_\delta \setminus Y_2) \leq c\varepsilon^{\frac{1}{1-\theta'}}$$

and  $t \notin 3T(J'_n)$  for  $n < N$ .

To estimate  $G(z)$  on  $\Omega$  we use Lemma 1 (with  $\nu = 0$  for  $n = N$  and  $\nu = \theta'$  for all  $1 \leq n \leq N-1$ ), Theorem V2 (see also (9)), and we recall that the induction procedure is finite ( $N \leq \frac{2}{\rho}$ ).

We get  $\|G\|_{4T_\delta} \leq A\delta^\rho$  and

$$\|G\|_\Omega \leq c\delta_N^\rho + \sum_{n=1}^N A_n \delta_n^\rho \cdot \delta_1^{\theta'} \leq A(\delta^\beta)^{\rho+\theta'}.$$

Since  $\|f\|_\Omega \leq A(\delta_n)^\rho \leq A\delta^\beta \leq A(\delta^\beta)^{\rho+\theta'}$ , (the case  $\rho = \frac{1}{2}$  or  $\theta' > \frac{1}{2}$  is evident), then

$$\|F\|_\Omega \leq A\delta^{\beta(\rho+\theta')}.$$

By Theorem V1 (as  $G(z) \in R(Y_2)$ ), we have

$$\left| \int_{\partial T_\delta} G dz \right| \leq A\delta^\rho \alpha(T_\delta \setminus Y_2) \leq A\delta^\rho \cdot \delta^\beta,$$

and by Theorem 0 (for  $\tau_1 = \tau/2$ ) we have  $|\int_{\partial T_\delta} F dz| \leq A(\tau, \rho) [\delta^{\beta(\rho+\theta')}]^{1-\frac{\tau}{2}} \cdot \delta$ . Hence

$$\left| \int_{\partial T_\delta} f dz \right| \leq a_1(\delta) [\delta^\beta + \delta^{\beta(\rho+\theta')(1-\tau)+1}],$$

and  $a_1(\delta) \rightarrow 0$  as  $\delta \rightarrow 0$ .

Finally, if  $\beta \leq \beta(\rho+\theta')(1-\tau)+1$  then (5') holds, but if  $\beta > \beta(\rho+\theta')(1-\tau)+1$ , that is,  $\beta > 1/[1+(\rho+\theta')(1-\tau)]$ , we also have (5'): Lemma 3 and Theorem 2 are proved.

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