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PURE BRAID SUBGROUPS OF BRAIDED THOMPSON'S GROUPS

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Abstract

We describe some properties of braided generalizations of Thompson's groups, introduced by Brin and Dehornoy. We give slightly different characterizations of the braided Thompson's groups BV and $\widehat{BV}$ which lead to natural presentations which emphasize one of their subgroup-containment properties. We consider pure braided versions of Thompson's group F. These groups, BF and $\widehat{BF}$, are subgroups of the braided versions of Thompson's group V. Unlike V, elements of F are order-preserving self-maps of the interval and we use pure braids together with elements of F thus again preserving order. We define these pure braided groups, give normal forms for elements, and construct infinite and finite presentations of these groups.

1. Introduction and definitions

Thompson's groups F and V have been studied from many perspectives. Both groups can be understood as groups of locally orientation-preserving piecewise-linear maps of the unit interval. In the case of F, these maps are homeomorphisms, and in the case of V the maps are right-continuous bijections. In both cases the breakpoints and discontinuities are restricted to be dyadic rational numbers, and the slopes, when defined, are powers of 2. Both groups can also be understood

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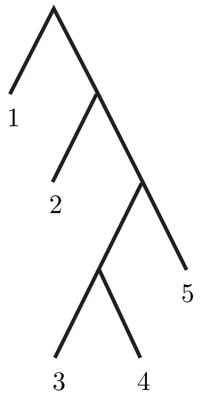
by means of rooted binary tree pair diagrams —order-preserving in the case of F. Cannon, Floyd and Parry [**7**] give an excellent introduction to these groups and several approaches to understanding their properties. Below, we will describe V in a manner which leads to natural descriptions of BV and $\widehat{BV}$, which we will show are naturally isomorphic to the braided Thompson groups described by Brin and Dehornoy.

A rooted binary tree is a finite tree where every node has valence three except the root, which has valence two, and the leaves, which have valence one. We usually draw such trees with the root on top and the nodes descending from it to the leaves along the bottom. The two nodes immediately below a node are its *children*. A node and its two children form a *caret*. A caret whose two children are leaves is called an *exposed caret*. We number the leaves of a rooted binary tree with $n-1$ carets and n leaves from 1 to n in left-to-right order.

A *tree pair diagram* is a triple (T_-, π, T_+), where T_- and T_+ are two binary trees with the same number of leaves n, and π is a permutation of the leaf numbers and is an element of S_n, regarded as the permutation group of the set of leaf numbers $\{1, 2, \ldots, n\}$. There are many equivalent tree pair diagrams representing the same group element, related by the notions of reduction and splittings. A *reduction* can be performed in a diagram if the left and right leaf numbers of an exposed caret in T_- are mapped by π to the left and right leaf numbers of an exposed caret in T_+, with the left leaf number of the exposed caret in T_- being mapped to the left leaf number of the exposed caret in T_+. In cases where a reduction is possible, we can replace each exposed caret with a leaf and renumber the leaves in both trees, giving an equivalent representative with a new permutation in S_{n-1} which pairs the leaves replacing the exposed carets in the trees in the natural way and pairs the leaves unaffected by the replacement in the same manner as before. The inverse operation of reduction is *splitting* —a tree pair diagram (T_-, π, T_+) which reduces to (S_-, π', S_+) is said to be a splitting of (S_-, π', S_+). A tree pair diagram is *reduced* if no reductions are possible. The set of binary tree pair diagrams thus admits an equivalence relation, whose classes consist of those diagrams which have a common reduced representative, with such reduced representatives being unique. Figure 1 shows a reduced diagram. The elements of V which are actually in F are precisely those elements for which the leaf number permutation is the identity.

Composition in V can be understood by means of these binary tree diagrams. If two elements in $t, s \in V$ are given by their reduced representative diagrams, (T_-, π, T_+) for t and (S_-, σ, S_+) for s, their composition st can be found by possible repeated splittings to find two tree diagrams



$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 4 & 3 & 1 \end{pmatrix}$$

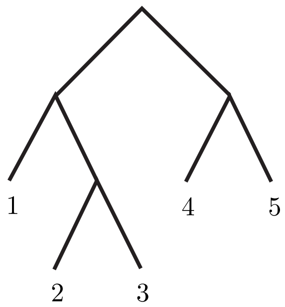


Figure 1. An element of V, expressed as the triple (T_-, π, T_+).

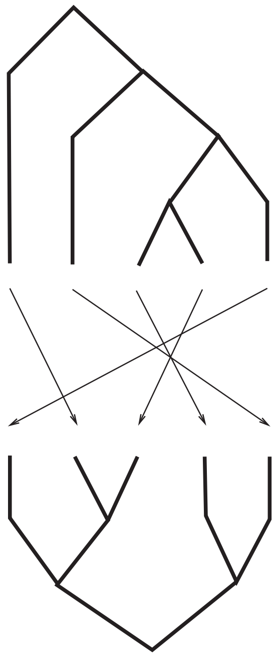


Figure 2. The same element, with the second tree upside down, and the arrows indicating the permutation of the leaves.

in the corresponding equivalence classes, (T'_-, π', T'_+) and (S'_-, σ', S'_+), such that $T'_+ = S'_-$. When this is achieved, the product st is represented by the diagram $(T'_-, \sigma'\pi', S'_+)$.

Brin [**2**], [**3**] and Dehornoy [**9**], [**8**] describe braided Thompson's groups, incorporating braids into tree pair diagrams. We construct braided tree pair diagrams by copying the construction of V, but using braids instead of permutations.

Definition 1.1. A *braided tree pair diagram* is a triple (T_-, b, T_+), where T_- and T_+ are binary rooted trees with the same number of leaves n, and b is a braid with n strands.

As in the case of V, an equivalence relation can be defined for braided tree pair diagrams using reductions and splittings.

Definition 1.2. If (T_-, b, T_+) is a braided tree pair diagram, where T_- and T_+ have n leaves and b is a braid on n strands, let $\phi_n(b) \in S_n$ be the usual permutation of $\{1, \dots, n\}$ induced by b. The diagram can be reduced if, for some i where $1 \le i \le n-1$, the i^{th} and $i+1^{\text{st}}$ strands are parallel and unbraided in the braid b, and the carets whose left and right leaves are numbered i and $i+1$ in the source tree, as well as the caret whose left and right leaves are numbered $[\phi_{n+1}(b)](i)$ and $[\phi_{n+1}(b)](i+1)$ respectively in the target tree, are both exposed. In this case, a *reduction* of (T_-, b, T_+) is a braided tree pair diagram (T'_-, b', T'_+), where T'_- is the tree T_- with the exposed caret whose leaves are numbered i and $i+1$ eliminated, T'_+ is obtained from T_+ by eliminating the caret whose leaves are numbered $[\phi_n(b)](i)$ and $[\phi_n(b)](i+1)$. The new braid b' is the braid on $n-1$ strands obtained from b by identifying the i^{th} and $i+1^{\text{th}}$ parallel strands. A *splitting* is the inverse process of a reduction, i.e. the replacement of the ith strand by two parallel untwisted ones to form a braid on one more strand, and enlarging the two trees by adding carets to the ith leaf of the source tree and leaf number $[\phi_{n-1}(b)](i)$ in the target tree.

See Figure 4 for an example of a splitting. The equivalence relation is then defined by relating two braided tree pair diagrams if there is a sequence of reductions and splittings that takes one to the other.

Multiplication of braided tree braid diagrams proceeds just as multiplication of tree pair diagrams does for V. If the target tree of the first diagram coincides with the source tree of the second diagram, the multiplication is done eliminating these trees:

$$(S, c, R)(T, b, S) = (T, cb, R)$$

where the braid multiplication is the usual one, since in this special case the two braids have the same number of strands. Now, given two equivalence classes, from the fact that two binary rooted trees always have a common subdivision, we see that there are always two representatives which satisfy the condition above and can be multiplied. Take the class of the product as a product of the classes. It is straightforward to see that this multiplication is well defined.

Definition 1.3. The group BV is the group of equivalence classes of braided tree pair diagrams, with the multiplication defined above.

For instance, the identity element in BV is the class of all diagrams of the form (T, id, T), and the inverse of the class of (T_-, b, T_+) is the class of (T_+, b^{-1}, T_-). From now on, we will abuse the language and omit "the class of" when we refer to elements of BV; that is, elements of BV will be referred to as braided tree pair diagrams, though it is understood that they are really equivalence classes of tree pair diagrams.

In Figure 3 an element of BV is depicted. We draw the rooted tree T_- with the root at the top, and the tree T_+ below with the root at the bottom, and then draw the braid between the leaves of the two trees as indicated.

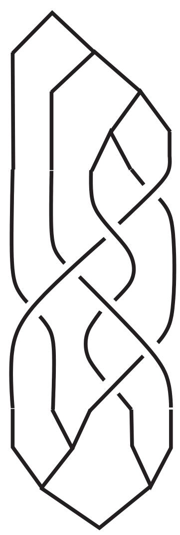


Figure 3. An element of BV, one of the preimages of the element drawn in Figures 1 and 2 under the map $\bar{\phi}$.

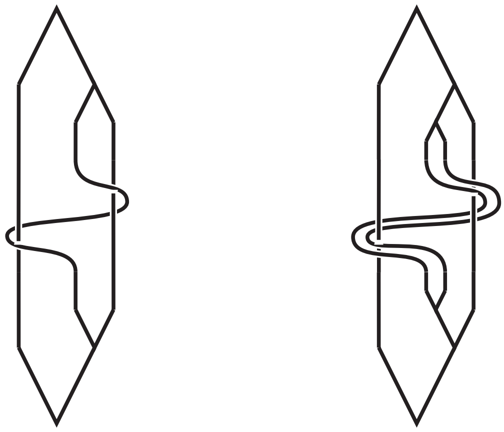


FIGURE 4. A splitting of an element of BV.

In [**2**], Brin describes a "larger" group $\widehat{BV}$; we describe how to modify our construction of the group BV to obtain a group isomorphic to Brin's $\widehat{BV}$. We again consider equivalence classes of triples, called braided forest pair diagrams (F_-, b, F_+), where F_- and F_+ are each sequences of binary trees of the form $(T_1, T_2, \dots)$ (we call this a forest, following Brin) in which all but finitely many of the T_i are trivial. The braid $b \in B_\infty$, that is, b is a braid on infinitely many strands which is eventually trivial. If we define multiplication, splittings, reductions, and equivalence classes just as before, the result is a group isomorphic to Brin's group $\widehat{BV}$.

It is fairly easy to see, from our descriptions of these groups, that each one sits inside the other in a natural way. To see that BV is a subgroup of $\widehat{BV}$, consider the subgroup of $\widehat{BV}$ consisting of elements in which only the first tree in the forest is nontrivial and the braid involves only the leaves in the first tree. It is clear that this is isomorphic to the group BV.

On the other hand, one can also see that $\widehat{BV}$ can be realized as a subgroup of BV. To do this, suppose we have an element (F_-, b, F_+) of $\widehat{BV}$; we describe how to view it as an element of BV. To form T_-, take an infinite all-right tree (a binary tree where each caret has a right child), and attach the root node of T_i in the source forest to leaf i of the all right tree. Do the same for T_+ using the target forest. There exists some i such that for any $j > i$, leaf j in both F_- and F_+ is simply a

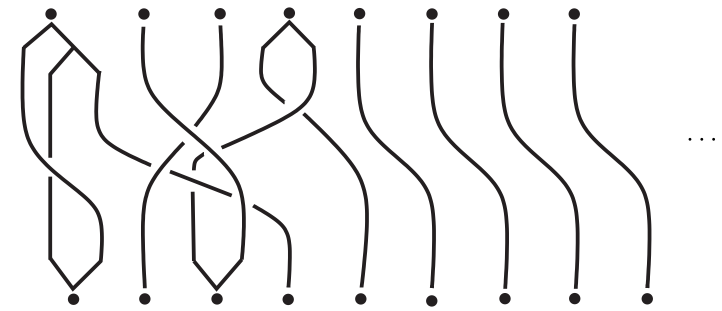


Figure 5. An element of $\widehat{BV}$, according to Brin's description.

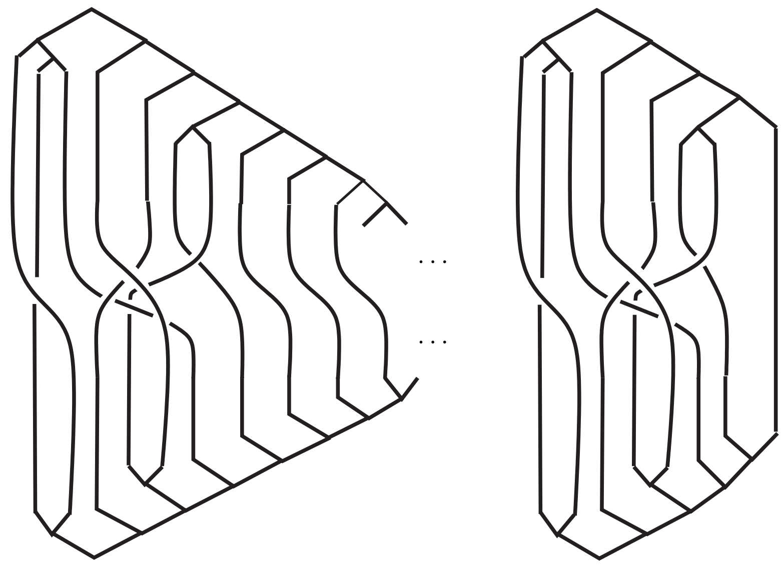


Figure 6. The element in Figure 5, shown inside BV, both before and after reducing the trivial braid at the end. Observe that elements in $\widehat{BV}$, by construction, will always produce a straight unbraided strand at the end. So we can identify $\widehat{BV}$ with the subgroup of BV of those elements where the last strand is unbraided.

trivial tree in both forests, and the braid b leaves strand j unbraided. Then we can remove the right subtree of the parent node of leaf i from the infinite source tree to obtain T_{-}, and remove the right subtree of the parent node of leaf $[\phi(b)](i)$ in the target tree to form T_- (where of course ϕ is the usual surjection from B_∞ to S_∞), in effect collapsing the subtrees into a single strand. Taking the obvious braid on n strands, where n is the number of leaves in the trees T_- and T_+, namely the braid which agrees with b on the first $n-1$ stands, and leaves the final strand unbraided, we obtain an element of BV. Thus, $\widehat{BV}$ sits inside BV as the subgroup of braided tree diagrams where the rightmost strand is always unbraided.

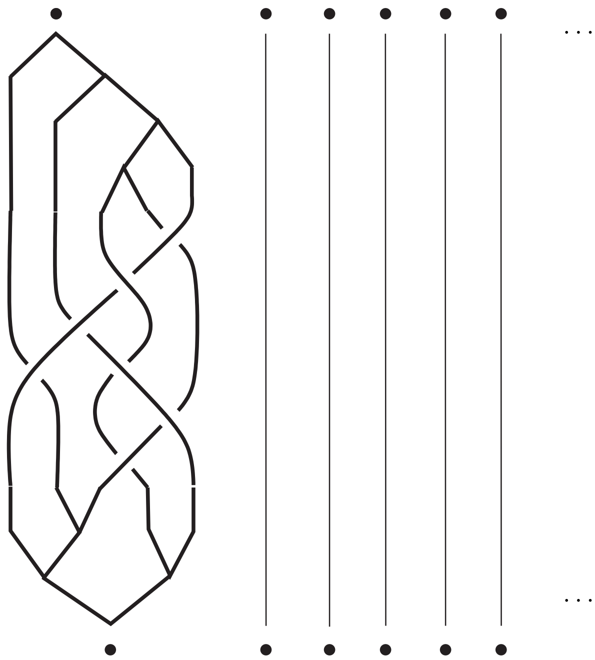


FIGURE 7. The element in BV from Figure 3 but now seen inside Brin's $\widehat{BV}$.

We begin this paper by providing finite and infinite presentations of BV which contain the presentations provided by Brin in [**2**] as subpresentations, highlighting the containment of BV in $\widehat{BV}$. Next, we describe subgroups BF of BV and $\widehat{BF}$ of $\widehat{BV}$. Just as F is the subgroup of V of order-preserving right continuous bijections of V, the groups BF and $\widehat{BF}$ are the subgroups of order-preserving elements of the braided versions of V. The order is preserved by using generators which come from F and generators which involve pure braids. We describe normal forms for elements in these subgroups and obtain infinite and finite presentations for these groups. Dehornoy [**9**] calls this pure braid subgroup $PB_\bullet$, the group of pure parenthesized braids.

2. The braided Thompson's group BV

In [**2**], an infinite presentation for $\widehat{BV}$ is given. The generators in this presentation are the generators in the standard infinite presentation for Thompson's group F, as well as the generators for B_∞. Here B_∞ is considered as a direct limit of the groups B_n, where B_n is included in B_{n+1} via adding one strand on the right. Now $\widehat{BV}$ sits naturally as a subgroup of BV; it is isomorphic to the subgroup of all elements represented by braided tree diagrams in which the rightmost strand is unbraided. Although presentations for BV, both finite and infinite, are given in [**3**], they are not related in a simple way to the presentation for $\widehat{BV}$. Instead, we give a presentation for BV which contains Brin's presentation for $\widehat{BV}$ as a subpresentation. First, we define the set of generators. Recall that any element of BV can be represented by a braided tree diagram (T_-, b, T_+) where both T_- and T_+ have n leaves and b is a braid in B_n. A single tree can be thought of as a *positive* element of Thompson's group F, by choosing it as the target tree, and choosing an *all-right* tree, which is a tree whose carets are all right children of their parent carets, as the source tree in the tree pair diagram. These positive elements correspond to elements which are positive words in F with respect to the infinite generating set $\{x_0, x_1, \dots\}$. Similarly, an element represented by a tree pair diagram in which the target tree is an all-right tree is a *negative* element of Thompson's group F. The correspondence between tree pair diagrams and normal forms with respect to the infinite generating set is given by the process of exponents of leaves, as described by Cannon, Floyd and Parry [**7**] and Fordham [**10**]. All-right trees have all leaf exponents zero, and thus the normal forms in F for tree pair diagrams which involve one all-right tree will be *purely negative* (involving only negative powers of the generators x_i) or *purely positive* (involving only positive powers of generators.) We will denote by R_n the all-right tree which has n leaves.

We can factor an element (T_-, b, T_+) into three pieces, using all-right trees of the appropriate number of leaves, in a manner similar to that done for elements of Thompson's group T by Burillo, Cleary, Stein and Taback [**5**]. The resulting three elements in this factorization are

$$(R_n, \mathrm{id}, T_+) \quad (R_n, b, R_n) \quad (T_-, \mathrm{id}, R_n),$$

and the product of the elements represented by these three diagrams yields the original group element. In general, these three tree pair diagrams will not be reduced; in order for each of them to have the same number of carets, we may need to take unreduced representatives for as many as two of the three terms. By enlarging trees in this manner, it is clear that every element of BV can be factored this way.

Hence, we can always think of an element of BV as if it were composed of two elements of Thompson's group F, one positive and one negative, and one braid. It makes sense then to consider, as a set of generators of BV, the set of generators for F and the set of generators for the braid groups, interpreted as braided tree pair diagrams between all-right trees.

The infinite set of generators for F consists of the elements x_i with $i \geq 0$. We can define the x_i as conjugates of x_1 by powers of x_0 as $x_i = x_0^{-i+1} x_1 x_0^{i-1}$. Figure 8 shows x_2 in both tree pair diagram form and in braided tree form. These generators from F are enough to produce the two elements (T_-, id, R_n) and (R_n, id, T_+) in BV.

FIGURE 8. The generator x_2 of F, in standard and in braided form.

We can consider the element (R_n, b, R_n) as an element in the appropriate braid group B_n. Now this copy of B_n is generated by $n-1$ elements, the i^{th} of which braids strand i over strand $i+1$. We do not, however, need to include all of these as generators for every B_n. The generators which do not involve braiding the last two strands can be obtained from the generators of the braid groups B_j with $j < n$ by splitting the last strand, possibly repeatedly. For example, once we have included the generator σ_2 in B_5 which crosses the second strand over the third strand of five strands, there is no need to include σ_2 in B_6 as there is a representative of the earlier σ_2 which exactly realizes the crossing of the second strand over the third strand of six strands, and which is obtained by performing a single splitting on the last strand. However, for braiding which involves the rightmost strand, splitting the last strand does not accomplish the same thing as adding a parallel, unbraided strand, in the typical way that B_{n-1} is included in B_n.

For this reason we must consider two sets of generating braids, one which leaves the rightmost strand unbraided and one which does not. We define σ_i to be the element represented by the braided tree diagram (R_{i+2}, a_i, R_{i+2}), where a_i is the braid on $i+2$ strands which crosses strand i over strand $i+1$. Similarly, τ_i is the element represented by the diagram (R_{i+1}, b_i, R_{i+1}), where b_i is the braid on $i+1$ strands which crosses strand i over stand $i+1$. Then the set $\{\sigma_1, \sigma_2, \dots, \sigma_{n-2}, \tau_{n-1}\}$ generates the copy of B_n containing all elements of BV represented by diagrams of the form (R_n, b, R_n). Notice that the x_i together with the σ_i generate the copy of $\widehat{BV}$ inside BV, and they correspond to Brin's generators. We have shown:

Proposition 2.1. *The elements x_i, for $i \geq 0$, σ_i for $i \geq 1$, and τ_i for $i \geq 1$ form a set of generators for BV.*

There are three types of natural relations among these generators. First, there are the relations involving only the generators of F, namely $x_j x_i = x_i x_{i+j}$ for $j > i$. These are the relations for the standard presentation for F. Next, we expect to need relations for each copy of B_n. These yield four types of relations:

- $\sigma_i \sigma_j = \sigma_j \sigma_i$, for $j \geq i+2$
- $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$
- $\sigma_i \tau_j = \tau_j \sigma_i$, for $j \geq i+2$
- $\sigma_i \tau_{i+1} \sigma_i = \tau_{i+1} \sigma_i \tau_{i+1}$.

Finally, there are relations governing the interactions between the generators for F and the generators for the braid groups B_n.

- $\sigma_i x_j = x_j \sigma_i$, for $i < j$
- $\sigma_i x_i = x_{i-1} \sigma_{i+1} \sigma_i$
- $\sigma_i x_j = x_j \sigma_{i+1}$, for $i \geq j+2$
- $\sigma_{i+1} x_i = x_{i+1} \sigma_{i+1} \sigma_{i+2}$
- $\tau_i x_j = x_j \tau_{i+1}$, for $i \geq j+2$
- $\tau_i x_{i-1} = \sigma_i \tau_{i+1}$
- $\tau_i = x_{i-1} \tau_{i+1} \sigma_i$.

In preparation for showing that the relations above give a presentation, we first introduce a special class of words in the generators. We would like to identify those words in the generators which could be identified easily with a triple of diagrams in BV. As noted earlier, any element of BV can be represented by a triple of braided tree diagrams of the form (R_n, id, T_+) (R_n, b, R_n) (T_-, id, R_n). Such a triple leads easily to a word in the generators as follows. The group element represented by the first diagram is a positive element a of $F \subset BV$, and may be expressed uniquely as a word of the form $x_{i_1}^{r_1} x_{i_2}^{r_2} \dots x_{i_k}^{r_k}$, where $i_1 < i_2 < \cdots < i_k$ and $r_m \geq 1$ for all m. Similarly, the group element $c \in F \subset BV$ represented by the third diagram can be uniquely expressed as a word of the form $x_{j_l}^{-s_l} \dots x_{j_2}^{-s_2} x_{j_1}^{-s_1}$, where $j_1 < j_2 < \cdots < j_l$ and $s_m \geq 1$ for all m. Now the group element represented by the middle diagram may be represented as some word in the generators $\sigma_1, \sigma_2, \dots, \sigma_{n-2}, \tau_{n-1}$, and their inverses. For convenience, we will call any word in this set of generators and their inverses *a word in the B_n generators*. Note that if such a word contains no τ generators, it can be considered a word in the B_n generators for many values of n. Notice that the minimum number of carets required in the trees for tree pair diagrams representing a and c respectively, is at most $n-1$. The concatenation of the three words described above yield a word which cannot serve as a normal form, since we have not specified preferred arrangements of the σ's and τ and furthermore, there are many different triples of tree pair diagrams representing any element. However, these words are nice in that any word of the above special form can be easily translated into a triple of diagrams, and we find them to be useful tools.

Given a word $w \in F$, denote by $N(w)$ the number of carets in either tree in the reduced binary tree diagram representing it. Here is the algebraic description of blocks, which are these words which come from a single triple of diagrams.

Definition 2.2. A word in the generators $x_i^{\pm 1}$, $\sigma_i^{\pm 1}$, $\tau_i^{\pm 1}$ is called a *block* if it is of the form $w_1 w_2 w_3^{-1}$ where

(1) w_1 is of the form $x_{i_1}^{r_1} x_{i_2}^{r_2} \dots x_{i_k}^{r_k}$, where $i_1 < i_2 < \cdots < i_k$ and $r_m \geq 1$ for all m.
(2) w_3 is of the form $x_{j_1}^{s_1} x_{j_2}^{s_2} \dots x_{j_l}^{s_l}$, where $j_1 < j_2 < \cdots < j_l$ and $s_m \geq 1$ for all m, and by w_3^{-1} we mean the word $x_{j_l}^{-s_l} \dots x_{j_2}^{-s_2} x_{j_1}^{-s_1}$.
(3) Let $N = \max(N(w_1), N(w_2))$. Then there exists an integer n, $n \geq N+1$, such that w_2 is a word in the B_n generators.

Then we have the following lemma:

Lemma 2.3. *A block $w_1 w_2 w_3^{-1}$ is the identity in BV if and only if w_1 and w_3 are the same word, and w_2 is the identity in the copy of the braid group B_n generated by $\tau_{n-1}^{\pm 1}$ and $\sigma_i^{\pm 1}$ where $1 \leq i \leq n-2$.*

Proof: The lemma follows directly from the fact that any word which is a block $w_1 w_2 w_3^{-1}$ can be represented by a braided tree diagram (T_-, b, T_+) where w_1 is represented by (T_-, id, R_n), w_2 is represented by (R_n, b, R_n), and w_3^{-1} is represented by (R_n, id, T_+), and from any such triple of diagrams a block can be read off, unique up to the choice of the word in B_n expressing b. Since the identity in BV can be represented by the diagram consisting of the tree with only one vertex, and the trivial braid on one strand, all other diagrams representing the identity result from splitting strands, and will always have two identical trees with the trivial braid. What is essential here is that the splitting and reduction operations which can be used to move within the equivalence class of braided tree pair diagrams have the property that any splitting or reduction of a non-trivial braid remains nontrivial and that any splitting or reduction of a trivial braid remains trivial. Such diagrams translate into blocks of the form described in the lemma. □

We will use these blocks to prove:

Theorem 2.4. *The group BV admits a presentation with generators:*

- x_i, *for* $i \geq 0$,
- σ_i, *for* $i \geq 1$,
- τ_i, *for* $i \geq 1$,

and relators

(A) $x_j x_i = x_i x_{j+1}$, *for* $j > i$

(B1) $\sigma_i \sigma_j = \sigma_j \sigma_i$, *for* $j - i \geq 2$

(B2) $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$

(B3) $\sigma_i \tau_j = \tau_j \sigma_i$, *for* $j - i \geq 2$

(B4) $\sigma_i \tau_{i+1} \sigma_i = \tau_{i+1} \sigma_i \tau_{i+1}$

(C1) $\sigma_i x_j = x_j \sigma_i$, *for* $i < j$

(C2) $\sigma_i x_i = x_{i-1} \sigma_{i+1} \sigma_i$

(C3) $\sigma_i x_j = x_j \sigma_{i+1}$, *for* $i \geq j + 2$

(C4) $\sigma_{i+1} x_i = x_{i+1} \sigma_{i+1} \sigma_{i+2}$

(D1) $\tau_i x_j = x_j \tau_{i+1}$, *for* $i - j \geq 2$

(D2) $\tau_i x_{i-1} = \sigma_i \tau_{i+1}$

(D3) $\tau_i = x_{i-1} \tau_{i+1} \sigma_i$.

This presentation appears without proof in J. Belk's thesis [**1**].

Proof: Let G be the abstract group given by the presentation above. We map G to BV via ϕ by sending each generator to the element of BV with the same name. All relations in the presentation hold in BV, so ϕ is a well-defined homomorphism. Proposition 2.1 shows that the map is surjective, so it remains only to show that ϕ is injective. To show this, we must show that any word in the generators which maps to the identity in BV is already the identity in G. Now by Lemma 2.3, this is true if the word in question happens to be a block. So we are done once we show that the relations in G are sufficient to transform any word into a block. But since any generator is itself a block, an arbitrary word of length k is trivially the product of k blocks. So to show ϕ is injective it is sufficient to prove that a word in G which is the product of two blocks can be rewritten, using the relations in G, as a single block. We first prove a series of three preliminary lemmas, from which we will deduce this fact in Lemma 2.8, which will complete the proof of the theorem. □

Our first lemma permits us to push the $x_i^{\pm 1}$ generators to the left or right of the braid generators, which helps move a word toward block form.

Lemma 2.5. *If w is a word in the B_n generators, and $i \leq n-2$, then $x_i^{-1}w$ is equivalent in G to either $\bar{w}x_{i'}^{-1}$ or $\bar{w}$, where $\bar{w}$ is a word in the B_{n+1} generators, and $i' \leq n-2$. Similarly, under the same conditions on all indices, a word wx_i may be replaced by either $x_{i'}\bar{w}$ or $\bar{w}$.*

Proof: We describe first how to push x_i^{-1} past the σ and τ generators. Pushing past σ type generators is always possible, but in order to push past τ's we must carefully keep track of the index of the x_i^{-1} as it moves along. Using the relations of type C, we may replace $x_k^{-1}\sigma_j^{\pm 1}$ by $w(\sigma)x_{k'}^{-1}$, where $w(\sigma)$ is a word in the σ generators and their inverses of length one or two. Furthermore, the maximum index appearing in $w(\sigma)$ is $j+1$. Now the index k' can, in general, be either k, $k-1$, or $k+1$. However, it only increases to $k+1$ in the case where $j = k+1$ also. So since the initial index i satisfies $i \leq n-2$ and $j \leq n-2$, even a series of such replacements results in the presence of $x_{i'}^{-1}$ with $i' \leq n-2$. This is important, since relations (D2) and (D3) allow replacement of $x_{n-2}^{-1}\tau_{n-1}^{\pm 1}$ by $\tau_n^{\pm 1}\sigma_{n-1}^{\pm 1}$, and relations (D1) allow us to replace $x_k^{-1}\tau_{n-1}^{\pm 1}$ by $\tau_n^{\pm 1}x_k^{-1}$ if $k \leq n-3$. Hence, $x_i^{-1}w$ can be replaced by either $\bar{w}x_{i'}^{-1}$ or simply $\bar{w}$ as claimed. The argument for wx_i is similar. □

Next we prove a lemma showing that a word in the B_n generators can always be pumped up to a word in the B_{n+1} generators at the possible expense of tacking on an x_i generator.

Lemma 2.6. *Let w be a word in the B_n generators. Then using the relators in G, w may be replaced by either $\bar{w}$ or $\bar{w}x_i^{-1}$ where $i \leq n-2$ and $\bar{w}$ is a word in the B_{n+1} generators. Similarly, w may also be replaced by either $\bar{w}$ or $x_i\bar{w}$ where $\bar{w}$ is a word in the B_{n+1} generators.*

Proof: Consider the leftmost occurrence of τ_{n-1} in the word w, that is, $w = w_1\tau_{n-1}^{\pm 1}w_2$, where w_1 has only σ generators. Using relations (D2) or (D3) depending on the exponent of τ_{n-1}, replace w by $w_1\sigma_{n-1}^{\pm 1}\tau_n^{\pm 1}x_{n-2}^{-1}w_2$, and then apply Lemma 2.5 to $x_{n-2}^{-1}w_2$ to replace it with either $\bar{w}_2x_i^{-1}$ or $\bar{w}_2$ with $i \leq n-2$ and where $\bar{w}_2$ is a word in the B_{n+1} generators. Then the desired $\bar{w}$ is $w_1\sigma_{n-1}^{\pm 1}\tau_n^{\pm 1}\bar{w}_2$. Similarly, working from the right, w can be replaced by either $x_i\bar{w}$ or $\bar{w}$. ☐

The two previous lemmas will now be used to show that the relators allow us to transform the product of two blocks to a new product of two blocks where the combined length of the middle two of the 6 subwords involved is reduced.

Lemma 2.7. *Let $w = w_1w_2w_3^{-1}$ and $v = v_1v_2v_3^{-1}$ be two blocks. Then the relations in G allow us to replace the word wv by $w_1'w_2'w_3'^{-1}v_1'v_2'v_3'^{-1}$, the product of two blocks $w' = w_1'w_2'w_3'^{-1}$ and $v' = v_1'v_2'v_3'^{-1}$, where $l(v_1') + l(w_3') < l(v_1) + l(w_3)$.*

Proof: Let x_{i_w} and x_{i_v} be the first letters in w_3 and v_1. If they are the same, we can delete the pair $x_{i_w}^{-1}x_{i_v}$ and we are done. If not, suppose $i_w < i_v$ (if $i_v < i_w$ a similar argument works, truncating v_3 and absorbing x_{i_v} into w). Let $w_1' = w_1$, $w_2' = w_2$, and let w_3' be w_3 with x_{i_w} deleted. Now we use the relations (A) to replace $x_{i_w}^{-1}v_1$ by $v_1'x_{i_w}^{-1}$. Note that v_1' and v_1 have the same length, but $N(v_1') = N(v_1)+1$, since each index in v_1 is increased by 1 as $x_{i_w}^{-1}$ moves past it (see Theorem 3 of [**4**]). Next, suppose v_2 is a word in $\tau_{n-1}^{\pm 1}$ and $\sigma_j^{\pm 1}$ with $1 \leq j \leq n-2$. Then $N(v_1) \leq n$ and $N(v_3) \leq n$, so $N(v_1') \leq n+1$. We must replace $x_{i_w}^{-1}v_2v_3^{-1}$ by $v_2'v_3'^{-1}$ so that $v_1'v_2'v_3'^{-1}$ is a block. We will consider two cases.

Case 1: If $i_w \leq n-2$, we use Lemma 2.5 to replace $x_{i_w}^{-1}v_2$ by $v_2'x_i^{-1}$ with $i \leq n-2$ and v_2' a word in $\tau_n^{\pm 1}$ and $\sigma_j^{\pm 1}$. Then $(v_3x_i)^{-1}$ can be rewritten using relations (A) as a word $(v_3')^{-1}$ so that v_3' is a positive word in the generators of F with increasing indices from left to right.

Then it again follows from [**4**] that $N(v_3') = \max(N(v_3)+1, i+2)$. Hence $N(v_3') \leq n+1$, since $N(v_3)+1 \leq n+1$ and $i+2 \leq n$, and this implies that $v_1'v_2'(v_3')^{-1}$ is a block as desired.

Case 2: If $i_w > n-2$, it is necessary to first use Lemma 2.6 to replace v_2 by either $\bar{v}_2$ or $\bar{v}_2 x_i^{-1}$ where $i \leq n-2$. If x_i^{-1} is present, we use relations (A) to replace $(v_3 x_i)^{-1}$ by $\bar{v}_3^{-1}$, and we see that $N(\bar{v}_3) = \max(N(v_3)+1, i+2) \leq n+1$. We continue applying Lemma 2.6 and absorbing any resulting x_i^{-1} letters into the v_3^{-1} part of the word in this manner, and after $i_w - (n-2)$ repetitions we have replaced $x_{i_w}^{-1} v_1 v_2 v_3^{-1}$ by $v_1' x_{i_w}^{-1} \bar{v}_2 \bar{v}_3^{-1}$, where $\bar{v}_2$ a word in the B_{i_w+2} generators, and $\bar{v}_3$ is a word in the x_i with indices increasing from left to right with $N(\bar{v}_3) \leq n + (i_w - (n-2)) = i_w + 2$. Now just as before we can apply Lemma 2.5 to replace $v_1' x_{i_w}^{-1} \bar{v}_2 \bar{v}_3^{-1}$ by $v_1' v_2' x_i^{-1} \bar{v}_3^{-1}$ where $i \leq i_w$, and v_2' is a word in the B_{i_w+3} generators. When we use relations (A) to replace $(\bar{v}_3 x_i)^{-1}$ by $v_3'^{-1}$, $N(v_3') = \max(N(\bar{v}_3)+1, i+2)$. But since $N(\bar{v}_3)+1 \leq n+(i_w-(n-2)) = i_w+3$ and $i+2 \leq i_w+2$, $N(v_3') \leq i_w+3$, and hence $v_1'v_2'v_3'$ is a block.

□

Now we are in a position to prove the final lemma which completes the proof of Theorem 2.4.

Lemma 2.8. *The product of two blocks may be rewritten, using the relations of G, as a single block.*

Proof: Let $w = w_1 w_2 w_3^{-1}$ and $v = v_1 v_2 v_3^{-1}$ be two blocks. We apply Lemma 2.7, at most $l(w_3) + l(v_1)$ times, to replace wv by $w_1' w_2' v_2' (v_3')^{-1}$ where w_2' is a word in the B_n generators, v_2' is a word in the $B_{n'}$ generators, $N(w_1') \leq n$, and $N(v_3') \leq n'$. If $n = n'$, declaring $u_1 = w_1'$, $u_2 = w_2' v_2'$, and $u_3 = v_3'$ shows that $u_1 u_2 u_3^{-1} = w_1' w_2' v_2' (v_3')^{-1}$ is a block. If not, say $n' < n$, we apply Lemma 2.6 $n - n'$ times to replace v_2' by a word $\bar{v}_2$, a word in the B_n generators, followed by some new x^{-1} generators, so that $v_2' v_3'^{-1}$ has been replaced by $\bar{v}_2 (v_3' x_{i_1} x_{i_2} \cdots x_{i_{n-n'}})^{-1}$ where $i_j \leq n'+j-3$ for $1 \leq j \leq n-n'$. Now $N(v_3' x_{i_1}) = \max(N(v_3)+1, i_1+2) \leq n'+1$. So inductively, we have that $N(v_3' x_{i_1} x_{i_2} \cdots x_{i_{n-n'}}) \leq n$, and hence declaring $u_1 = w_1'$, $u_2 = w_2' \bar{v}_2$, and $u_3 = \bar{v}_3$, which shows that $u_1 u_2 u_3^{-1} = w_1' w_2' \bar{v}_2 \bar{v}_3^{-1}$ is a block. Of course, if $n' > n$, a similar argument works, pumping up indices in w_2' instead. □

We remark that an infinite presentation of Thompson's group V is easily obtained from the presentation for BV by adding two more infinite families of relators, $\sigma_i^2 = 1$ and $\tau_i^2 = 1$ for all i.

3. A finite presentation for BV

It is common for these types of infinite presentations for Thompson-type groups to reduce to finite presentations. For example, in [**7**] an inductive argument is spelled out which obtains the standard two generator-two relator presentation for F from the standard infinite presentation. Brin uses similar arguments to obtain finite presentations for BV and $\widehat{BV}$ from his infinite ones. In a similar manner, the infinite presentation for BV in the previous section reduces to a finite presentation with 4 generators and only 18 relators, an improvement over the presentation in [**3**], which has 4 generators and 26 relators.

Theorem 3.1. *The group BV admits a finite presentation with generators x_0, x_1, σ_1, τ_1 and relators*

(a) $x_2x_0 = x_0x_3$, $x_3x_1 = x_1x_4$

(c1) $\sigma_1x_2 = x_2\sigma_1$, $\sigma_1x_3 = x_3\sigma_1$, $\sigma_2x_3 = x_3\sigma_2$, $\sigma_2x_4 = x_4\sigma_2$

(c3) $\sigma_2x_0 = x_0\sigma_3$, $\sigma_3x_1 = x_1\sigma_4$

(c4) $\sigma_1x_0 = x_1\sigma_1\sigma_2$, $\sigma_2x_1 = x_2\sigma_2\sigma_3$

(d1) $\tau_2x_0 = x_0\tau_3$, $\tau_3x_1 = x_1\tau_4$

(d2) $\tau_1x_0 = \sigma_1\tau_2$, $\tau_2x_1 = \sigma_2\tau_3$

(b1) $\sigma_1\sigma_3 = \sigma_3\sigma_1$

(b2) $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$

(b3) $\sigma_1\tau_3 = \tau_3\sigma_1$

(b4) $\sigma_1\tau_2\sigma_1 = \tau_2\sigma_1\tau_2$

where the letters in the relators not in the set of 4 *generators are defined inductively by $x_{i+2} = x_i^{-1}x_{i+1}x_i$ for $i \geq 0$, $\sigma_{i+1} = x_{i-1}^{-1}\sigma_ix_i\sigma_i^{-1}$ for $i \geq 1$, and $\tau_{i+1} = x_{i-1}^{-1}\tau_i\sigma_i^{-1}$ for $i \geq 1$.*

Proof: That the two (a) relators yield inductively all (A) relators in the infinite presentation is a standard argument, given in [**7**]. Notice that the relators (C2) and (D3) in the infinite presentation are precisely the relations used to inductively define the higher index generators in the infinite presentation. Now a straightforward induction yields the (C1) relators in the infinite presentation from the (c1) relators, then the (C3) relators from the (c3) relators, and so on, in the order the groups of relators are listed in the finite presentation above. As an example, we spell out the induction for the (B3) relators. So suppose we have (b3), or $\sigma_1\tau_3 = \tau_3\sigma_1$. Then suppose inductively that we have established $\sigma_1\tau_i = \tau_i\sigma_1$ for $3 \le i < k$, where $k \ge 4$. Then $\sigma_1\tau_k = \sigma_1(x_{k-2}^{-1}\tau_{k-1}\sigma_{k-1}^{-1})$ using the relator defining τ_k. Now we can move the σ_1 to the right, first using the (C1) relators, then the inductive hypothesis, and finally the (B1) relators, and then use the defining relation for τ_k in the other direction, to obtain the relator $\sigma_1\tau_k = \tau_k\sigma_1$. Therefore, by induction, $\sigma_1\tau_k = \tau_k\sigma_1$ for all $k \ge 3$. Now suppose that we have $\sigma_i\tau_j = \tau_j\sigma_i$ for $j - i \ge 2$, $1 \le i < k$, and $k \ge 2$. Then it follows that $\sigma_k\tau_j = \tau_j\sigma_k$ for $j - k \ge 2$. We replace σ_k in the word $\sigma_k\tau_j$ by $x_{k-2}^{-1}\sigma_{k-1}x_{k-1}\sigma_{k-1}^{-1}$, and then moves the τ_j to the left, first using the inductive hypothesis to obtain $x_{k-2}^{-1}\sigma_{k-1}x_{k-1}\tau_j\sigma_{k-1}^{-1}$, then using the (D1) relators to obtain $x_{k-2}^{-1}\sigma_{k-1}\tau_{j-1}x_{k-1}\sigma_{k-1}^{-1}$, and finally using the inductive hypothesis again to obtain $x_{k-2}^{-1}\tau_{j-1}\sigma_{k-1}x_{k-1}\sigma_{k-1}^{-1}$. Now use (D1) relators to move τ_{j-1} left, to obtain $\tau_j x_{k-2}^{-1}\sigma_{k-1}x_{k-1}\sigma_{k-1}^{-1}$. But now the rightmost four letters can be replaced by σ_k using the defining relation for σ_k in reverse, showing that $\sigma_k\tau_j = \tau_j\sigma_k$. Hence, by induction, all (B3) relators hold. $\square$

For the corresponding finite presentation for V, we note that the relations $\tau_i^2 = 1$ and $\sigma_i^2 = 1$ for $i \ge 2$ can be deduced inductively from the two relations $\tau_1^2 = \sigma_1^2 = 1$, using the (C2) and (C3) relators in the case of σ_i, and the (D2) and (D3) relators for τ_i. This yields a presentation for V with 4 generators and 20 relators, not quite as efficient as the presentation in [**7**] with 14 relators.

4. The group *PBV*

The pure braid groups P_n are the groups of braids where the ith strand is braided with the other strands but returns to the ith position. There are several possible ways to construct analogous subgroups of BV. One way is by considering the standard short exact sequences for the braid groups, involving the pure braid groups and the permutation groups. For each n, we have:

$$1 \longrightarrow P_n \longrightarrow B_n \xrightarrow{\phi_n} S_n \longrightarrow 1$$

which maps a braid to its permutation, and whose kernel is the pure braid group P_n. This family of maps ϕ_n collectively induces a map

$$\bar{\phi}\colon BV \longrightarrow V$$

defined by $\bar{\phi}(T_-, b, T_+) = (T_-, \phi_n(b), T_+)$, where we use the appropriate ϕ_n for the number of leaves in either tree.

Let $PBV = \ker \bar{\phi}$. By definition, a diagram (T_-, b, T_+) represents an element in PBV if it maps to the identity in V, that is, if $\phi(b) = \mathrm{id}$, and if $T_- = T_+$. Hence, PBV is the subgroup of BV which consists of those elements which admit a representative (T, p, T) on which the two trees are the same and the braid is pure. If an element admits one representative where the two trees are equal, then every representative will have the trees being equal. So PBV is a subgroup, because the product of such two elements also has representatives where the two trees are equal. Note that it is crucial in this construction that the braid is a pure braid.

The main result concerning the group PBV is the following.

Theorem 4.1. *The group PBV is not finitely generated.*

Proof: Given two elements of PBV by their diagrams (T, p, T) and (S, q, S), their product always admits a representative diagram (R, r, R), where R is the *least common multiple* of S and T —that is, the minimal tree which contains both S and T as subtrees. Hence, if PBV were to be generated by a finite set (T_i, p_i, T_i), for $i = 1, \ldots, k$, every tree in PBV would admit a representative whose tree would be the least common multiple of the T_i. There are elements whose smallest representatives are of increasing size and thus PBV cannot be finitely generated. $\square$

5. The braided Thompson's group *BF*

From the map $\bar{\phi}$ defined above, since F is the subgroup of V of those elements whose permutation is the identity, we can define the group $BF = \bar{\phi}^{-1}(F)$. The group BF is the subgroup of BV of those elements which admit a representative (T_-, p, T_+), where p is a pure braid. Note the contrast with PBV —here the two trees are not necessarily equal. In fact, PBV is a subgroup of BF, and observing the restriction to BF of the map $\bar{\phi}$ above, it is easy to see that PBV is also the kernel of $\bar{\phi}\big|BF$. Thus the diagram below is commutative:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & PBV & \longrightarrow & BF & \xrightarrow{\bar{\phi}|BF} & F & \longrightarrow & 1 \\
 & & \Big\| & & \Big\downarrow & & \Big\downarrow & & \\
1 & \longrightarrow & PBV & \longrightarrow & BV & \xrightarrow{\bar{\phi}} & V & \longrightarrow & 1
\end{array}$$

The main goal of the remainder of this paper is to prove that BF is finitely presented and to find both finite and infinite presentations.

Finding generators for BF is not difficult. Just as for BV, an element of BF is given by a triple (T_-, p, T_+), where this time p is a pure braid. Again, we factor the element into three pieces

$$(T_-, \mathrm{id}, R_n) \quad (R_n, p, R_n) \quad (R_n, \mathrm{id}, T_+),$$

where the individual diagrams may not be reduced. Hence, we can always think of an element of BF as if it were composed of two elements of Thompson's group F, one positive and one negative, and one pure braid. Again, just as for BV, we take as a set of generators of BF, the set of generators for F and the set of generators for the pure braid groups, interpreted as braided tree pair diagrams between all-right trees. We consider the element (R_n, p, R_n) as an element of the appropriate group P_n of pure braids. To generate these groups P_n we would like to use the braids A_{ij}, for $i < j$, which wrap the ith strand around the jth one. See Hansen [**11**] for details of these generating sets. The process of obtaining the generators of BF from the generators A_{ij} of P_n is the same than the process specified above for BV from the standard braid generators.

We will denote by α_{ij} the element $(R_{j+1}, A_{ij}, R_{j+1})$, and by β_{ij} the element (R_j, A_{ij}, R_j). As in BV, the differences between these two families of generators are whether or not the last strand is involved in the braiding. Figure 9 shows an example of the two generators of BF corresponding to a generator A_{ij}.

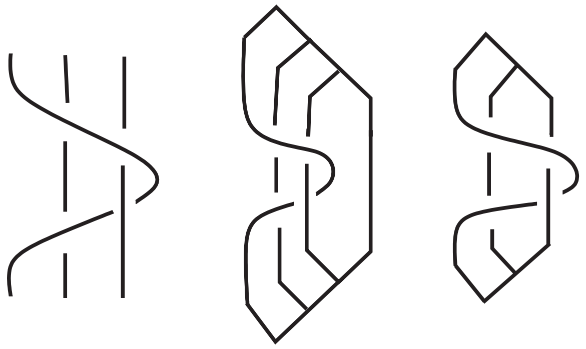


Figure 9. The generator A_{13} in P_3, with its corresponding generators α_{13} and β_{13} in BF.

The proof of the following proposition is analogous to the proof of Proposition 2.1.

Proposition 5.1. *The elements x_i, for $i \geq 0$, α_{ij}, for $1 \leq i < j$, and β_{ij}, for $1 \leq i < j$, form a set of generators of BF.*

In the next theorem we will give a presentation for the group BF. The relators are going to be divided into four families. The family (A) is obtained from the relators of F. The family (B) is obtained from the presentation of the pure braid group:

- $A_{rs}^{-1} A_{ij} A_{rs} = A_{ij}$, if $1 \leq r < s < i < j \leq n$ or $1 \leq i < r < s < j \leq n$
- $A_{rs}^{-1} A_{ij} A_{rs} = A_{rj} A_{ij} A_{rj}^{-1}$, if $1 \leq r < s = i < j \leq n$
- $A_{rs}^{-1} A_{ij} A_{rs} = (A_{ij} A_{sj}) A_{ij} (A_{ij} A_{sj})^{-1}$, if $1 \leq r = i < s < j \leq n$
- $A_{rs}^{-1} A_{ij} A_{rs} = (A_{rj} A_{sj} A_{rj}^{-1} A_{sj}^{-1}) A_{ij} (A_{rj} A_{sj} A_{rj}^{-1} A_{sj}^{-1})^{-1}$, if $1 \leq r < i < s < j \leq n$

obtained from [**11**]. The family (D) reflects the interactions between generators of F and pure braids. In [**2**], Brin constructs these relators using the structure of Zappa-Szép product of the monoid associated to $\widehat{BV}$. This construction is not possible here because the presentations for the groups P_∞ are not monoid presentations. In fact, the monoid of pure positive braids is not finitely generated as shown by Burillo, Gutierrez, Krstić and Nitecki [**6**].

The family (C) of relators is given by the special way that the pure braid groups are embedded into each other inside BF. To embed P_n into P_{n+1} we split the last strand in two. If the last strand is not braided, this does not affect the element, but if the last strand takes part in the actual braiding, then these elements in P_n change when embedded in P_{n+1}. When a generator β_{ij} has its last strand split, now the ith strand wraps around two strands (the jth and the $(j+1)$th), and the element is now a product of two generators.

Theorem 5.2. *The group BF admits a presentation with generators:*

- x_i, *for* $i \geq 0$,
- α_{ij}, *for* $1 \leq i < j$,
- β_{ij}, *for* $1 \leq i < j$,

and relators:

(A) $x_j x_i = x_i x_{j+1}$, *if* $i < j$

(B1) $\alpha_{rs}^{-1}\alpha_{ij}\alpha_{rs} = \alpha_{ij}$, *if* $1 \le r < s < i < j$ *or* $1 \le i < r < s < j$

(B2) $\alpha_{rs}^{-1}\alpha_{ij}\alpha_{rs} = \alpha_{rj}\alpha_{ij}\alpha_{rj}^{-1}$, *if* $1 \le r < s = i < j$

(B3) $\alpha_{rs}^{-1}\alpha_{ij}\alpha_{rs} = (\alpha_{ij}\alpha_{sj})\alpha_{ij}(\alpha_{ij}\alpha_{sj})^{-1}$, *if* $1 \le r = i < s < j$

(B4) $\alpha_{rs}^{-1}\alpha_{ij}\alpha_{rs} = (\alpha_{rj}\alpha_{sj}\alpha_{rj}^{-1}\alpha_{sj}^{-1})\alpha_{ij}(\alpha_{rj}\alpha_{sj}\alpha_{rj}^{-1}\alpha_{sj}^{-1})^{-1}$, *if* $1 \le r < i < s < j$

(B5) $\alpha_{rs}^{-1}\beta_{ij}\alpha_{rs} = \beta_{ij}$, *if* $1 \le r < s < i < j$ *or* $1 \le i < r < s < j$

(B6) $\alpha_{rs}^{-1}\beta_{ij}\alpha_{rs} = \beta_{rj}\beta_{ij}\beta_{rj}^{-1}$, *if* $1 \le r < s = i < j$

(B7) $\alpha_{rs}^{-1}\beta_{ij}\alpha_{rs} = (\beta_{ij}\beta_{sj})\beta_{ij}(\beta_{ij}\beta_{sj})^{-1}$, *if* $1 \le r = i < s < j$

(B8) $\alpha_{rs}^{-1}\beta_{ij}\alpha_{rs} = (\beta_{rj}\beta_{sj}\beta_{rj}^{-1}\beta_{sj}^{-1})\beta_{ij}(\beta_{rn}\beta_{sj}\beta_{rj}^{-1}\beta_{sj}^{-1})^{-1}$, *if* $1 \le r < i < s < j$

(C) $\beta_{ij} = \beta_{i,j+1}\alpha_{ij}$, *if* $i < j$

(D1) $\alpha_{ij}x_k = x_k\alpha_{i+1,j+1}$, *if* $k < i-1$

(D2) $\alpha_{ij}x_k = x_k\alpha_{i+1,j+1}\alpha_{i,j+1}$, *if* $k = i-1$

(D3) $\alpha_{ij}x_k = x_k\alpha_{i,j+1}$, *if* $i-1 < k < j-1$

(D4) $\alpha_{ij}x_k = x_k\alpha_{i,j+1}\alpha_{ij}$, *if* $k = j-1$

(D5) $\alpha_{ij}x_k = x_k\alpha_{ij}$, *if* $k > j-1$

(D6) $\beta_{ij}x_k = x_k\beta_{i+1,j+1}$, *if* $k < i-1$

(D7) $\beta_{ij}x_k = x_k\beta_{i+1,j+1}\beta_{i,j+1}$, *if* $k = i-1$

(D8) $\beta_{ij}x_k = x_k\beta_{i,j+1}$, *if* $i-1 < k < j-1$

(D9) $\beta_{ij}x_k = x_k\beta_{ij}$, *if* $k \ge j-1$.

Proof: As in the proof of Theorem 2.4, we consider the algebraic and the geometric group, establish a homomorphism between them, which is well-defined and surjective. For instance, one needs to check geometrically the relators to see it is well defined. See Figure 10 for an example. After this, it only remains to check the injectivity of the map.

To prove that the homomorphism is injective, we consider an element of G, given as a word $w(x_i, \alpha_{ij}, \beta_{ij})$, and imagine that it is mapped to the identity in G. We need to prove that it is consequence of the relators listed above.

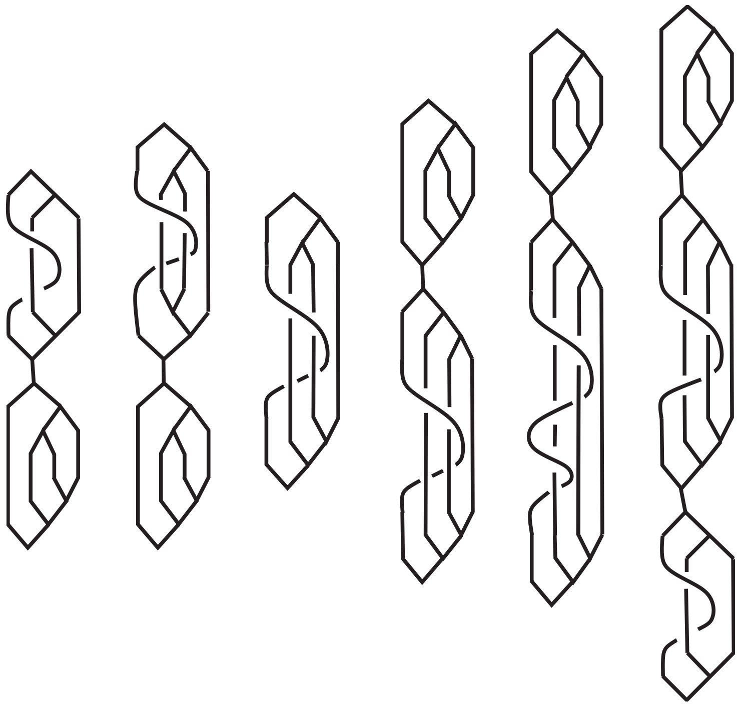


Figure 10. The process of checking the relator (D4). It starts on the left with $\alpha_{12}x_1$ and the following steps are the multiplication processes which transform it into $x_1\alpha_{13}\alpha_{12}$.

As a first step, we can see that the relators (D1) to (D9) and (A) can be used to transform any such word into the product of three words,

$$w_1(x_i)\, w_2(\alpha_{ij}, \beta_{ij})\, w_3(x_i)^{-1}.$$

We can arrange this in such a way that the words w_1 and w_3 contain only generators x_i and not their inverses.

This special expression is then particularly useful for studying the element, because it corresponds easily to the representative (T_-, p, T_+). We will use now the following lemma whose proof is analogous to the proof of Lemma 2.3:

Lemma 5.3. *A triple (T_-, p, T_+) in BF represents the identity element if and only if the braid p is the trivial braid and the element (T_-, T_+) represents the identity in F.*

With this lemma, we can assume now that the two words $w_1w_3^{-1}$ and w_2 map to the identity, and we must prove that they are consequence of the relators. The word $w_1w_3^{-1}$ lies in the subgroup isomorphic to F. So if it is the identity, it is consequence of the relators (A).

The word w_2 is a product of some α and β generators. We would like to consider this word inside some P_n, for a fixed n. The image in BF of β_{ij} has j strands, and the image of α_{ij} has $j+1$ strands. So the appropriate n to use is the maximum of the following set:

$$\{j \mid \beta_{ij} \text{ appears in } w_2\} \cup \{j+1 \mid \alpha_{ij} \text{ appears in } w_2\}.$$

If we have a β_{ij} with $j < n$, we use the relators (C) to increase the second index of that β_{ij} to n. This way, the only generators involved are α_{ij}, for $1 < i < j < n$, and β_{in}, for $1 < i < n$, which generate a copy of P_n inside BF. So, we can use the relators (C) to have our word expressed in this small set of generators and assume that it is a word in P_n. So if the word is the identity, it is consequence of the relators of P_n. But these relators correspond to the relators (B1) to (B8). □

6. A finite presentation for *BF*

As is common in the groups of the Thompson family, the infinite presentations are interesting and useful because of their symmetry and associated normal forms, but often it turns out that there are finite presentations from which the infinitely many generators and relations can be constructed and deduced. In this section, we construct a finite presentation for BF.

Thompson's group F admits a finite presentation which is merely the first two generators x_0 and x_1 and the first two non-trivial relations. We construct x_n from x_0 and x_1 as $x_n = x_0^{-n+1}x_1x_0^{n-1}$ and from the two first two non-trivial relations we can deduce all of the relations in (A) above. This is the first building block for our finite presentation.

In a similar way we can construct all generators α_{ij} from a few ones. The generators needed are α_{12}, α_{13}, α_{23}, α_{24}. The idea is that conjugating a braid with x_k has the effect of splitting the $(k-1)$th strand, so from these four generators and the generators for F, we can construct any generator α_{ij} by the process of splitting as many strands as necessary to produce the strands before the ith and between the ith and jth. This process is as follows:

- Given $\alpha_{i+1,j+1}$, with $j \geq i+3$, we use the relators (D3) to decrease the distance between i and j until 2:

$$\alpha_{i,j+1} = x_{j-2}^{-1}\alpha_{ij}x_{j-2}.$$

If $i = 1$ then this process brings any generator α_{1j} down to α_{13}. For any other value of i it reduces to $\alpha_{i,i+2}$.

- We reduce the generators $\alpha_{i+1,i+3}$ with $i \geq 2$ to α_{24} with relators of type (D1):
$$\alpha_{i+1,i+3} = x_{i-2}^{-1}\alpha_{i,i+2}x_{i-2}.$$
- And finally, we reduce generators of type $\alpha_{i+1,i+2}$ to α_{23} by again using (D1):
$$\alpha_{i+1,i+2} = x_{i-2}^{-1}\alpha_{i,i+1}x_{i-2}.$$

The generators β are constructed in exactly the same way, where we replace α by β, and with the same constraints on indices.

To see which relators to include in the finite presentation, we see which can be used to get the full families. We consider the relators (D) first and we will use those to help with the other families.

In each of the relators of (D1), there are three strands which are important: the strands labelled i, j and k. The idea is that between those, we only need to have one strand, because by splitting it, we can get to any number of strands in that position. Thus, we can get all of the relators (D1) from the following

(d1.1) $\alpha_{34} = x_0^{-1}\alpha_{23}x_0$

(d1.2) $\alpha_{35} = x_0^{-1}\alpha_{24}x_0$

(d1.3) $\alpha_{45} = x_0^{-1}\alpha_{34}x_0$

(d1.4) $\alpha_{46} = x_0^{-1}\alpha_{35}x_0$

(d1.5) $\alpha_{45} = x_1^{-1}\alpha_{34}x_1$

(d1.6) $\alpha_{46} = x_1^{-1}\alpha_{35}x_1$

(d1.7) $\alpha_{56} = x_1^{-1}\alpha_{45}x_1$

(d1.8) $\alpha_{57} = x_1^{-1}\alpha_{46}x_1$.

Every relator of the type (D1) is a consequence of the definitions above and of these eight relators. As an example, we will show the relator $\alpha_{i,i+1}x_0 = x_0\alpha_{i+1,i+2}$. If $i = 2$, the relator is (d1.1) above. If $i > 2$, then we use the definitions:
$$\alpha_{i+1,i+2} = x_{i-2}^{-1}x_{i-3}^{-1}\dots x_2^{-1}\alpha_{45}x_2\dots x_{i-3}x_{i-2}.$$
And using (d1.3) we get to
$$x_{i-2}^{-1}x_{i-3}^{-1}\dots x_2^{-1}\left(x_0^{-1}\alpha_{34}x_0\right)x_2\dots x_{i-3}x_{i-2}$$
which is
$$x_0^{-1}x_{i-3}^{-1}x_{i-4}^{-1}\dots x_1^{-1}\alpha_{34}x_1\dots x_{i-4}x_{i-3}x_0$$

finally equal to

$$x_0^{-1}\alpha_{i,i+1}x_0.$$

All the other relators of type (D) are very similar to this case, and we leave the details to the reader as they are straightforward but tedious.

The families of relators (D1) and (D3) are especially important, because they are used to split or combine adjacent strands which are not involved in the braiding. Any two adjacent strands which are not braided can be joined using a relator from one of these families. This is useful for the families (B1) to (B9).

For instance, the relators (B1) show that two braids α_{ij} and α_{rs} commute if $1 \leq r < s < i < j$. This relation only involves the strands r, s, i and j. If there are strands in between, they are uninvolved in the braiding. If there is more than one strand, we can apply the conjugating relations (D1) or (D3) to bring the relators down to a simple one where there is just one strand between. For instance, we show the relator

$$\alpha_{14}\alpha_{56} = \alpha_{56}\alpha_{14}$$

in Figure 11.

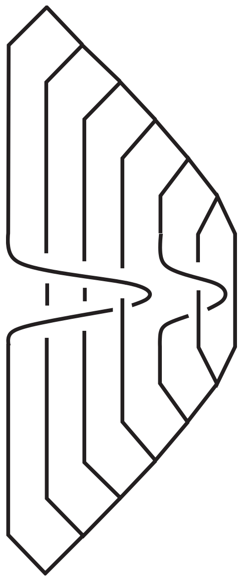


Figure 11. The element involved in the relator $\alpha_{14}\alpha_{56} = \alpha_{56}\alpha_{14}$. The commutativity is apparent.

We see that the two strands between the first and the fourth are straight. It is clear they can be obtained by splitting a single strand from an analogous relator whose braided strands are 1, 3, 4 and 5. A conjugation then by x_1 brings it down, according to the definitions above. Now we have that

$$\alpha_{14} = x_1^{-1} \alpha_{13} x_1,$$

which is of the type (D3), and

$$\alpha_{56} = x_1^{-1} \alpha_{45} x_1,$$

which is of the type (D1). So the relator is a consequence of the relator

$$\alpha_{13}\alpha_{45} = \alpha_{45}\alpha_{13},$$

using only (D1) and (D3) relators. In this way we see that the only relators that we need to construct all the relators in (B1) are those that have either zero or one strand at the beginning, or between the i, j, r, and sth strands. These are:

- $\alpha_{12}\alpha_{34} = \alpha_{34}\alpha_{12}$
- $\alpha_{12}\alpha_{35} = \alpha_{35}\alpha_{12}$
- $\alpha_{12}\alpha_{45} = \alpha_{45}\alpha_{12}$
- $\alpha_{12}\alpha_{46} = \alpha_{46}\alpha_{12}$
- $\alpha_{13}\alpha_{45} = \alpha_{45}\alpha_{13}$
- $\alpha_{13}\alpha_{46} = \alpha_{46}\alpha_{13}$
- $\alpha_{13}\alpha_{56} = \alpha_{56}\alpha_{13}$
- $\alpha_{13}\alpha_{57} = \alpha_{57}\alpha_{13}$
- $\alpha_{23}\alpha_{45} = \alpha_{45}\alpha_{23}$
- $\alpha_{23}\alpha_{46} = \alpha_{46}\alpha_{23}$
- $\alpha_{23}\alpha_{56} = \alpha_{56}\alpha_{23}$
- $\alpha_{23}\alpha_{57} = \alpha_{57}\alpha_{23}$
- $\alpha_{24}\alpha_{56} = \alpha_{56}\alpha_{24}$
- $\alpha_{24}\alpha_{57} = \alpha_{57}\alpha_{24}$
- $\alpha_{24}\alpha_{67} = \alpha_{67}\alpha_{24}$
- $\alpha_{24}\alpha_{68} = \alpha_{68}\alpha_{24}$.

The other families, including the ones which involve β generators, are completely analogous.

We are left only with the family (C). The relators are all of the type $\beta_{ij} = \beta_{i,j+1}\alpha_{ij}$, with $i < j$. As before, it is clear that if $i \geq 3$, we can conjugate by x_0 to get a relator with lower indices. So only ones with $i = 1$ or $i = 2$ are needed. Again we see that all are consequences of a fundamental finite set of relations:

- $\beta_{12} = \beta_{13}\alpha_{12}$
- $\beta_{13} = \beta_{14}\alpha_{13}$
- $\beta_{14} = \beta_{15}\alpha_{14}$
- $\beta_{23} = \beta_{24}\alpha_{23}$
- $\beta_{24} = \beta_{25}\alpha_{24}$
- $\beta_{25} = \beta_{26}\alpha_{25}$.

For example, we see that

$$\beta_{1j} = x_{j-3}^{-1}\beta_{1,j-1}x_{j-3} = x_{j-3}^{-1}\beta_{1j}\alpha_{1,j-1}x_{j-3} = \beta_{1,j+1}\alpha_{1j}$$

if $j \geq 5$. And the cases for $i = 2$ are exactly similar.

Hence, we have proved the following theorem:

Theorem 6.1. *The group BF is finitely presented.*

The finite presentation for BF is the following:

Generators: x_0, x_1, α_{12}, α_{13}, α_{23}, α_{24}, β_{12}, β_{13}, β_{23}, β_{24}.

Relators:

(A) $x_j x_i = x_i x_{j+1}$, for $(i,j) = (1,2), (1,3)$

(B1) $\alpha_{rs}^{-1}\alpha_{ij}\alpha_{rs} = \alpha_{ij}$, for

$$(r,s,i,j) = \begin{cases} (1,2,3,4) & (1,2,3,5) & (1,2,4,5) & (1,2,4,6) \\ (1,3,4,5) & (1,3,4,6) & (1,3,5,6) & (1,3,5,7) \\ (2,3,4,5) & (2,3,4,6) & (2,3,5,6) & (2,3,5,7) \\ (2,4,5,6) & (2,4,5,7) & (2,4,6,7) & (2,4,6,8) \\ (2,3,1,4) & (2,3,1,5) & (2,4,1,5) & (2,4,1,6) \\ (3,4,1,5) & (3,4,1,6) & (3,5,1,6) & (3,5,1,7) \\ (3,4,2,5) & (3,4,2,6) & (3,5,2,6) & (3,5,2,7) \\ (4,5,2,6) & (4,5,2,7) & (4,6,2,7) & (4,6,2,8) \end{cases}$$

(B2) $\alpha_{rs}^{-1}\alpha_{ij}\alpha_{rs} = \alpha_{rj}\alpha_{ij}\alpha_{rj}^{-1}$, for

$$(r,s,i,j) = \begin{cases} (1,2,2,3) & (1,2,2,4) & (1,3,3,4) & (1,3,3,5) \\ (2,3,3,4) & (2,3,3,5) & (2,4,4,5) & (2,4,4,6) \end{cases}$$

(B3) $\alpha_{rs}^{-1}\alpha_{ij}\alpha_{rs} = (\alpha_{ij}\alpha_{sj})\alpha_{ij}(\alpha_{ij}\alpha_{sj})^{-1}$, for

$$(r,s,i,j) = \begin{cases} (1,2,1,3) & (1,2,1,4) & (1,3,1,4) & (1,3,1,5) \\ (2,3,2,4) & (2,3,2,5) & (2,4,2,5) & (2,4,2,6) \end{cases}$$

(B4) $\alpha_{rs}^{-1}\alpha_{ij}\alpha_{rs} = (\alpha_{rj}\alpha_{sj}\alpha_{rj}^{-1}\alpha_{sj}^{-1})\alpha_{ij}(\alpha_{rj}\alpha_{sj}\alpha_{rj}^{-1}\alpha_{sj}^{-1})^{-1}$, for

$$(r,s,i,j) = \begin{cases} (1,3,2,4) & (1,3,2,5) & (1,4,2,5) & (1,4,2,6) \\ (1,4,3,5) & (1,4,3,6) & (1,5,3,6) & (1,5,3,7) \\ (2,4,3,5) & (2,4,3,6) & (2,5,3,6) & (2,5,3,7) \\ (2,5,4,6) & (2,5,4,7) & (2,6,4,7) & (2,6,4,8) \end{cases}$$

(B5) $\alpha_{rs}^{-1}\beta_{ij}\alpha_{rs} = \beta_{ij}$, for

$$(r,s,i,j) = \begin{cases} (1,2,3,4) & (1,2,3,5) & (1,2,4,5) & (1,2,4,6) \\ (1,3,4,5) & (1,3,4,6) & (1,3,5,6) & (1,3,5,7) \\ (2,3,4,5) & (2,3,4,6) & (2,3,5,6) & (2,3,5,7) \\ (2,4,5,6) & (2,4,5,7) & (2,4,6,7) & (2,4,6,8) \\ (2,3,1,4) & (2,3,1,5) & (2,4,1,5) & (2,4,1,6) \\ (3,4,1,5) & (3,4,1,6) & (3,5,1,6) & (3,5,1,7) \\ (3,4,2,5) & (3,4,2,6) & (3,5,2,6) & (3,5,2,7) \\ (4,5,2,6) & (4,5,2,7) & (4,6,2,7) & (4,6,2,8) \end{cases}$$

(B6) $\alpha_{rs}^{-1}\beta_{ij}\alpha_{rs} = \beta_{rj}\beta_{ij}\beta_{rj}^{-1}$, for

$$(r,s,i,j) = \begin{cases} (1,2,2,3) & (1,2,2,4) & (1,3,3,4) & (1,3,3,5) \\ (2,3,3,4) & (2,3,3,5) & (2,4,4,5) & (2,4,4,6) \end{cases}$$

(B7) $\alpha_{rs}^{-1}\beta_{ij}\alpha_{rs} = (\beta_{ij}\beta_{sj})\beta_{ij}(\beta_{ij}\beta_{sj})^{-1}$, for

$$(r,s,i,j) = \begin{cases} (1,2,1,3) & (1,2,1,4) & (1,3,1,4) & (1,3,1,5) \\ (2,3,2,4) & (2,3,2,5) & (2,4,2,5) & (2,4,2,6) \end{cases}$$

(B8) $\alpha_{rs}^{-1}\beta_{ij}\alpha_{rs} = (\beta_{rj}\beta_{sj}\beta_{rj}^{-1}\beta_{sj}^{-1})\beta_{ij}(\beta_{rn}\beta_{sj}\beta_{rj}^{-1}\beta_{sj}^{-1})^{-1}$, for

$$(r,s,i,j) = \begin{cases} (1,3,2,4) & (1,3,2,5) & (1,4,2,5) & (1,4,2,6) \\ (1,4,3,5) & (1,4,3,6) & (1,5,3,6) & (1,5,3,7) \\ (2,4,3,5) & (2,4,3,6) & (2,5,3,6) & (2,5,3,7) \\ (2,5,4,6) & (2,5,4,7) & (2,6,4,7) & (2,6,4,8) \end{cases}$$

(C) $\beta_{ij} = \beta_{i,j+1}\alpha_{ij}$, for

$$(i,j) = (1,2),(1,3),(1,4),(2,3),(2,4),(2,5)$$

(D1) $\alpha_{ij}x_k = x_k\alpha_{i+1,j+1}$, for

$$(i,j,k) = \begin{cases} (2,3,0) & (2,4,0) & (3,4,0) & (3,5,0) \\ (3,4,1) & (3,5,1) & (4,5,1) & (4,6,1) \end{cases}$$

(D2) $\alpha_{ij}x_k = x_k\alpha_{i+1,j+1}\alpha_{i,j+1}$, for

$$(i,j,k) = (1,2,0),(1,3,0),(2,3,1),(2,4,1)$$

(D3) $\alpha_{ij}x_k = x_k\alpha_{i,j+1}$, for

$$(i,j,k) = \begin{cases} (1,3,1) & (1,4,1) & (1,4,2) & (1,5,2) \\ (2,4,2) & (2,5,2) & (2,5,3) & (2,6,3) \end{cases}$$

(D4) $\alpha_{ij}x_k = x_k\alpha_{i,j+1}\alpha_{ij}$, for

$$(i,j,k) = (1,2,1),(1,3,2),(2,3,2),(2,4,3)$$

(D5) $\alpha_{ij}x_k = x_k\alpha_{ij}$, for

$$(i,j,k) = \begin{cases} (1,2,2) & (1,2,3) & (1,3,3) & (1,3,4) \\ (2,3,3) & (2,3,4) & (2,4,4) & (2,4,5) \end{cases}$$

(D6) $\beta_{ij}x_k = x_k\beta_{i+1,j+1}$, for

$$(i,j,k) = \begin{cases} (2,3,0) & (2,4,0) & (3,4,0) & (3,5,0) \\ (3,4,1) & (3,5,1) & (4,5,1) & (4,6,1) \end{cases}$$

(D7) $\beta_{ij}x_k = x_k\beta_{i+1,j+1}\beta_{i,j+1}$, for

$$(i,j,k) = (1,2,0),(1,3,0),(2,3,1),(2,4,1)$$

(D8) $\beta_{ij}x_k = x_k\beta_{i,j+1}$, for

$$(i,j,k) = \begin{cases} (1,3,1) & (1,4,1) & (1,4,2) & (1,5,2) \\ (2,4,2) & (2,5,2) & (2,5,3) & (2,6,3) \end{cases}$$

(D9) $\beta_{ij}x_k = x_k\beta_{ij}$, for

$$(i,j,k) = (1,2,1),(1,3,2),(2,3,2),(2,4,3).$$

This gives a total of 10 generators and 192 relators.

7. The braided Thompson group $\widehat{BF}$

In the previous section we constructed a presentation of BF. Brin [**2**] described both BV and $\widehat{BV}$. Brin describes the group $\widehat{BV}$ via a Zappa-Szép product of F and B_∞, and describes BV as a subgroup of $\widehat{BV}$. The group $\widehat{BV}$ is the group of braided forest diagrams, where all but finitely many of the forests are trivial, and the group BV is the subgroup of $\widehat{BV}$ where all of the trees in the forest pairs are trivial except the first pair, which have the same number of leaves. Not only is BV a subgroup of $\widehat{BV}$, but also $\widehat{BV}$ is a subgroup of BV, as described by Brin [**2**]. We take the standard identification of the real line with the unit interval which is compatible with the relevant dyadic subdivisions which sends the interval $[i, i+1]$ of $\mathbf{R}$ with the interval $[1-2^{-i}, 1-2^{-i-1}]$ and then we see that $\widehat{BV}$ is the subgroup of BV in which the last strand is not braided with any other strands. Similarly, we have the group $\widehat{BF}$ which can either be regarded as the supergroup of BF of pure braided forest diagrams, or as a subgroup of BF where the braiding does not involve the last strand.

Here, we easily describe the subgroup $\widehat{BF}$ of BF by omitting the generators and relations from BF which involve braiding the last strand. So we obtain presentations for $\widehat{BF}$ which are sub-presentations of the infinite and finite presentations for BF given in the earlier sections above.

Proposition 7.1. *The elements x_i, for $i \geq 0$, α_{ij}, for $1 \leq i < j$ form a set of generators of $\widehat{BF}$.*

Proof: The proof is similar to the case for BF. Here we note that $\widehat{BF}$ is exactly the subgroup where the last strand is not braided with any previous strands, and by omitting the β generators we guarantee that the last strand is not braided. An argument similar to the earlier one for BF shows that these generate $\widehat{BF}$. □

To find relators for a presentation of $\widehat{BF}$ we use the same sets of generators and relators as for BF, deleting the generators in the β family and deleting all relations which include any of the β_{ij}.

We thus obtain the following:

Theorem 7.2. *The group $\widehat{BF}$ admits a presentation with generators:*

- x_i, *for* $i \geq 0$,
- α_{ij}, *for* $1 \leq i < j$,

and relators:

(A) $x_j x_i = x_i x_{j+1}$, *if* $i < j$

(B1) $\alpha_{rs}^{-1}\alpha_{ij}\alpha_{rs} = \alpha_{ij}$, *if* $1 \leq r < s < i < j$ *or* $1 \leq i < r < s < j$

(B2) $\alpha_{rs}^{-1}\alpha_{ij}\alpha_{rs} = \alpha_{rj}\alpha_{ij}\alpha_{rj}^{-1}$, *if* $1 \leq r < s = i < j$

(B3) $\alpha_{rs}^{-1}\alpha_{ij}\alpha_{rs} = (\alpha_{ij}\alpha_{sj})\alpha_{ij}(\alpha_{ij}\alpha_{sj})^{-1}$, *if* $1 \leq r = i < s < j$

(B4) $\alpha_{rs}^{-1}\alpha_{ij}\alpha_{rs} = (\alpha_{rj}\alpha_{sj}\alpha_{rj}^{-1}\alpha_{sj}^{-1})\alpha_{ij}(\alpha_{rj}\alpha_{sj}\alpha_{rj}^{-1}\alpha_{sj}^{-1})^{-1},$ *if* $1 \leq r < i < s < j$

(D1) $\alpha_{ij}x_k = x_k\alpha_{i+1,j+1},$ *if* $k < i-1$

(D2) $\alpha_{ij}x_k = x_k\alpha_{i+1,j+1}\alpha_{i,j+1},$ *if* $k = i-1$

(D3) $\alpha_{ij}x_k = x_k\alpha_{i,j+1},$ *if* $i-1 < k < j-1$

(D4) $\alpha_{ij}x_k = x_k\alpha_{i,j+1}\alpha_{ij},$ *if* $k = j-1$

(D5) $\alpha_{ij}x_k = x_k\alpha_{ij},$ *if* $k > j-1$.

Proof: Using the interpretation of the geometric group $\widehat{BF}$ as the subgroup of BF where braiding never involves the last strand, the same proof used in the section 5 to establish the presentation for BF goes through in this situation. The only difference is that since we have no β generators, once we rearrange the word to have all α generators in the middle, they are already generators for one copy of the pure braids on n strands with the rightmost strand unbraided, so there is no need for the step using relators of type (C). □

Note that the group $\widehat{BF}$ will also be finitely presented. The arguments needed to see this are similar to those for BF. The finite presentation for $\widehat{BF}$ can be easily obtained from the finite presentation for BF by deleting all generators β_{ij} and the relations where they appear.

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