G$_2$ and the rolling ball

Abstract: Understanding the exceptional Lie groups as the symmetry groups of simpler objects is a long-standing program in mathematics. Here, we explore one famous realization of the smallest exceptional Lie group, G$_2$: Its Lie algebra $\mathfrak{g}_2$ acts locally as the symmetries of a ball rolling on a larger ball, but only when the ratio of radii is 1:3. Using the split octonions, we devise a similar, but more global, picture of G$_2$: it acts as the symmetries of a 'spinorial ball rolling on a projective plane', again when the ratio of radii is 1:3. We describe the incidence geometry of both systems, and use it to explain the mysterious 1:3 ratio in simple, geometric terms.