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ON THE ANGULAR LIMITS OF BLOCH FUNCTIONS

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Abstract

This paper contains a method to associate to each function f in the little Bloch space another function f^* in the Bloch space in such way that f has a finite angular limit where f^* is radially bounded. The idea of the method comes from the theory of the lacunary series. An application to conformal mapping from the unit disc to asymptotically Jordan domains is given.

1. Introduction and main results

Let $\mathbb{D}$ denote the unit disk and $\mathbb{T} = \partial\mathbb{D}$. A Bloch function [1] [8, p. 268] is a function f analytic in $\mathbb{D}$ such that

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup_{|z|<1} (1-|z|^2)|f'(z)| < \infty. \tag{1}$$

With this norm, the Bloch functions form a Banach space $\mathcal{B}$. The closure in $\mathcal{B}$ of the polynomials is a subspace $\mathcal{B}_0$ that consists of all $f \in \mathcal{B}$ such that

$$(1-|z|^2)|f'(z)| \to 0 \text{ as } |z| \to 1. \tag{2}$$

For Bloch functions, radial and angular limits are identical [7] [8, p. 268], that is,

$$f(r\zeta) \to a\ (r \to 1) \Rightarrow f(z) \to a\ (z \to \zeta,\ z \in \Delta(\zeta))$$

holds for each $\zeta \in \mathbb{T}$ where $\Delta(\zeta)$ is any triangle in $\mathbb{D}$ with vertex ζ. Furthermore [8, p. 269]

$$\sup_{0<r<1} |f(r\zeta)| < \infty \Rightarrow \sup_{z\in\Delta(\zeta)} |f(z)| < \infty.$$

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Each bounded analytic function belongs to $\mathcal{B}$ but not always to $\mathcal{B}_0$. Things are very simple for the special case of Hadamard gap series

$$(3) \qquad f(z) = \sum_{k=0}^{\infty} b_k z^{n_k}, \ \frac{n_{k+1}}{n_k} \geq \lambda > 1 \quad (k = 0, 1, \dots).$$

In this case [1] [13, vol. I, p. 247]

$$\begin{aligned} f \in \mathcal{B} &\iff \sup |b_k| < \infty, \\ f \in \mathcal{B}_0 &\iff b_k \to 0 \quad (k \to \infty), \\ f \in H^\infty &\iff \sum_k |b_k| < \infty. \end{aligned}$$

If a gap series has radial limits on a set of positive measure then $\sum_k |b_k|^2 < \infty$ [13, vol. I, p. 203]. It follows that

$$(4) \qquad f_0(z) = \sum_{k=1}^{\infty} k^{-1/2} z^{2^k} \quad (z \in \mathbb{D})$$

belongs to $\mathcal{B}_0$ but has angular limits almost nowhere on $\mathbb{T}$.

There is a close connection with conformal mappings [8, p. 269]. If g is an (injective) conformal mapping of $\mathbb{D}$ then $f = c \log g' \in \mathcal{B}$ holds for all $c \in \mathbb{C}$. Conversely if $f \in \mathcal{B}$ and $|c| < 1/\|f\|_{\mathcal{B}}$, then the function g defined by $f = c \log g'$ maps $\mathbb{D}$ conformally onto a domain bounded by a Jordan curve J. Furthermore f belongs to $\mathcal{B}_0$ if and only if [9] the curve J is asymptotically conformal, i.e. if

$$\max_{w \in J(a,b)} \frac{|b - w| + |w - a|}{|b - a|} \to 1 \text{ as } |a - b| \to 0, \ a, b \in J$$

where $J(a, b)$ is the (smaller) arc of J between a and b.

We shall describe a method to reduce the existence problem of finite radial (=angular) limits for $\mathcal{B}_0$ to the problem of radial boundedness for $\mathcal{B}$.

Theorem. *If $f \in \mathcal{B}_0$ then there is a function $f^* \in \mathcal{B}_0 \subset \mathcal{B}$ such that, for all $\zeta \in \mathbb{T}$,*

$$\sup_r |f^*(r\zeta)| < \infty \Rightarrow \lim_{r \to 1} f(r\zeta) \text{ exists } \neq \infty.$$

This generalizes a result on Hadamard gap series by Gnuschke [5]. We shall develop every Bloch function into a series of polynomials that is analogous to a gap series.

Using a method of Noshiro and T. Wolff [11], it can be shown [3] that each Bloch function is radially bounded on a set that has positive capacity on every arc of $\mathbb{T}$. Hence we obtain from the theorem:

Corollary 1. *If $f \in \mathcal{B}_0$ then there is a set $E \subset \mathbb{T}$ with cap $(E \cap I) > 0$ for every arc I of $\mathbb{T}$ such that*

$$\lim_{r \to 1} f(r\zeta) \textit{ exists } \neq \infty \textit{ for } \zeta \in E. \tag{5}$$

Previously it was only known [6] that a function in $\mathcal{B}_0$ has finite radial limits on an uncountably dense set. The present method, however, does not imply the fact [6] that the image set of angular limits has always positive linear measure.

It was asked in [4] whether all $f \in \mathcal{B}$ satisfy

$$\dim \{\zeta \in \mathbb{T} : \sup_r |f(r\zeta)| < \infty\} = 1,$$

where dim denotes the Hausdorff dimension. If the answer turns out to be positive, then the theorem would imply that all $f \in \mathcal{B}_0$ have finite angular limits on a set of Hausdorff dimension 1. This would be much stronger than our corollary because already $\dim E > 0$ implies cap $E > 0$. Note that it is not possible to replace dimension 1 by positive (Lebesgue) measure as the function f_0 defined by (4) shows.

Much more is known about infinite angular limits. Recently J.M. Anderson and L.D. Pitt [2] have proved that each Bloch function has either finite radial limits on a set of positive measure or satisfies

$$\dim \{\zeta \in \mathbb{T} : \operatorname{Re} f(r\zeta) \to +\infty \text{ as } r \to 1\} = 1.$$

This implies that every conformal map has a finite angular derivative (possibly $=0$) on a set of dimension 1.

Corollary 1 implies a result on the unrestricted boundary derivative for univalent functions; see [6] for the corresponding weaker result.

Corollary 2. *Let g map $\mathbb{D}$ conformally onto the inner domain of an asymptotically conformal Jordan curve. Then there is a set $E \subset \mathbb{T}$ with cap $(E \cap I) > 0$ for every arc I of $\mathbb{T}$ such that*

$$g'(\zeta) = \lim_{z \to \zeta, z \in \mathbb{D}} \frac{g(z) - g(\zeta)}{z - \zeta} \textit{ exists } \neq 0, \infty \textit{ for } \zeta \in E.$$

2. A series expansion of Bloch functions

We consider an analytic function

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ for } z \in \mathbb{D} \tag{6}$$

and define polynomials $p_0(z) = a_0 + a_1 z + a_2 z^2$ and

$$(7) \qquad p_k(z) = \sum_{n=2^{k-1}+2}^{2^k} \frac{2n - 2^k - 2}{n-1} a_n z^n + \sum_{n=2^k+1}^{2^{k+1}} \frac{2^{k+1} - n + 1}{n-1} a_n z^n$$

for $k = 1, 2, \ldots$. Induction shows that

$$\sum_{k=0}^{m} p_k(z) = \sum_{n=0}^{2^m} a_n z^n + \sum_{n=2^m+1}^{2^{m+1}} \frac{2^{m+1} - n + 1}{n-1} a_n z^n,$$

and since $\limsup |a|^{1/n} \le 1$ it follows that

$$(8) \qquad f(z) = \sum_{k=0}^{\infty} p_k(z) \text{ for } z \in \mathbb{D}.$$

This expansion shares many properties of lacunary power series; see for instance (16) and Proposition 3 below.

The next two results are essentially known. They are implicit in the work of Zygmund [**12**][**13**, vol. I, p. 115 ff] and actually hold in a slightly different form in the more general context of Besov spaces. For convenience we shall give proofs.

Proposition 1. *If $f \in \mathcal{B}$ then*

$$(9) \qquad \|p_k\|_\infty \equiv \sup_{|z|\le 1} |p_k(z)| \le 6\|f\|_{\mathcal{B}} \text{ for } k = 0, 1, \ldots,$$

and if $f \in \mathcal{B}_0$ then

$$(10) \qquad \|p_k\|_\infty \to 0 \text{ as } k \to \infty.$$

Proof: We may assume that $\|f\|_{\mathcal{B}} = 1$. It easily follows from (1) that $|a_0| \le 1$, $|a_1| \le 1$ and $|a_2| < 2$ so that $|p_0(\zeta)| \le 4$ for $|\zeta| \le 1$. For $m = 1, 2, \ldots$, we consider now the polynomial

$$(11) \qquad q_m(z) = \left(\frac{1 - z^m}{1 - z}\right)^2 = \sum_{\nu=0}^{m-1} (\nu + 1) z^\nu + \sum_{\nu=m}^{2m-2} (2m - \nu - 1) z^\nu.$$

We see from (6) that

$$\frac{1}{2\pi i} \int_{|z|=r} f'(z) q_m(\bar{z}\zeta) z^{-2} dz = \sum_{\nu=0}^{m-1} (\nu+1)(\nu+2) a_{\nu+2} r^{2\nu} \zeta^\nu + \\ + \sum_{\nu=m}^{2m-2} (2m - \nu - 1)(\nu + 2) a_{\nu+2} r^{2\nu} \zeta^\nu.$$

A simple calculation therefore shows that

$$\frac{1}{\pi}\iint_{\mathbf{D}}(1-|z|^2)f'(z)q_m(\zeta\bar{z})z^{-2}\,dxdy = \\ = \sum_{n=2}^{m+1} a_n\zeta^{n-2} + \sum_{n=m+2}^{2m} \frac{2m-n+1}{n-1}a_n\zeta^{n-2}.$$

Hence it follows from (7) that, for $k = 1, 2, \ldots$,

$$p_k(\zeta) = \frac{1}{\pi}\iint_{\mathbf{D}}(1-|z|^2)f'(z)\left(\frac{\zeta}{z}\right)^2 [q_{2^k}(\zeta\bar{z}) - q_{2^{k-1}}(\zeta\bar{z})]\,dxdy. \tag{12}$$

We write $y(r) = \max_{|z|=r}(1-|z|^2)|f'(z)|$. Since

$$|q_{2m}(z) - q_m(z)| = |2z^m + z^{2m}|\left|\frac{1-z^m}{1-z}\right|^2$$

by (11), we see from (12) that

$$|p_k(\zeta)| \le \frac{3}{\pi}\int_0^1 y(r)r^{m-1}\left(\int_0^{2\pi}|1+re^{it}+\cdots+r^{m-1}\,e^{i(m-1)t}|^2dt\right)dr \le \\ \le 6\int_0^1 y(r)r^{m-1}m\,dr < 6,$$

with $m = 2^{k-1}$, because $y(r) \le 1$. If $f \in \mathcal{B}_0$ and $\varepsilon > 0$ then, by (2), there is $\rho < 1$ such that $y(r) < \varepsilon$ for $\rho \le r < 1$. Hence the last integral is less than $m\rho^{m+1} + \varepsilon < 2\varepsilon$ for large m which implies (10). ■

Proposition 2. *If $\|p_k\|_\infty$ is bounded then $f \in \mathcal{B}$ and*

$$\|f\|_{\mathcal{B}} \le 16\sup_{k\ge 0}\|p_k\|_\infty. \tag{13}$$

If $\|p_k\|_\infty \to 0$ as $k \to \infty$ then $f \in \mathcal{B}_0$.

Thus the Bloch norm is equivalent to the norm $\sup_k \|p_k\|_\infty$ where the polynomials p_k are defined by (7). In the case of a lacunary series this norm is essentially the same as $\sup|b_k|$.

Proof: Let $n_k = 2^{k-1}$. If $k > 0$ we can write

$$p_k(z) = z^{n_k+1}g_k(z),\ \deg(g_k) \le 3n_k - 1,\ \|g_k\|_\infty = \|p_k\|_\infty. \tag{14}$$

Hence it follows from Bernstein's inequality [**13**, vol. II, p. 11] that, for $|z| \le r < 1$,

$$|p_k'(z)| = |(n_k+1)z^{n_k}g_k(z) + z^{n_k+1}g_k'(z)| \le \\ \le [(n_k+1)r^{n_k} + (3n_k-1)r^{n_k+1}]\|g_k\|_\infty \le 4n_k r^{n_k}\|p_k\|_\infty. \tag{15}$$

Therefore we deduce from (1) and (8) that

$$\|f\|_{\mathcal{B}} \leq 2\|p_0\|_\infty + \sup_{0\leq r<1} (1-r^2) \sum_{k=1}^{\infty} 4n_k r^{n_k} \|p_k\|_\infty. \tag{16}$$

Since (this is a standard estimate for gap series)

$$\frac{1}{1-r}\sum_{k=1}^{\infty} n_k r^{n_k} = \sum_{m=1}^{\infty}\left(\sum_{n_k\leq m} n_k\right) r^m \leq 2\sum_{m=1}^{\infty} m r^m = \frac{2r}{(1-r)^2}$$

we conclude that

$$\|f\|_{\mathcal{B}} \leq 2\|p_0\|_\infty + 16 \sup_{k\geq 1} \|p_k\|_\infty$$

which inplies 813). The final assertion of Proposition 2 is deduced in a similar way from (14). ■

Proposition 3. *Let $f \in \mathcal{B}$. If $s_k = p_0 + p_1 + \cdots + p_k$ and $r_k = 1 - 2^{-k}$ then*

$$|f(r_k z) - s_k(z)| \leq 30\|f\|_{\mathcal{B}} \quad \textit{for} \quad |z| \leq 1. \tag{17}$$

Proof: We may assume that $\|f\|_{\mathcal{B}} \leq 1$. Then $\|p_j\|_\infty \leq 6$ by Proposition 1. We see from (8) that, for $|z| \leq 1$,

$$f(r_k z) - s_k(z) = \sum_{j=0}^{k} (p_j(r_k z) - p_j(z)) + \sum_{j=k+1}^{\infty} p_j(r_k z).$$

The first sum is bounded by

$$\sum_{j=0}^{k} (1-r_k) \max_{|\zeta|\leq 1} |p_k'(\zeta)| \leq 2^{-k} \sum_{j=0}^{k} 4 \cdot 2^{j-1} \cdot 6 < 24$$

because of (15), and we see from (14) that the second sum is bounded by

$$6\sum_{j=k+1}^{\infty} r_k^{2^{j-1}} < 6\sum_{j=k+1}^{\infty} \exp(-2^{j-1-k}) = 6\sum_{\nu=0}^{\infty} \exp(-2^{\nu}) < 6. \quad ■$$

3. Proof of the main results

Proof of the theorem: Let $f \in \mathcal{B}_0$. We obtain from Proposition 1 that there is a decreasing sequence (ε_k) such that

$$\|p_k\|_\infty < \varepsilon_k^2 \quad \text{for} \quad k = 0, 1, \ldots \tag{18}$$

where p_k is given by (7). We define

$$f^* = \sum_{k=0}^{\infty} p_k^*, \quad p_k^* = \varepsilon_k^{-1} p_k \qquad (k = 0, 1, \dots). \tag{19}$$

This coincides with the expansion (8) of f^*.

Since $\|p_k^*\|_\infty < \varepsilon_k$ by (18), we conclude from Proposition 2 that $f^* \in \mathcal{B}_0$.

Writing $s_k^* = p_0^* + p_1^* + \dots + p_k^*$, a partial summation gives

$$\sum_{k=0}^{N} \varepsilon_k p_k^* = \varepsilon_N s_N^* + \sum_{k=0}^{N-1} (\varepsilon_k - \varepsilon_{k+1}) s_k^*.$$

Let now $|f^*(r\zeta)|$ be bounded in $0 \le r < 1$ for some $\zeta \in \mathbf{T}$. Proposition 3 implies that $|s_k^*(\zeta)|$ is also bounded in k. Since $s_k^*(r\zeta)$ is continuous in $0 \le r < 1$ for each k and since $\varepsilon_k - \varepsilon_{k+1} \ge 0$, we easily deduce that

$$f(r\zeta) = \sum_{k=0}^{\infty} (\varepsilon_k - \varepsilon_{k+1}) s_k^*(r\zeta)$$

is uniformly continuous in $0 \le r < 1$ and therefore has a finite limit as $r - 1$. ■

Proof of the Corollary: The function $f = \log g'$ belongs to $\mathcal{B}_0$ and therefore has a finite radial limit on a set $E \subset \mathbf{T}$ with cap $(E \cap I) > 0$ for every arc I of $\mathbf{T}$.

Let now $\zeta \in E$. Then g' has a finite nonzero radial limit at ζ and it follows [8, p. 305] that

$$\lim_{r \to 1} \frac{g(r\zeta) - g(\zeta)}{(r-1)\zeta} \ne 0, \infty$$

exists. Since J is asymptotically conformal, we conclude from a theorem of Warschawski [10, Satz II] or from [9, Corollary 3] that the unrestricted derivative exists. ■

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