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INVARIANT SUBSPACES ON MULTIPLY CONNECTED DOMAINS

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Abstract

The lattice of invariant subspaces of several Banach spaces of analytic functions on the unit disk, for example the Bergman spaces and the Dirichlet spaces, have been studied recently. A natural question is to what extent these investigations carry over to analogously defined spaces on an annulus. We consider this question in the context of general Banach spaces of analytic functions on finitely connected domains Ω. The main result reads as follows: Assume that B is a Banach space of analytic functions satisfying some conditions on the domain Ω. Assume further that $M(B)$ is the set of all multipliers of B. Let Ω_1 be a domain obtained from Ω by adding some of the bounded connectivity components of $\mathbb{C}\setminus\Omega$. Also, let B_1 be the closed subspace of B of all functions that extend analytically to Ω_1. Then the mapping $I \mapsto \mathrm{clos}(I \cdot M(B))$ gives a one-to-one correspondence between a class of multiplier invariant subspaces I of B_1, and a class of multiplier invariant subspaces J of B. The inverse mapping is given by $J \mapsto J \cap B_1$.

0. Introduction

The study of the lattice of invariant subspaces of a given Banach space B consisting of analytic functions in a domain Ω of the complex plane $\mathbb{C}$ has attracted much attentions in recent years. This is mainly due to the applications to spectral theory as well as to polynomial approximation theory. Let Ω be a domain, an open and connected subset of the complex plane, and let $B = B(\Omega)$ be a Banach space of analytic functions defined on Ω. We shall need three different notions of invariant subspaces. A closed subspace M of the Banach space B is said to be *invariant* provided that $zM \subset M$, where z stands for the

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coordinate function $z(\lambda) = \lambda$ for every λ in Ω (rather than the usual complex variable). Invariant subspaces are sometimes called *simply invariant*, or *polynomially invariant* subspaces. Similarly, we call a closed subspace M of B *rationally invariant* provided that $rM \subset M$ for every rational function r with poles outside $\overline{\Omega}$ (the closure will always be taken in the extended complex plane $\mathbb{C}\cup\{\infty\}$, whenever we have an unbounded set). Some authors use the phrase *fully invariant* for rationally invariant subspaces. An analytic function φ defined on Ω is said to be a *multiplier* for the Banach space B, if $\varphi B \subset B$. The set of all multipliers of B is denoted by $M(B)$. A closed subspace M of B is said to be *multiplier invariant* or *hyperinvariant* provided that $\varphi B \subset B$, for every $\varphi \in M(B)$. We shall write $M(B)$-invariant for a multiplier invariant subspace when we need to emphasize that M is a multiplier invariant subspace in B (and not in some other space). In this article, we shall mainly deal with multiplier invariant subspaces. It may happen that some of these concepts of invariance coincide for a given space B.

Throughout this paper, for Ω an open and connected subset of $\mathbb{C}\cup\{\infty\}$ with $\overline{\Omega} \neq \mathbb{C} \cup \{\infty\}$, we shall assume that all Banach spaces $B = B(\Omega)$ under consideration satisfy the following axioms:

(0-1) B is a linear subspace of the space of all analytic functions on Ω.

(0-2) The functional of evaluation at λ is continuous for all $\lambda \in \Omega$.

(0-3) If $f \in B$, then so is rf, for every rational function r with poles in $(\mathbb{C} \cup \{\infty\}) \setminus \overline{\Omega}$.

(0-4) If $f \in B$, $\lambda \in \Omega$, and $f(\lambda) = 0$, then $\exists g \in B$ such that $(z-\lambda)g = f$.

Such conditions have appeared in the literature before. For example, one can find similar set-ups in L. Brown and A. L. Shields [**9**], and in S. Richter [**27**].

Axiom (0-3) implies that $M(B)$ contains rational functions with poles outside $\overline{\Omega}$. In case Ω is a bounded domain, z is regarded as a rational function with pole at infinity so that the polynomials are included in $M(B)$. From the first three axioms together with the closed graph theorem one can obtain the continuity of the operator of multiplication by z, that is, $M_z = M_z[B] : B \to B$ defined by $f \mapsto zf$. We denote by $\sigma(M_z)$ the spectrum of the operator M_z, that is, the set of all complex numbers λ for which the operator $\lambda - M_z$ is not invertible. Assuming that the Banach space B is nontrivial, we conclude by exploiting the last axiom that $\Omega \subset \sigma(M_z)$. Note that $\sigma(M_z)$ consists of all $\lambda \in \mathbb{C}$ for which $(\lambda - z)^{-1} \notin M(B)$. If $\overline{\Omega}$ contains the point at infinity, then M_z is not a bounded operator, but we may still talk of $\sigma(M_z)$ as the set of all complex numbers λ for which $(\lambda-z)^{-1} \notin M(B)$. It can be shown to be closed in $\mathbb{C}$.

More generally, we say that a closed subspace M of B is invariant for an operator T, defined on B, if $TM \subset M$. It is obvious that polynomially invariant subspaces of B are those which are invariant for the operator M_z. In contrast, multiplier invariant subspaces of B are those which are invariant under all the bounded linear operator $M_\varphi : f \mapsto \varphi f$ from B into B, where $\varphi \in M(B)$. An invariant subspace of B is called *hyperinvariant* for the operator M_z, if it is invariant for all operators which commute with M_z. For Banach spaces of analytic functions satisfying the axioms (0-1)-(0-4), it is known that the operator T on B commutes with M_z if and only if $T = M_\varphi$ for some $\varphi \in M(B)$ (see [**27**, Proposition 2.4]). It follows that the multiplier invariant subspaces of B are precisely the hyperinvariant subspaces for M_z.

Examples of spaces. Before advancing any further, let us take a look at some familiar Banach spaces of analytic functions which satisfy the above mentioned axioms.

As a first example we mention the classical Hardy spaces of analytic functions on the unit disk $H^p = H^p(\mathbb{D})$, $1 \leq p \leq +\infty$ (for definition see [**12**]). Note that for $p = +\infty$, we get H^∞, the Banach algebra of bounded analytic functions in the unit disk equipped with the supremum norm. It is easy to see that the Hardy spaces satisfy our axioms. The invariant subspaces of the Hilbert space H^2 were classified by A. Beurling [**8**]. Later K. deLeeuw and W. Rudin showed that Beurling's result is true for $p = 1$, and finally, T. P. Srinivasan extended Beurling's theorem to H^p, $1 \leq p \leq +\infty$ [**22**, p. 28]. Beurling's theorem is even valid in the context of the spaces H^p, for $0 < p < 1$, which are not Banach spaces (however, they are quasi-Banach spaces [**25**]; we use this term to denote a complete p-normed space, for some p, $0 < p \leq 1$). Beurling's theorem asserts that for $0 < p < +\infty$, every invariant subspace M of H^p, other than $\{0\}$, is of the type $M = uH^p$ where u is some inner function. Furthermore, if $p = +\infty$, the statement remains true if the word "closed" is replaced by the expression "weak-star closed" (see [**13**, p. 132]). It is not difficult to extend Beurling's description to general simply connected domains, but the problem of describing the invariant subspaces on mutiply connected domains is more complicated. However, there is a characterization, similar to Beurling's, of rationally invariant subspaces of the Hardy spaces on multiply connected domains. This was observed by D. Sarason [**32**] for the annulus, and by M. Hasumi [**15**] and M. Voichick [**34**], [**35**] for more general domains (for a brief proof, see the paper of H. L. Royden [**29**]). Some effort has been made to describe the simply invariant (but not rationally invariant) subspaces of the Hardy spaces on multiply connected domains. We refer the reader to the works of H. L. Royden [**29**], D. Hitt [**23**], V. Yakubovich [**36**], and A. Aleman and S. Richter [**3**].

Another example of such a Banach space of analytic functions is the Bergman space $L^2_a = L^2_a(\mathbb{D})$, consisting of those analytic functions in the unit disk which are square integrable with respect to Lebesgue area measure. More precisely, $f \in L^2_a$ means that f is holomorphic on $\mathbb{D}$, and has finite norm (we write, for $z = x + iy$, $dA(z) = \pi^{-1}\,dx\,dy$):

$$\|f\|_{L^2_a} = \left(\int_{\mathbb{D}} |f(z)|^2\,dA(z)\right)^{1/2} < +\infty.$$

The Bergman space L^2_a is a Hilbert space with the inner product

$$\langle f, g\rangle_{L^2_a} = \int_{\mathbb{D}} f(z)\overline{g(z)}\,dA(z), \quad f,\, g \in L^2_a(\mathbb{D}).$$

We define $L^p_a(\Omega)$, for $0 < p < +\infty$, the Bergman space with exponent p of a general domain Ω, to be the space of all holomorphic functions f on Ω such that

$$\|f\|_{L^p_a(\Omega)} = \left(\int_{\Omega} |f(z)|^p\,dA(z)\right)^{1/p} < +\infty,$$

where $dA(z)$ denotes the area measure on Ω. More generally, we define the weighted Bergman spaces $L^p_a(\Omega, w)$ where the weight function $w : \Omega \to \mathbb{C}$ is strictly positive and continuous on Ω: for $0 < p < +\infty$, the space $L^p_a(\Omega, w)$ consists of those analytic functions f on Ω for which

$$\|f\|_{p,w} = \left(\int_{\Omega} |f(z)|^p w(z)\,dA(z)\right)^{1/p} < +\infty.$$

For $1 \le p < +\infty$, this is a norm which makes $L^p_a(\Omega, w)$ a Banach space. For $w \equiv 1$, $\Omega = \mathbb{D}$, and $p = 2$, we get the previously encountered Bergman space $L^2_a(\mathbb{D})$. For $1 \le p < +\infty$, the weighted Bergman spaces satisfy our axioms (0-1)-(0-4). For instance, the continuity of point evaluation functionals follows from the sub-mean value property for subharmonic functions. As we mentioned earlier, Beurling's theorem describes the invariant subspace structure of the Hardy spaces H^p. In contrast, the invariant subspace structure of the Bergman spaces has never been completely understood and is known to be extremely complicated. However, Aleman, Richter, and Sundberg have recently obtained a genuine analog of Beurling's theorem for L^2_a [**5**].

As a final example of spaces that satisfy our axioms, we would like to mention the spaces D_α, for $-\infty < \alpha < +\infty$. Consider an analytic function f in the unit disk, with Taylor series $f(z) = \sum_{n=0}^{\infty} a_n z^n$. The

space D_α consists of those analytic functions in the unit disk for which the following sum is finite:

$$\sum_{n=0}^{\infty}(n+1)^\alpha |a_n|^2 < +\infty.$$

It is a Hilbert space with the inner product

$$\langle f, g\rangle = \sum_{n=0}^{\infty}(n+1)^\alpha a_n \overline{b}_n,$$

where a_n and b_n are the coefficients in the Taylor series expansions of f and g, respectively. The space D_0 is the familiar Hardy space H^2, D_1 is the Dirichlet space, and D_{-1} is the Bergman space L^2_a. The Dirichlet space consists of those analytic functions $\mathbb{D} \to \mathbb{C}$ that map the unit disk onto a Riemann surface of finite area $\int_{\mathbb{D}} |f'|^2\, dA$. Clearly, $f \in D_1$ if and only if $f' \in L^2_a(\mathbb{D})$. We refer the reader to [**9**], [**33**], and [**27**] for a review of known results on the spaces D_α. ■

In this paper, we study how the lattice of multiplier invariant subspaces changes with the connectivity of the underlying domain. It follows from our main result that to characterize all multiplier invariant subspaces (of index one) of a Bergman space $L^p_a(\Omega)$ on a finitely connected smoothly bordered domain Ω, one only needs to solve the problem for the Bergman spaces $L^p_a(\Omega')$, where Ω' is a simply connected smoothly bordered domain. We should point out that the study of $L^p_a(\Omega')$ is essentially the same as that of $L^p_a(\mathbb{D})$.

Although we shall mainly be interested in a general geometric situation, we start by describing the particular case of an annulus.

We consider a bounded planar domain Ω, and keep in mind the annulus $A_\rho = \{z \in \mathbb{C} : \rho < |z| < 1\}$ with $0 < \rho < 1$ as a typical example of Ω. Now assume that Ω_1 and Ω_2 are two other planar domains satisfying the following conditions:

(1) Ω_1 is bounded,

(2) Ω_2 is unbounded and non-dense in the complex plane,

(3) $\Omega_1 \cup \Omega_2 = \mathbb{C}$ and $\Omega = \Omega_1 \cap \Omega_2$.

For example, one can think of Ω_1 as $\{z \in \mathbb{C} : |z| < 1\}$, and of Ω_2 as $\{z \in \mathbb{C} : |z| > \rho\}$, where $0 < \rho < 1$. Suppose that B is a Banach space of analytic functions on Ω (satisfying the axioms). We assume that B can be written as the direct sum of $B = B_1 \oplus B_2^0$, where B_1 and B_2^0 are

Banach spaces of analytic functions on Ω_1 and Ω_2, respectively. More precisely, $B_1 = B \cap \mathcal{O}(\Omega_1)$ and $B_2^0 = B \cap \mathcal{O}_0(\Omega_2)$. Here, $\mathcal{O}_0(\Omega_2)$ denotes the subspace of $\mathcal{O}(\Omega_2)$ consisting of those functions f such that $f(z) \to 0$ as $|z| \to +\infty$. We shall also assume that the multiplier space $M(B)$ can be written as the direct sum of $M(B_1)$ and $M_0(B_2)$, where this latter consists of those elements of $M(B_2)$ having the value 0 at infinity. We wish to study the lattices of multiplier invariant subspaces of B and of B_1. Our main objective is to compare the lattice structure of multiplier invariant subspaces of B with that of B_1. Once this is done, we would simultaneously have a knowledge of the lattice structure of multiplier invariant subspaces of B_2 as well, because we can easily interchange B_1 and B_2 via a Möbius transformation. The advantage of this kind of comparison theorem is due to the fact that B_1 is defined on a domain with fewer holes, and so it is likely that its lattice structure is better known than that of B.

Suppose that J is an $M(B)$-invariant subspace; then the *contraction* $J \cap B_1$ is an $M(B_1)$-invariant subspace. Conversely, if I is an $M(B_1)$-invariant subspace, then the *extension* $\operatorname{clos}(I \cdot M(B))$ is an $M(B)$-invariant subspace. Here "clos" indicates closure, and $I \cdot M(B)$ stands for

$$I \cdot M(B) = \operatorname{span}\{f\varphi : f \in I,\, \varphi \in M(B)\},$$

that is, the set of finite linear combinations of products of the form $f\varphi$, $f \in I$, and $\varphi \in M(B)$. The extension $\operatorname{clos}(I \cdot M(B))$ is the smallest multiplier invariant subspace of B containing I. It turns out that for a large class of subspaces I and J, multiplier invariant in B_1 and B, respectively, the following equalities hold:

$$I = \operatorname{clos}\big(I \cdot M(B)\big) \cap B_1 \quad \text{and} \quad J = \operatorname{clos}\big((J \cap B_1) \cdot M(B)\big).$$

These equalities guarantee a one-to-one correspondence between parts of the two lattices of multiplier invariant subspaces of B and B_1. To make the correspondence work, we need to assume that $\sigma(M_z[B_1/I]) \subset \Omega_2$ and $\sigma(M_z[B/J]) \subset \Omega_2$. Here, the notation $\sigma(M_z[B/J])$ is used for the spectrum of the operator $M_z[B/J] : B/J \to B/J$ defined by

$$f + J \mapsto zf + J,$$

where J is an invariant subspace of B. Similarly, one defines the operator $M_z[B_1/I]$ and its spectrum $\sigma(M_z[B_1/I])$. It turns out that it is possible to describe the structure of all multiplier invariant subspaces of index one (see Definition 2.1) in B in terms of the corresponding lattices of multiplier invariant subspaces of B_1 and B_2.

This paper generalizes the methods introduced in [**16**] to establish a correspondence of this type between the closed ideals of B_1 and those of B in the Banach algebra setting. Related comparison theorems regarding the structure of closed ideals in some Banach algebras can be found in [**17**], [**18**]. The results of this paper can be generalized to quasi-Banach spaces (see [**1**]). In particular, the results apply to the Bergman spaces $L^p_a(\Omega)$, $0 < p < 1$, as well.

1. Preliminaries

For an unbounded domain Ω, we mean by $\mathcal{O}_0(\Omega)$ the subspace of $\mathcal{O}(\Omega)$ consisting of those functions $f(z)$ which tend to 0 as $|z| \to \infty$ within Ω. Note that here the notation $\mathcal{O}(\Omega)$ is used to indicate the vector space of all analytic functions on the domain Ω regardless of whether Ω is bounded or unbounded.

Given an invariant subspace J of the Banach space B of analytic functions defined on Ω, we consider the operator $M_z[B/J] : B/J \to B/J$ defined by the following relation:

$$M_z[B/J](f + J) = zf + J, \quad f \in B.$$

As we shall see later on, this operator and especially its spectrum plays a significant role in our future work. Recall that the notation $\sigma(M_z[B/J])$ stands for the spectrum of this operator, that is, the set of all $\lambda \in \mathbb{C}$ for which the operator $\lambda - M_z[B/J]$ acting on B/J, is not invertible. We note that the spectrum $\sigma(M_z[B/J])$ is a compact non-empty subset of $\overline{\Omega}$ (see [**19**]). Similarly, if J is a multiplier invariant subspace of B and if $\varphi \in M(B)$, then we define the operator $M_\varphi[B/J] : B/J \to B/J$ by

$$f + J \mapsto \varphi f + J, \quad f \in B.$$

Our first three axioms together with the closed graph theorem imply that M_φ is a bounded linear operator on B/J. We would like to mention that the multiplier space $M(B)$ can be made into a Banach space by putting $\|\varphi\|_{M(B)} = \|M_\varphi\|_{\mathcal{L}(B)}$. It follows immediately that

$$\|\varphi \cdot f\|_B \leq \|\varphi\|_{M(B)} \cdot \|f\|_B, \quad \text{for every} \quad \varphi \in M(B),\, f \in B.$$

Because of its very important role in our future disscusions, we would like to recall some holomorphic functional calculus. Let $\mathcal{O}\big(\sigma(M_z[B/J])\big)$ denote the algebra of all functions analytic in some neighbourhood of the spectrum of the operator $M_z[B/J]$. Note that every such function h is defined and analytic in a corresponding neighbourhood U_h of the compact

set $\sigma(M_z[B/J])$; strictly speaking, $\mathcal{O}\big(\sigma(M_z[B/J])\big)$ is thus an algebra of germs of analytic functions. Consider $h \in \mathcal{O}\big(\sigma(M_z[B/J])\big)$ and let D_h be a compact subset of U_h such that its interior contains the spectrum $\sigma(M_z[B/J])$. Now let γ_h be the positively oriented boundary of the compact set D_h consisting of a finite number of piecewise smooth curves, each homeomorphic to the unit circle $\mathbb{T}$. For $\lambda \in \mathbb{C} \setminus \sigma(M_z[B/J])$ define the operator $H_J : \mathcal{O}\big(\sigma(M_z[B/J])\big) \to \mathcal{L}(B/J)$ by

$$h \mapsto H_J(h) = \frac{1}{2\pi i} \int_{\gamma_h} h(\lambda)(\lambda - M_z[B/J])^{-1}\, d\lambda.$$

Here, we identify the complex number λ and the operator $\lambda \cdot id$, where id stands for the identity operator on B/J. We also note that in fact $H_J(h) = h(M_z[B/J])$, the image of the operator $M_z[B/J]$ under the holomorphic function h, under what is customarily called the holomorphic functional calculus. The mapping $h \mapsto H_J(h)$ from $\mathcal{O}\big(\sigma(M_z[B/J])\big)$ into $\mathcal{L}(B/J)$, the Banach algebra of all bounded linear operators on B/J, is linear and multiplicative (see [**26**, p. 84] or for an interesting discussion of general spectral theory [**11**]). That is, for functions h and g analytic in a neighbourhood of the spectrum of the operator $M_z[B/J]$, we have $H_J(hg) = H_J(h)H_J(g)$. Note that for $\lambda \notin \sigma(M_z[B/J])$, the inverse operator

$$(\lambda - M_z[B/J])^{-1} : B/J \longrightarrow B/J$$

exists, and that by the analyticity in the λ variable the integral defining $H_J(h)$ does not depend on the particular choice of γ_h, so that there is no ambiguity in the above integral formula. Assume that the constant function $1 \in B$, so that $1 + J \in B/J$. Associate to each $S \in \mathcal{L}(B/J)$, the element $S(1 + J)$ which is in B/J. This defines a mapping $P_J : \mathcal{L}(B/J) \to B/J$, which sends

$$S \mapsto P_J(S) = S(1 + J).$$

We then have a mapping $L_J = P_J \circ H_J$ from $\mathcal{O}\big(\sigma(M_z[B/J])\big)$ into B/J; in fact we have the following chain of mappings

$$\mathcal{O}\big(\sigma(M_z[B/J])\big) \to \mathcal{L}(B/J) \to B/J.$$

More precisely, we define the mapping $L_J : \mathcal{O}\big(\sigma(M_z[B/J])\big) \to B/J$ by

$$h \mapsto P_J(H_J(h)) = (H_J(h))(1 + J).$$

Let φ be a multiplier for the Banach space B of analytic functions, which contains the constant function 1 and satisfies our axioms. Since for $f \in B$, both f and φf are analytic functions on Ω, it follows that φ itself is an analytic function. We would like to mention that if we do not assume that $1 \in B$, then our axiom (0-4) implies that φ is an analytic function. It turns out that the mapping from $(M(B), B)$ into B, sending the pair (φ, f) to φf, induces a module structure on B. In other words, B is an $M(B)$-module. For this reason, we call a module-homomorphism any bounded linear mapping on B which preserves this module structure of B. More precisely, we make the following definition, in which we shall temporarily use the notation A for a Banach space of analytic functions on a domain Ω.

Definition 1.1. Let A be a Banach space of analytic functions on Ω, and let Y be a Banach space. A bounded linear mapping $L : A \to Y$ is said to be a module-homomorphism provided that there exists a continuous homomorphism $L_M : M(A) \to \mathcal{L}(Y)$ such that

$$L(\varphi f) = L_M(\varphi)L(f),$$

for every $\varphi \in M(A)$ and every $f \in A$.

Remark 1.2. If L is a module-homomorphism $A \to Y$, then the kernel $\ker L$ is an $M(A)$-invariant subspace of A. To see this, let $f \in \ker L$ and $\varphi \in M(A)$; the space of multipliers of A. We then have $L(\varphi f) = L_M(\varphi)L(f)$. But $L(f) = 0$, therefore $L(\varphi f) = L_M(\varphi)(0) = 0$, which shows that $\varphi f \in \ker L$. Moreover, if the linear mapping L is onto then the associated homomorphism L_M is unique. To see this, assume that L_M^* is another continuous homomorphism satisfying the requirements of above definition. Then, for a fixed $\varphi \in M(A)$ and any $f \in A$, we have

$$L_M(\varphi)L(f) = L_M^*(\varphi)L(f).$$

Since $L(f)$ can be any element of the Banach space Y, we get $L_M = L_M^*$. When we need to emphasize the associated homomorphism L_M, we shall speak of the pair (L, L_M) as a module-homomorphism pair.

In the special case that the target Banach space is the complex plane, the associated homomorphism L_M sends multipliers to the complex numbers, because $\mathcal{L}(\mathbb{C}) = \mathbb{C}$.

Definition 1.3. Suppose that A is a Banach space of analytic functions on Ω containing 1 and that $M(A)$ is the set of all multipliers of A. We mean by a module-functional on A, a linear functional $\phi : A \to \mathbb{C}$ with $\phi(1) = 1$ such that ϕ is a module-homomorphism $A \to \mathbb{C} : \phi(\varphi f) = \phi_M(\varphi)\phi(f)$, for every $\varphi \in M(A)$ and every $f \in A$.

We now return to our previous notation B for the Banach space of analytic functions on Ω.

Proposition 1.4. *Assume that B is a Banach space of analytic functions on Ω satisying the axioms (0-1)-(0-4), and that z denotes the coordinate function. If $1 \in B$, then for each λ, the point evaluation is the only module-functional ϕ which satisfies $\phi(z) = \lambda$.*

Proof: Having fixed $\lambda \in \Omega$, we want to show that $\phi(f) = f(\lambda)$ for every $f \in B$. First, we note that

$$1 = \phi(1 \cdot 1) = \phi_M(1) \cdot \phi(1).$$

Therefore, $\phi_M(1) = 1$. Secondly, $\phi_M(z) = \lambda$, because

$$\lambda = \phi(z) = \phi(z \cdot 1) = \phi_M(z) \cdot \phi(1) = \phi_M(z).$$

Let $f \in B$ be arbitrarily chosen. Using axiom (0-4), we can write

$$f = f(\lambda) + (z - \lambda)g \quad \text{for some} \quad g \in B.$$

Note that $z - \lambda$ belongs to the multiplier space $M(B)$. Hence

$$\begin{aligned} \phi(f) &= \phi(f(\lambda)) + \phi_M(z - \lambda)\phi(g) \\ &= f(\lambda)\phi(1) + \big(\phi_M(z) - \lambda\phi_M(1)\big)\phi(g) \\ &= f(\lambda), \end{aligned}$$

and the proof is complete. ■

We shall bring this section to an end by the following proposition on the essential ingredients of this manuscript.

Proposition 1.5. *Assume that B is a non-trivial Banach space of analytic functions satisfying the axioms (0-1)-(0-4) on the bounded domain Ω. Then the following hold:*

(a) $\sigma(M_z) = \overline{\Omega}$.

(b) $\mathcal{O}(\overline{\Omega}) \subset M(B) \subset H^\infty(\Omega)$, *and if* $1 \in B$, *then we have*

$$\mathcal{O}(\overline{\Omega}) \subset M(B) \subset H^\infty(\Omega) \cap B \subset B \subset \mathcal{O}(\Omega).$$

Proof: To prove part (a), we fix $\lambda \in \Omega$. Since B is non-trivial, there exists a function $f \in B$ which is not identically zero. We may assume that $f(\lambda) \neq 0$. Suppose that $\lambda \in \mathbb{C} \setminus \sigma(M_z)$, then $\lambda - M_z$ as an operator on B is invertible and in particular, it is onto. That is, there exists a function $g \in B$ such that $(\lambda - z)g = f$. But this is not possible, because $f(\lambda) \neq 0$. Thus $\Omega \subset \sigma(M_z)$, and since the spectrum of M_z is compact, we get $\overline{\Omega} \subset \sigma(M_z)$. For the reverse inclusion, suppose that $\lambda \notin \overline{\Omega}$; then by axiom (0-3), we have $1/(\lambda - z) \in M(B)$. Now it is easily seen that the operator $\lambda - M_z$ is invertible, so $\lambda \notin \sigma(M_z)$ which proves the desired inclusion. Thus part (a) is proved.

Proof of (b): For the first inclusion we shall use the holomorphic functional calculus. Let $\varphi \in \mathcal{O}\big(\sigma(M_z)\big)$, and let $f \in B$. We intend to show that $\varphi f \in B$. Using the holomorphic functional calculus, we get $\varphi(M_z) \in \mathcal{L}(B)$. Note that here, we use the assumption that φ is holomorphic in a neighbourhood of the spectrum of the operator M_z. Choosing an arbitrary function $f \in B$, we have $\varphi(M_z)f \in B$. What therefore remains to be proved is that $\varphi(M_z)f = \varphi f$. To this end, we first recall that

$$\varphi(M_z) = \frac{1}{2\pi i} \int_{\partial U} \varphi(\lambda)(\lambda - M_z)^{-1}\, d\lambda,$$

where U is a finitely connected smoothly bordered domain containing $\sigma(M_z)$ and such that $\overline{U}$ lies in the domain of analyticity of φ, and ∂U is oriented in the positive direction. Hence

$$\begin{aligned} \varphi(M_z)f &= \frac{1}{2\pi i} \int_{\partial U} \varphi(\lambda)(\lambda - M_z)^{-1} f\, d\lambda \\ &= \frac{1}{2\pi i} \int_{\partial U} \varphi(\lambda)(\lambda - z)^{-1} f\, d\lambda \\ &= \varphi f. \end{aligned}$$

So far the first inclusion is proved. The facts that $M(B) \subset H^\infty(\Omega)$ and that if $1 \in B$, then $M(B) \subset H^\infty(\Omega) \cap B$, are the contents of Proposition 2.6 of [**27**]. ■

2. Basic Concepts

Let M be an invariant subspace of B and let $\lambda \in \Omega$. It is known that the dimension of the quotient space $M/(z - \lambda)M$ does not depend on λ [**27**]. Therefore, we can make the following definition.

Definition 2.1. We say that the invariant subspace $M \neq \{0\}$ of B has index n if $\dim(M/(z - \lambda)M) = n$. In particular, if M is the trivial null subspace, then the index of M is 0.

The invariant subspaces of index n are also known as invariant subspaces having the codimension n property. Before advancing any further, we turn to a class of examples of invariant subspaces of index one. As we mentioned earlier, we shall assume that the multiplier invariant subspace J satisfies the condition $\sigma(M_z[B/J]) \subset \Omega_2$. This entails that J has index one (see Corollary 2.4). Hence we need to have a reasonable class of examples.

Examples of index one invariant subspaces. For a subset S of B, the notation $[S]$ is used for the smallest invariant subspace of B which contains S. If S consists of just one single element f, then the notation $[f]$ is used for the smallest invariant subspace of B containing f. Such invariant subspaces are called *cyclic* or *singly generated*. Note that $[f]$ is the closure of the polynomial multiples of f in B. Similarly, one may consider $M[f]$, the closure in B of multiplier multiples of f. Invariant subspaces of this kind always have index one ([**27**, Corollary 3.3]). On the other hand, by Beurling's theorem, every invariant subspace of the Hardy space H^2 is cyclic (see [**14**, p. 82] or [**24**, p. 99]). In fact, unless it is $\{0\}$, it is of the type $uH^2 = [u]$ for some inner function u. Thus every invariant subspace of H^2 has index one. It is also known that every nontrivial invariant subspace of the Dirichlet space D_1 has index one. This was noticed by S. Richter and A. L. Shields [**28**]. This result remains true for weighted Dirichlet spaces, as was observed by A. Aleman [**2**]. Moreover, by recent work of Aleman, Richter, and Ross [**4**], many other Banach spaces of analytic functions share this property. However, given $\alpha < 0$, and $1 \le n \le \infty$, there exists an invariant subspace M_n of D_α having index n, as was observed by Apostol, Bercovici, Foiaş, and Pearcy [**6**]. The Bergman space L^2_a is the special case $\alpha = -1$, $L^2_a = D_{-1}$, [**6**], [**7**]. The proofs presented for the existence of invariant subspaces of a given index in both of the last two examples have abstract natures involving the axiom of choice. A concrete construction of such invariant subspaces was first found by H. Hedenmalm [**20**] for the Bergman space L^2_a, and then by H. Hedenmalm, S. Richter, and K. Seip [**21**] for the Bergman spaces L^p_a.

In any Banach space of analytic functions satisfying our axioms, every invariant subspace of index larger than 1 contains an invariant subspace of index 1. Also every closed ideal in a unital Banach algebra, for instance H^∞, has index one [**27**]. The spaces D_α are algebras for $\alpha > 1$. Since the polynomials are dense in D_α, every invariant subspace of D_α for $\alpha > 1$ has index one [**27**]. This is so even for $0 < \alpha < 1$ (see [**4**, p. 260]). Finally, we note that the invariant subspaces of index 1 form a complete lattice with respect to intersection and span [**27**].

Definition 2.2. The collection of all $\lambda \in \mathbb{C}$ for which the operator $\lambda - M_z[B/J]$ is not onto is denoted by $\sigma'_J = \sigma'(M_z[B/J])$ and is called the weak spectrum of the operator $M_z[B/J]$.

It is clear that the weak spectrum σ'_J is a subset of the spectrum $\sigma_J \subset \overline{\Omega}$. For an invariant subspace J we have in general $\sigma'_J \cap \Omega = Z(J)$, where the notation $Z(J)$ stands for the common zero set of J in Ω, that is, the intersection of zero sets of all the elements of J (see [**27**, Proposition 4.4]). So far we know that $\sigma'_J \cap \Omega \subset \sigma_J \cap \Omega$. Now, assume that the invariant subspace J has index one and that $\lambda \in \Omega \setminus \sigma'_J$, so that the operator $\lambda - M_z[B/J]$ is onto. We claim that it is also one-to-one. To see this, let $(\lambda - M_z[B/J])(f + J) = 0 + J$, which implies that $(\lambda - z)f \in J$. Now, we use Lemma 3.1 of [**27**] to obtain $f \in J$, or equivalently $f + J = 0 + J$ which in turn proves that $\lambda - M_z[B/J]$ is one-to-one. Therefore, $\lambda \in \Omega \setminus \sigma_J$, making $\sigma'_J \cap \Omega = \sigma_J \cap \Omega$ (see [**27**, Theorem 4.5]). An alternative way to define the weak spectrum is the following: $\lambda \in \mathbb{C} \setminus \sigma'_J$ if and only if $(\lambda - z)B + J = B$. The following proposition sheds some light on the relationship between the spectrum of the operator $M_z[B/J]$ and the index of J.

Proposition 2.3. *Assume that B is a Banach space of analytic functions satisfying the axioms (0-1)-(0-4) on Ω. If J is an invariant subspace of B such that* $\operatorname{index}(J) > 1$, *then we have $\sigma(M_z[B/J]) = \overline{\Omega}$.*

Proof: We know that for $\lambda \in \Omega$, $\dim(J/(\lambda - z)J) = \operatorname{index}(J)$ and also

$$\sigma(M_z[B/J]) \subset \sigma(M_z) = \overline{\Omega}.$$

For the reverse inclusion, it suffices to show that $\Omega \setminus Z(J) \subset \sigma(M_z[B/J])$. Assume that $\lambda - M_z[B/J]$ is invertible, for $\lambda \in \Omega \setminus Z(J)$. Then it is one-to-one, so $(\lambda - M_z[B/J])(f + J) = 0 + J$ must imply $f \in J$. But the above equality is equivalent to $(\lambda - z)f \in J$, therefore this latter must imply $f \in J$. Since we have

$$(\lambda - z)f \in (\lambda - z)B = \{g \in B : g(\lambda) = 0\} = M_\lambda,$$

we obtain $(\lambda - z)f \in J \cap M_\lambda$. But $f \in J$ if and only if $(\lambda - z)f \in (\lambda - z)J$. Now, the question is the same as asking whether we have

$$(\lambda - z)J = J \cap M_\lambda.$$

This is true for $\lambda \in \Omega \setminus Z(J)$ precisely for invariant subspaces J of index 1 [**27**, Lemma 3.1]. This means that $\sigma(M_z[B/J])$ cannot be smaller than $\overline{\Omega}$. The proof is complete. ■

Corollary 2.4. *The only possibility for the invariant subspace J of the Banach space B of analytic functions on Ω to satisfy the condition $\sigma(M_z[B/J]) \neq \overline{\Omega}$ is that J has index one.*

Proposition 2.5. *Assume that B is a Banach space of analytic functions satisfying the axioms (0-1)-(0-4) on Ω, and that J is a multiplier invariant subspace of B. If $\sigma_J = \sigma(M_z[B/J]) \subset \Omega$, then there exists a polynomial p with zeros in Ω such that $J = pB = \{pf : f \in B\}$.*

Proof: As we have seen before, $\sigma_J \cap \Omega = Z(J)$, so that $\sigma_J = Z(J)$. It is evident that $Z(J)$ consists of finitely many points of Ω, otherwise it would accumulate at some point in Ω, in which case the unicity theorem for analytic functions would force $J = \{0\}$, making $B/J = B$, whence $\sigma_J = \sigma(M_z[B]) = \overline{\Omega}$, contradicting our assumption. Let p be a polynomial with zeros in $Z(J)$, such that the multiplicity of each zero equals the least common multiplicity of all the functions in J. As a result of the axiom (0-4) we know that pB can be described as those functions in B vanishing to the required multiplicities on $Z(J)$. Note that our axiom (0-2) (the continuity of point evaluation at each $\lambda \in \Omega$) implies that pB is closed in B. It is easily seen that pB is a multiplier invariant subspace of B containing J. We claim that $J = pB$. Put $J/p = J_1$. It is clear that J_1 is a multiplier invariant subspace of B, and that $J = pJ_1 \subset J_1$. We first note that our assumption on σ_J entails that J has index one (see Corollary 2.4), and secondly if we temporarily accept that

$$\sigma_{J_1} = \sigma(M_z[B/J_1]) \subset \sigma(M_z[B/J]) \subset \Omega,$$

then using the fact that $Z(J_1) = \emptyset$ we conclude that $\sigma_{J_1} = Z(J_1) \cap \Omega = \emptyset$. Therefore $B/J_1 = \{0\}$, or equivalently $J_1 = B$ and finally $J = pJ_1 = pB$. Now, the only thing which requires verification is $\sigma_{J_1} \subset \sigma_J$. To this end, define $\Psi : pB/J \to B/J_1$ by $pf + J \mapsto f + J_1$, for $f \in B$. First note that for f and g in B, the equality $pf + J = pg + J$ is equivalent to $(f - g) \in J/p = J_1$, or $f + J_1 = g + J_1$, so that the mapping Ψ is well-defined and one-to-one. The surjectivity of Ψ is also clear, making it an isomorphism. Now, assume that T_1 and T_2 are the operators of multiplication by z on B/J_1 and pB/J, respectively. We then have

$$\begin{aligned}\Psi(T_2(pf + J)) = \Psi(zpf + J) = zf + J_1 &= T_1(f + J_1) \\ &= T_1(\Psi(pf + J)).\end{aligned}$$

Therefore $\Psi T_2 = T_1 \Psi$, or equivalently $T_2 = \Psi^{-1} T_1 \Psi$. Similarly, we can obtain $\lambda - T_2 = \Psi^{-1}(\lambda - T_1)\Psi$ for every $\lambda \in \mathbb{C}$, which in turn shows

that $\sigma(T_1) = \sigma(T_2)$. Now, let $\lambda \in \mathbb{C} \setminus \sigma_J$, and define the operator $A_\lambda : pB/J \to pB/J$ by the following:

$$A_\lambda(pf + J) = p(M_z[B/J])(\lambda - M_z[B/J])^{-1}(f + J), \quad f \in B.$$

We first note that for every natural number n, and every $f \in B$ we have

$$(M_z[B/J])^n(f + J) = (M_z[B/J])^{n-1}(zf + J),$$

so that

$$(M_z[B/J])^n(f + J) = z^n f + J = M_{z^n}[B/J](f + J).$$

Therefore $p(M_z[B/J]) = M_p[B/J]$ takes B/J onto pB/J. A straightforward calculation shows that for $\lambda \in \mathbb{C} \setminus \overline{\Omega}$, one has

$$(\lambda - M_z[pB/J])A_\lambda(f + J) = f + J,$$

that is, $(\lambda - M_z[pB/J])A_\lambda$ equals the identity operator for every $\lambda \in \mathbb{C} \setminus \overline{\Omega}$. It is easy to see that for such λ, the operator A_λ commutes with $\lambda - M_z[pB/J]$. But A_λ is analytic in the λ variable in $\mathbb{C} \setminus \sigma_J$, and also $\lambda - M_z[pB/J]$ is an entire function of λ, therefore their composition is analytic in $\mathbb{C} \setminus \sigma_J$. It follows from the uniqueness theorem for vector valued analytic functions that in fact $(\lambda - M_z[B/J])A_\lambda$ equals the identity operator for all $\lambda \in \mathbb{C} \setminus \sigma_J$, which means that A_λ is the inverse operator to $\lambda - M_z[pB/J]$ for all $\lambda \in \mathbb{C} \setminus \sigma_J$, and hence

$$\sigma_{J_1} = \sigma(T_1) = \sigma(T_2) = \sigma(M_z[pB/J]) \subset \sigma_J.$$

The proof is complete. ■

3. The main result

This section aims to carry out the following plan. Suppose Ω is a bounded planar domain and B a Banach space of analytic functions on Ω which contains 1 and satisfies the axioms (0-1)-(0-4). Let B_1 be a closed subspace of B containing both the constant function 1 and the coordinate function z, and let I be a proper multiplier invariant subspace of B_1. Consider the mapping $H_I : \mathcal{O}\big(\sigma(M_z[B_1/I])\big) \to \mathcal{L}(B_1/I)$ defined by

$$f \mapsto \frac{1}{2\pi i} \int_{\partial U} f(\lambda)(\lambda - M_z[B_1/I])^{-1}\, d\lambda,$$

where U is some finitely connected smoothly bordered domain surrounding the spectrum $\sigma(M_z[B_1/I])$ such that $\overline{U}$ lies in the domain of analyticity of f, and ∂U is oriented in the positive direction. Note that by Cauchy formula the above integral does not depend on the choice of U, moreover, if $\mathcal{O}\big(\sigma(M_z[B_1/I])\big)$, the algebra of all analytic functions on $\sigma(M_z[B_1/I])$, is endowed with compact open topology, then the above mapping is continuous. Aside from these properties, this mapping has the following specifications:

(1) It is a homomorphism, because it is multiplicative, and

(2) It maps unit to unit, and z to $M_z[B_1/I]$.

Therefore, with some abuse of notation, we have a module-homomorphism L_I, from $\mathcal{O}\big(\sigma(M_z[B_1/I])\big)$ into B_1/I which takes unit to unit and z to $z+I$. We recall that the action of L_I on an element $h \in \mathcal{O}\big(\sigma(M_z[B_1/I])\big)$ is defined to be the action of $H_I(h)$ on the element $1+I$ of B_1. We shall at times refer to this mapping as HFC for the involvement of the holomorphic functional calculus. However, we notice that this mapping is in fact $L_I = P_I \circ H_I$ as defined earlier. One of the most important features of HFC as defined above is its compatibility in the following manner: If $B = B_1 \oplus B_2^0$ then the canonical surjective module-homomorphism $B_1 \to B_1/I$, the module-homomorphism which sends f to $f+I$, and the HFC module-homomorphism are compatible in such a way that they define a module-homomorphism $L_I : B \to B_1/I$. Furthermore, the quotient Banach spaces B_1/I and $B/\ker L_I$ are canonically module-isomorphic; this means that $\tilde{L}_I : B/\ker L_I \to B_1/I$ is injective and surjective, furthermore, its inverse is also a module-homomorphism. It is desirable to reveal the relationship between I and $\ker L_I$. The objective of this paper is achieved whenever this is done for a reasonable class of $M(B_1)$-invariant subspaces.

For a bounded planar domain Ω, we can find two other planar domains Ω_1 and Ω_2 such that:

(1) Ω_1 is bounded,

(2) Ω_2 is unbounded and non-dense in the complex plane,

(3) $\Omega = \Omega_1 \cap \Omega_2$, and $\Omega_1 \cup \Omega_2 = \mathbb{C}$.

Define the projection $P_1 : \mathcal{O}(\Omega) \to \mathcal{O}(\Omega_1)$ by the following formula

$$P_1 f(z) = \frac{1}{2\pi i} \int\limits_{\partial V_n} (\zeta - z)^{-1} f(\zeta)\, d\zeta \quad z \in V_n,$$

where $\{V_n\}_{n=1}^{\infty}$ is an increasing sequence of finitely connected smoothly bordered subdomains of Ω_1 satisfying the relations $V_1 \supset \mathbb{C} \setminus \Omega_2$ and $\cup_{n=1}^{\infty} V_n = \Omega_1$. As usual in the theory of one complex variables, the boundary of V_n is oriented in the positive direction. The above formula does not depend on the choice of V_n, and furnishes us with a continuous projection $P_1 : \mathcal{O}(\Omega) \to \mathcal{O}(\Omega_1)$. Similarly one can find a continuous projection $P_2 : \mathcal{O}(\Omega) \to \mathcal{O}_0(\Omega_2)$ with the condition that $P_1 + P_2 =$ identity. Note that by Liouville's theorem both P_1 and P_2 are uniquely determined. We also note that P_1 and P_2 map B into $\mathcal{O}(\Omega_1)$ and $\mathcal{O}_0(\Omega_2)$, respectively. Keeping in mind that B is a Banach space of analytic functions on Ω which contains 1 and satisfies the axioms (0-1)-(0-4), we put $B_1 = B \cap \mathcal{O}(\Omega_1)$, $B_2 = B \cap \mathcal{O}(\Omega_2)$, and $B_2^0 = B \cap \mathcal{O}_0(\Omega_2)$. It is evident that B_1 and B_2^0 are two closed subspaces of B. Let $f \in B_1 \cap B_2^0$, then $f \in \mathcal{O}(\mathbb{C})$ and $f(z) \to 0$ as $|z| \to \infty$; so that by Liouville's theorem, f is identically 0. It is also obvious that $B \supset B_1 \oplus B_2^0$, the direct sum of B_1 and B_2^0. We shall assume that B can be written as the direct sum of B_1 and B_2^0, and that B_1 and B_2^0 satisfy the axioms (0-1)-(0-4) on Ω_1 and Ω_2, respectively. We emphasize that B_1 and B_2^0 automatically satisfy the first three axioms. Furthermore, B_1 satisfies the fourth axiom for every $\lambda \in \Omega_1$ except those on the boundary of Ω_2. The same is true for B_2^0 also. Note that our assumption that $B = B_1 \oplus B_2^0$ is equivalent to either of $P_1 B \subset B$ or $P_2 B \subset B$.

Now, suppose that I is a proper multiplier invariant subspace of B_1 and recall the HFC module-homomorphism $\mathcal{O}\big(\sigma(M_z[B_1/I])\big) \to B_1/I$ given by

$$f \mapsto \left(\frac{1}{2\pi i} \int_{\partial U} f(\lambda)(\lambda - M_z[B_1/I])^{-1} d\lambda \right) (1 + I),$$

where U is some finitely connected smoothly bordered domain surrounding the spectrum of the operator $M_z[B_1/I] : B_1/I \to B_1/I$. If we assume that $\sigma(M_z[B_1/I]) \subset \Omega_2$, then HFC module-homomorphism furnishes us with a continuous module-homomorphism $B_2^0 \to B_1/I$, because of

$$B_2^0 \subset \mathcal{O}_0(\Omega_2) \subset \mathcal{O}\big(\sigma(M_z[B_1/I])\big).$$

Now we are in a place which allows us to construct our desired surjective module-homomorphism which we again write

$$L_I : B = B_1 \oplus B_2^0 \to B_1/I,$$

defined to be the canonical module-homomorphism on B_1, that is, $L_I(f) = f + I$ for $f \in B_1$ and the HFC module-homomorphism on B_2^0. The following lemma tells us more about L_I.

Before proceeding, let us fix once again our assumptions on B. We know that B is a Banach space of analytic functions defined on a bounded domain Ω in the extended complex plane. Furthermore, B contains 1, satisfies the axioms (0-1)-(0-4), and has $B = B_1 \oplus B_2^0$. We shall also make the assumption that $M(B) = M(B_1) \oplus M_0(B_2)$, where $M_0(B_2) = M(B_2) \cap \mathcal{O}_0(\Omega_2)$.

Lemma 3.1. *Let I be an $M(B_1)$-invariant subspace of B_1 satisfying $\sigma(M_z[B_1/I]) \subset \Omega_2$. If $\mathcal{O}_0(\overline{\Omega}_2)$ is dense in $M_0(B_2)$ and in B_2^0, then L_I is a surjective module-homomorphism $B = B_1 \oplus B_2^0 \to B_1/I$.*

Proof: The surjectivity of L_I is obvious and the continuity is a consequence of the closed graph theorem. We just need to verify that for every $\varphi \in M(B)$ and every $f \in B$, the following equality holds

$$L_I(\varphi f) = L_{I,M}(\varphi) L_I(f),$$

where $L_{I,M} : M(B) \to \mathcal{L}(B_1/I)$ is an appropriate continuous homomorphism. Since L_I is onto, it follows from Remark 1.2 that there exists at most one such homomorphism. Let $\varphi \in M(B)$ be written as $\varphi = \varphi_1 + \varphi_2$ where $\varphi_1 \in M(B_1)$ and $\varphi_2 \in M_0(B_2)$. Put

$$L_{I,M}(\varphi) = L_{I,M}(\varphi_1 + \varphi_2) = M_{\varphi_1}[B_1/I] + \varphi_2(M_z[B_1/I]).$$

Here, $M_{\varphi_1}[B_1/I]$ stands for the operator on B_1/I which sends $f + I$ to $\varphi f + I$. Note that the second operator in the definition of $L_{I,M}$ comes from the holomorphic functional calculus. It is not difficult to see that $L_{I,M}$ is a homomorphism; the important fact which requires verification is to show that for $\varphi_1 \in M(B_1)$ and $\varphi_2 \in M_0(B_2)$, we have

$$L_{I,M}(\varphi_1\varphi_2) = L_{I,M}(\varphi_1) L_{I,M}(\varphi_2) = L_{I,M}(\varphi_2) L_{I,M}(\varphi_1).$$

We first check this for $\varphi_2(z) = (\lambda - z)^{-1}$, with $\lambda \in \mathbb{C} \setminus \overline{\Omega}_2$. In this case

$$\frac{\varphi_1(z)}{\lambda - z} = \frac{\varphi_1(z) - \varphi_1(\lambda)}{\lambda - z} + \frac{\varphi_1(\lambda)}{\lambda - z}. \tag{3-1}$$

Note that the first term on the right hand side of (3-1) is an element of $M(B_1)$. To see this, put $T_\lambda\varphi_1(z) = \frac{\varphi_1(\lambda) - \varphi_1(z)}{\lambda - z}$, and let $\lambda \in \Omega_1$, $f \in B_1$. We first note that by axiom (0-4), $T_\lambda\varphi_1 \in B_1$. We then have

$$\begin{aligned} T_\lambda\varphi_1(z) \cdot f(z) &= \frac{\varphi_1(\lambda) - \varphi_1(z)}{\lambda - z} \cdot f(z) \\ &= T_\lambda(\varphi_1(z) f(z)) + \frac{\varphi_1(\lambda) f(z) - \varphi_1(\lambda) f(\lambda)}{\lambda - z} \\ &= T_\lambda(\varphi_1(z) f(z)) - \varphi_1(\lambda) T_\lambda f(z). \end{aligned}$$

We now use the axiom (0-4) to get $T_\lambda\varphi_1(z) \in M(B_1)$. We also note that $\varphi_1(\lambda)(\lambda - z)^{-1} \in M_0(B_2)$, because it is a rational function with pole outside $\overline{\Omega}_2$. Hence, by (3-1),

$$L_{I,M}\big(\varphi_1 \cdot (\lambda - z)^{-1}\big) = M_{T_\lambda\varphi_1}[B_1/I] + \varphi_1(\lambda)(\lambda - M_z[B_1/I])^{-1}.$$

We claim that the right hand side is equal to $M_{\varphi_1}[B_1/I](\lambda - M_z[B_1/I])^{-1}$ and to $(\lambda - M_z[B_1/I])^{-1}M_{\varphi_1}[B_1/I]$. To prove the claim, we multiply both sides by $(\lambda - M_z[B_1/I]) = M_{\lambda-z}[B_1/I]$, from the right in one case and from the left in the other. So that the question is whether

$$M_{T_\lambda\varphi_1}[B_1/I](\lambda - M_z[B_1/I]) + \varphi_1(\lambda) = M_{\varphi_1}[B_1/I]. \tag{3-2}$$

Since in general, for φ and ψ in $M(B_1)$ we have $M_\varphi[B_1/I]M_\psi[B_1/I] = M_{\varphi\psi}[B_1/I]$, the question transforms into whether $M_{\varphi_1(z)-\varphi_1(\lambda)}[B_1/I] + \varphi_1(\lambda) = M_{\varphi_1(z)}[B_1/I]$, which is true. Therefore the claim is proved, and consequently

$$L_{I,M}(\varphi_1 \cdot (\lambda - z)^{-1}) = L_{I,M}(\varphi_1)L_{I,M}((\lambda - z)^{-1}).$$

Since finite linear combinations of $z \mapsto (\lambda - z)^{-1}$ for $\lambda \in \mathbb{C} \setminus \overline{\Omega}_2$ are dense in $\mathcal{O}_0(\overline{\Omega}_2)$, we obtain

$$L_{I,M}(\varphi_1\varphi_2) = L_{I,M}(\varphi_1)L_{I,M}(\varphi_2), \quad \text{for} \quad \varphi_1 \in M(B_1) \quad \text{and} \quad \varphi_2 \in \mathcal{O}_0(\overline{\Omega}_2). \tag{3-3}$$

Finally because of our assumption that $\mathcal{O}_0(\overline{\Omega}_2)$ is dense in $M_0(B_2)$, the equality (3-3) holds for every $\varphi_1 \in M(B_1)$ and every $\varphi_2 \in M_0(B_2)$. We now turn to prove that L_I is a module-homomorphism. To do this, we consider four cases.

Case 1: $\varphi \in M(B_1)$ and $f \in B_1$; we have $\varphi f \in B_1$, therefore

$$L_I(\varphi f) = \varphi f + I = M_\varphi[B_1/I](f + I) = L_{I,M}(\varphi)L_I(f).$$

Case 2: $\varphi \in M(B_1)$ and $f \in B_2^0$; it suffices to treat the case when $f = (\lambda - z)^{-1}$ for $\lambda \in \mathbb{C} \setminus \overline{\Omega}_2$, because $\mathcal{O}_0(\overline{\Omega}_2)$ is dense in B_2^0. We then have the decomposition

$$\varphi f = \frac{\varphi}{\lambda - z} = \frac{\varphi - \varphi(\lambda)}{\lambda - z} + \frac{\varphi(\lambda)}{\lambda - z},$$

where the first term is in $M(B_1) \subset B_1$ (see above) and the second one in $M_0(B_2) \subset B_2^0$. Hence

$$\begin{aligned} L_I(\varphi f) &= (T_\lambda \varphi + I) + \varphi(\lambda)(\lambda - M_z[B_1/I])^{-1}(1+I) \\ &= M_{T_\lambda \varphi}(1+I) + \varphi(\lambda)(\lambda - M_z[B_1/I])^{-1}(1+I) \\ &= \big[M_{T_\lambda \varphi} + \varphi(\lambda)(\lambda - M_z[B_1/I])^{-1}\big](1+I). \end{aligned}$$

We now use (3-2) to obtain

$$L_I(\varphi f) = M_\varphi[B_1/I](\lambda - M_z[B_1/I])^{-1}(1+I) = L_{I,M}(\varphi) L_I(f).$$

Case 3: $\varphi \in M_0(B_2)$ and $f \in B_2^0$; the desired equality is a consequence of the holomorphic functional calculus, since $\varphi f \in B_2^0$.

Case 4: $\varphi \in M_0(B_2)$ and $f \in B_1$; we may assume that $\varphi(z) = (\lambda - z)^{-1}$ for $\lambda \in \mathbb{C} \setminus \overline{\Omega}_2$, because $\mathcal{O}_0(\overline{\Omega}_2)$ is dense in $M_0(B_2)$. This time we use the following decomposition of φf;

$$\varphi f = \frac{f(z) - f(\lambda)}{\lambda - z} + \frac{f(\lambda)}{\lambda - z} = -T_\lambda f(z) + \frac{f(\lambda)}{\lambda - z}. \tag{3-4}$$

By the axiom (0-4), the first term is in B_1 and the second one is in $M_0(B_2) \subset B_2^0$. For $\lambda \in \Omega_1$, it is clear that $T_\lambda \in \mathcal{L}(B_1)$, which associates to a given $f \in B_1$, the element $T_\lambda f \in B_1$. There is an associated operator $T_\lambda[B_1/I]$ on B_1/I, which maps a given $f + I \in B_1/I$ onto the element $T_\lambda f + I \in B_1/I$. Therefore, on B_1 we have $(\lambda - M_z) T_\lambda f = f(\lambda) - f$, and on B_1/I we have

$$\big((\lambda - M_z[B_1/I]) T_\lambda[B_1/I]\big) f = f(\lambda) - f + I,$$

or equivalently

$$T_\lambda[B_1/I] f = (\lambda - M_z[B_1/I])^{-1}\big(f(\lambda) - f + I\big). \tag{3-5}$$

Finally, from (3-4) and (3-5) we have

$$\begin{aligned} L_I(\varphi f) &= -T_\lambda[B_1/I] f + f(\lambda)(\lambda - M_z[B_1/I])^{-1}(1+I) \\ &= (\lambda - M_z[B_1/I])^{-1}\big(f - f(\lambda) + I\big) + (\lambda - M_z[B_1/I])^{-1}\big(f(\lambda) + I\big) \\ &= (\lambda - M_z[B_1/I])^{-1}(f + I) \\ &= L_{I,M}(\varphi) L_I(f). \end{aligned}$$

The proof is now complete. ■

In the proof of Lemma 3.1, we used the assumptions that $\mathcal{O}_0(\overline{\Omega}_2)$ is dense in both $M_0(B_2)$ and B_2^0. In the following, we shall try to get rid of these assumptions. Let $\Omega_{2,\sharp}$ be a finitely connected smoothly bordered domain such that

$$(\mathbb{C} \setminus \Omega_1) \cup \sigma(M_z[B_1/I]) \subset \Omega_{2,\sharp} \subset \overline{\Omega}_{2,\sharp} \subset \Omega_2. \tag{3-6}$$

Lemma 3.2. *Let I be an $M(B_1)$-invariant subspace of B_1 satisfying $\sigma(M_z[B_1/I]) \subset \Omega_2$. Then there exists a pair $(L_I, L_{I,M})$ with the following properties:*

$L_I : B_1 \oplus \mathcal{O}_0(\overline{\Omega}_{2,\sharp}) \to B_1/I$ and $L_{I,M} : M(B_1) \oplus \mathcal{O}_0(\overline{\Omega}_{2,\sharp}) \to \mathcal{L}(B_1/I)$, such that $L_I(\varphi f) = L_{I,M}(\varphi)L_I(f)$ for every $\varphi \in M(B_1) \oplus \mathcal{O}_0(\overline{\Omega}_{2,\sharp})$ and every $f \in B_1 \oplus \mathcal{O}_0(\overline{\Omega}_{2,\sharp})$, and such that $L_{I,M}(\varphi_1\varphi_2) = L_{I,M}(\varphi_1)L_{I,M}(\varphi_2)$ for every φ_1 and φ_2 in $M(B_1) \oplus \mathcal{O}_0(\overline{\Omega}_{2,\sharp})$.

Proof: The proof is exactly as that of Lemma 3.1, if we notice that finite linear combinations of rational functions of the form $(\lambda - z)^{-1}$ for $\lambda \in \mathbb{C} \setminus \overline{\Omega}_{2,\sharp}$ are dense in $\mathcal{O}_0(\overline{\Omega}_{2,\sharp})$, and that $M(B_1) \oplus \mathcal{O}_0(\overline{\Omega}_{2,\sharp})$ is an algebra of analytic functions. To this end, it suffices to verify that for every $\varphi \in M(B_1)$ and every $f \in \mathcal{O}_0(\overline{\Omega}_{2,\sharp})$ we have $\varphi f \in M(B_1) \oplus \mathcal{O}_0(\overline{\Omega}_{2,\sharp})$. We may assume that $f(z) = (\lambda - z)^{-1}$ for $\lambda \in \mathbb{C} \setminus \overline{\Omega}_{2,\sharp}$. Let us first write

$$\varphi(z)f(z) = -T_\lambda\varphi(z) + \frac{\varphi(\lambda)}{\lambda - z}.$$

We have observed in the proof of Lemma 3.1 that $-T_\lambda\varphi(z) \in M(B_1)$, hence $\varphi(z)f(z) \in M(B_1) \oplus \mathcal{O}_0(\overline{\Omega}_{2,\sharp})$. The last thing which requires verification is the following

$$L_{I,M}(\varphi_1\varphi_2) = L_{I,M}(\varphi_1)L_{I,M}(\varphi_2), \quad \varphi_1 \in M(B_1),\ \varphi_2 \in \mathcal{O}_0(\overline{\Omega}_{2,\sharp}).$$

Since φ_2 is analytic in a neighbourhood of $\overline{\Omega}_{2,\sharp}$, we can then apply the Cauchy integral formula to get

$$\varphi_2(z) = \frac{1}{2\pi i}\int_{\partial U} \varphi_2(\lambda)(\lambda - z)^{-1}d\lambda, \quad z \in \Omega_{2,\sharp},$$

where U is some smoothly bordered domain containing $\overline{\Omega}_{2,\sharp}$, where φ_2 is analytic. Now we have

$$\varphi_1(z)\varphi_2(z) = \frac{1}{2\pi i}\int_{\partial U} \varphi_2(\lambda)\varphi_1(z)(\lambda - z)^{-1}d\lambda \quad z \in \Omega_1 \cap \Omega_{2,\sharp},$$

and finally

$$\begin{aligned} L_{I,M}(\varphi_1\varphi_2) &= \frac{1}{2\pi i}\int_{\partial U} \varphi_2(\lambda) L_{I,M}(\varphi_1) L_{I,M}\Big((\lambda - z)^{-1}\Big)\, d\lambda \\ &= L_{I,M}(\varphi_1)\bigg(\frac{1}{2\pi i}\int_{\partial U} \varphi_2(\lambda)(\lambda - M_z[B_1/I])^{-1}\, d\lambda\bigg) \\ &= L_{I,M}(\varphi_1) L_{I,M}(\varphi_2). \end{aligned}$$

The proof is complete. ■

Restricting to $B = B_1 \oplus B_2^0$, we obtain the following consequence.

Proposition 3.3. *Let I be an $M(B_1)$-invariant subspace of B_1 satisfying the condition $\sigma(M_z[B_1/I]) \subset \Omega_2$ and let L_I be as in Lemma 3.2. Consider the restriction $L_I : B = B_1 \oplus B_2^0 \to B_1/I$. Then L_I is a module-homomorphism.*

Proof: We need to note that such an $\Omega_{2,\sharp}$ satisfying (3-6) always exists, and that $B = B_1 \oplus B_2^0 \subset B_1 \oplus \mathcal{O}_0(\overline{\Omega}_{2,\sharp})$, and finally that $M(B) = M(B_1) \oplus M_0(B_2) \subset M(B_1) \oplus \mathcal{O}_0(\overline{\Omega}_{2,\sharp})$. ■

Proposition 3.4. *Let I be an $M(B_1)$-invariant subspace of B_1 satisfying the condition $\sigma(M_z[B_1/I]) \subset \Omega_2$ and let L_I be as in Proposition 3.3. If $J = \ker L_I$, then*

(a) $\operatorname{clos}(I \cdot M(B)) \subset J$,

(b) $J \cap B_1 = I$, *and*

(c) $\sigma(M_z[B_1/I]) = \sigma(M_z[B/J])$.

Proof: For ease of notation, we put $\Lambda(I) = \operatorname{clos}(I \cdot M(B))$. Suppose that $f \in \Lambda(I)$, then there exists a sequence $f_n \to f$, $f_n \in I \cdot M(B)$. Put $f_n = \sum_{k=1}^m h_{n,k}\, g_{n,k}$ where $h_{n,k} \in M(B)$ and $g_{n,k} \in I$. Now, we have

$$L_I(f_n) = \sum_{k=1}^m L_{I,M}(h_{n,k}) L_I(g_{n,k}) = \sum_{k=1}^m L_{I,M}(h_{n,k}) 0 = 0,$$

which proves $L_I(f) = 0$, by continuity. This shows that $\Lambda(I) \subset J$.

To prove part (b), we first note that the inclusion $I \subset J \cap B_1$ is obvious. For the reverse inclusion, let $f \in J \cap B_1$, then from the definitions of L_I and J we have $L_I(f) = f + I = 0 + I$ or equivalently $f \in I$, hence $J \cap B_1 \subset I$ and finally

$$J \cap B_1 = (\ker L_I) \cap B_1 = I.$$

The last thing to be proved is the equality

$$\sigma(M_z[B/J]) = \sigma(M_z[B_1/I]),$$

for all $M(B_1)$-invariant subspaces I with $\sigma(M_z[B_1/I]) \subset \Omega_2$. To prove this, we first recall the connection between the mapping $L_I : B \to B_1/I$ and the operator $M_z[B_1/I]$ defined on B_1/I. It is immediate that for every $f \in B_1$ we have $L_I(zf) = M_z[B_1/I]L_I(f)$. Moreover, if $f \in B_2^0$ then $L_I = P_I \circ H_I$ (see section 1) and we have

$$\begin{aligned} L_I(zf) = P_I(H_I(zf)) = P_I(H_I(z)H_I(f)) &= H_I(z)H_I(f)(1+I) \\ &= M_z[B_1/I]L_I(f). \end{aligned}$$

Therefore the equality $L_I(zf) = M_z[B_1/I]L_I(f)$ holds for every $f \in B$. We know that $J = \ker L_I$, therefore the surjective module-homomorphism L_I induces an isomorphism $\bar{L_I} : B/J \to B_1/I$ defined by $\bar{L_I}(f+J) = L_I(f)$, hence

$$\begin{aligned} \bar{L_I}\big(M_z[B/J](f+J)\big) = \bar{L_I}(zf+J) = L_I(zf) = M_z[B_1/I]L_I(f) \\ = M_z[B_1/I]\bar{L_I}(f+J). \end{aligned}$$

So that $\bar{L_I}$ carries over $M_z[B/J]$ to $M_z[B_1/I]$; more precisely we have $\bar{L_I}M_z[B/J] = M_z[B_1/I]\bar{L_I}$. Now it is easily seen that if $(\lambda - M_z[B_1/I])^{-1}$ exists for some complex number λ, then $(\lambda - M_z[B/J])^{-1}$ exists too. In fact we have the following equality

$$(\lambda - M_z[B_1/I])^{-1} = \bar{L_I}(\lambda - M_z[B/J])^{-1}\bar{L_I}^{-1},$$

since $M_z[B_1/I] = \bar{L_I}(M_z[B/J])\bar{L_I}^{-1}$. This argument shows that the two spectra are the same, hence the proof is complete. ■

As we shall see later on, Proposition 3.4 will constitute a considerable part of our main result, Theorem 3.10. What we intend to show is equality in part (a), instead of inclusion. Let us temporarily assume that we have been able to demonstrate the equality in part (a). Thus, we would have a mapping, say Λ, from the collection of all $M(B_1)$-invariant subspaces I with $\sigma(M_z[B_1/I]) \subset \Omega_2$ into the collection of all $M(B)$-invariant subspaces J with the same condition on the spectrum $\sigma(M_z[B/J])$. The image of I under the mapping Λ is defined to be

$$\Lambda(I) = \operatorname{clos}(I \cdot M(B)) = \ker L_I = J.$$

Part (b) of the above proposition tells us that the mapping Λ is injective. Now, assume that J is an $M(B)$-invariant subspace of B satisfying $\sigma(M_z[B/J]) \subset \Omega_2$. Concerning the surjectivity of Λ, we will look at $J \cap B_1$ as a natural candidate, as part (b) of the above proposition suggests. Unfortunately, it might happen that $J \cap B_1$ collapses to the trivial null subspace, which in turn will destroy the equality $\Lambda(J \cap B_1) = J$. To prevent this event, we follow Hedenmalm [**16**] and make the assumption that the Banach space B of analytic functions on Ω satisfies the shrinking domain condition (see Definition 3.5). This assumption makes it possible to prove a factorization theorem which turns out to be crucial both to the proof of the equality in part (a) of Proposition 3.4 and to the proof of the surjectivity of the mapping Λ. Given $f \in B$, it is not difficult to find two functions f_1 and f_2, analytic in Ω_1 and Ω_2 respectively, such that $f = f_1 \cdot f_2$. But, in general, there is no guarantee that $f_1 \in B_1$ and $f_2 \in B_2$. The "shrinking domain" condition which is easy to check seems to be what is needed to get the proof of the factorization theorem to go through (see Lemma 3.7). We mention that the authors in [**10**] used some rather similar arguments to factor out analytic functions in the Dirichlet space.

Let us fix two finitely connected smoothly bordered domains V_1 and V_2 such that:

(1) $\mathbb{C} \setminus \Omega_2 \subset V_1 \subset\subset \Omega_1$,

(2) $\mathbb{C} \setminus \Omega_1 \subset V_2 \subset \overline{V}_2 \subset \Omega_2$ and

(3) $\overline{V}_1 \cap \overline{V}_2 = \emptyset$.

We now make the following definition.

Definition 3.5. Let $B = B_1 \oplus B_2^0$ be a Banach space of analytic functions on Ω satisfying the axioms (0-1)-(0-4). We say that B satisfies the shrinking domain condition, with respect to the pair (V_1, V_2), if there exist a Banach space of analytic functions $B^{(1)}$ on $\Omega_1 \cap V_2$ and a Banach space of analytic functions $B^{(2)}$ on $V_1 \cap \Omega_2$ such that

(a) $B^{(1)}$ and $B^{(2)}$ satisfy the axioms (0-1)-(0-4) with respect to the domains $\Omega_1 \cap V_2$ and $\Omega_2 \cap V_1$, respectively,

(b) $B^{(1)} \cap \mathcal{O}(\Omega_1) = B_1$, and $B^{(2)} \cap \mathcal{O}_0(\Omega_2) = B_2^0$,

(c) $B^{(1)} = B_1 \oplus B_{2,0}^{(1)}$, where $B_{2,0}^{(1)} = B^{(1)} \cap \mathcal{O}_0(V_2)$, and $B^{(2)} = B_1^{(2)} \oplus B_2^0$, where $B_1^{(2)} = B^{(2)} \cap \mathcal{O}(V_1)$,

(d) $M(B^{(1)}) = M(B_1) \oplus M_0(B_2^{(1)})$, where $M_0(B_2^{(1)}) = M(B_2^{(1)}) \cap \mathcal{O}_0(V_2)$, and $M(B^{(2)}) = M(B_1^{(2)}) \oplus M_0(B_2)$.

Remark 3.6. The conditions of the previous definition are somewhat technical. It is important to realize that in the examples we have mentioned, they are indeed satisfied. For instance, if $B = L^2_a(\Omega)$, so that $B_1 = L^2_a(\Omega_1)$ and $B_2 = L^2_a(\Omega_2)$ (here, we use area measure on the Riemann sphere rather than the complex plane to define these Bergman spaces), we can then put $B^{(1)} = L^2_a(\Omega_1 \cap V_2)$ and $B^{(2)} = L^2_a(\Omega_2 \cap V_1)$. These spaces split as direct summands, and so do the corresponding multiplier spaces, as in this case the multiplier space is H^∞ on the respective domain.

Note that by the holomorphic functional calculus, Proposition 1.5(b), we know that

$$B_1 \subset \mathcal{O}(\Omega_1) \subset \mathcal{O}(\overline{V}_1) \subset \mathcal{O}(\overline{\Omega_2 \cap V_1}) \subset B^{(2)}.$$

On the other hand, $B_2^0 \subset B^{(2)}$, hence $B \subset B^{(2)}$. Similar argument shows that $B \subset B^{(1)}$.

In the sequel we shall need a factorization theorem for a function $f \in B$. To do this, we have to recall the notion of zero set of a function $f \in \mathcal{O}(\Omega)$ where Ω stands for a planar domain. Define

$$Z_\Omega(f) = \{z \in \Omega : f(z) = 0\}, \quad f \in \mathcal{O}(\Omega).$$

The following lemma has an important role in the factorization procedure.

Lemma 3.7. *Assume that B satisfies the shrinking domain condition. Let f be a non-identically vanishing function in B. Then for every set E, $\overline{V}_2 \subset E \subset \mathbb{C} \setminus \overline{V}_1$, there exist $f_1 \in B_1$ and $f_2 \in B_2$ such that $f = f_1 \cdot f_2$, furthermore $Z_{\Omega_1}(f_1) = Z_\Omega(f) \cap E$, and $Z_{\Omega_2}(f_2) = Z_\Omega(f) \setminus E$.*

Proof: First of all we note that $Z_\Omega(f)$ is a countable subset of Ω without any limit point, since f is not identically zero. In view of Weierstrass theorem (see [**30**, p. 303]), there exist $\varphi_1 \in \mathcal{O}(\Omega_1)$ and $\varphi_2 \in \mathcal{O}(\Omega_2 \cup \{\infty\})$ such that $Z_{\Omega_1}(\varphi_1) = Z_\Omega(f) \cap E$ and $Z_{\Omega_2}(\varphi_2) = Z_\Omega(f) \setminus E$, moreover, the zeros of $\varphi_1 \cdot \varphi_2$ have the same multiplicities as those of f. Hence $f/(\varphi_1 \cdot \varphi_2) \in \big(\mathcal{O}(\Omega)\big)^{-1}$, the set of all invertible elements of $\mathcal{O}(\Omega)$. Let U_1 and U_2 be two finitely connected smoothly bordered domains satisfying the following inclusions, respectively,

$$\overline{V}_1 \subset U_1 \subset\subset \Omega_1, \quad \overline{V}_2 \subset U_2 \subset \overline{U}_2 \subset \Omega_2,$$

and such that $U_1 \cup U_2 = \mathbb{C}$ (note that V_1 and V_2 have already been fixed before Definition 3.5). Now, we see that $U = U_1 \cap U_2$ is a finitely

connected subdomain of Ω having $\partial U = \partial U_1 \cup \partial U_2$ as its boundary. Recalling the projections P_1 and P_2 as introduced at the begining of section 3, we notice that P_1 and P_2 extend continuously to projections $\mathcal{O}(U) \to \mathcal{O}(U_1)$ and $\mathcal{O}(U) \to \mathcal{O}_0(U_2)$, respectively, adding up to identity. Since U is a finitely connected smoothly bordered domain, we can find rational functions $h_1 \in \big(\mathcal{O}(U_1)\big)^{-1}$ and $h_2 \in \big(\mathcal{O}(U_2 \cup \{\infty\})\big)^{-1}$ such that the winding number of $f/(\varphi_1\varphi_2 h_1 h_2)$ as an element of $(\mathcal{O}(U))^{-1}$ is zero around each component of $\mathbb{C}\setminus U$. This allows us to write $g = \log(f/(\varphi_1\varphi_2 h_1 h_2)) \in \mathcal{O}(U)$, therefore we have the following factorization of $f \in \mathcal{O}(U)$;

$$\begin{aligned} f = \varphi_1\varphi_2 h_1 h_2 \exp(g) &= \varphi_1\varphi_2 h_1 h_2 \exp(P_1 g + P_2 g) \\ &= \varphi_1 h_1 \exp(P_1 g) \cdot \varphi_2 h_2 \exp(P_2 g) \\ &= f_1 \cdot f_2 \end{aligned}$$

where $f_1 \in \mathcal{O}(U_1)$ and $f_2 \in \mathcal{O}(U_2 \cup \{\infty\})$. Since f_2 has no zeros in $\Omega_1 \setminus U_1$, we have in fact $f_1 = f/f_2 \in \mathcal{O}(\Omega_1)$. To see that $f_2 = f/f_1 \in \mathcal{O}(\Omega_2)$, we note that f_1 has no zeros in $\Omega_2 \setminus E$, and that f and f_1 have the same zeros with the same multiplicities in $\Omega_2 \cap E$. Moreover,

$$Z_{\Omega_1}(f_1) = Z_\Omega(\varphi_1) = Z_\Omega(f) \cap E, \quad \text{and} \quad Z_{\Omega_2}(f_2) = Z_{\Omega_2}(\varphi_2) = Z_\Omega(f) \setminus E.$$

What therefore has to be shown is $f_1 \in B_1$ and $f_2 \in B_2$. By our choice of E, it is clear that $f_1 \in (\mathcal{O}(\overline{V}_1))^{-1}$, and $f_2 \in (\mathcal{O}(\overline{V}_2))^{-1}$, thus, in view of Proposition 1.5(b), we obtain $f_1 \in (M(B^{(2)}))^{-1}$ and $f_2 \in (M(B^{(1)}))^{-1}$. Taking into account the fact that $f \in B \subset B^{(1)}$, we get

$$f_1 = \frac{1}{f_2} \cdot f \in B^{(1)} \cap \mathcal{O}(\Omega_1) = B_1.$$

A similar argument reveals that $f_2 \in B_2$. ■

The next lemma which has spectral theory flavour constitutes a half of ingredients of a proposition regarding the factorization of invariant subspaces of B with index one which in turn will prove to be useful.

Lemma 3.8. *Let J be any multiplier invariant subspace of B of index 1 and let E be the set defined in the formulation of the previous lemma. Then there exist two multiplier invariant subspaces J_1 and J_2 of B such that the following hold:*

(a) $J = J_1 \cap J_2$,
(b) $\sigma(M_z[B/J_1]) = \sigma(M_z[B/J]) \cap E$,
(c) $\sigma(M_z[B/J_2]) = \sigma(M_z[B/J]) \setminus E$.

Proof: To simplify the notation, let us temporarily denote by T the operator $M_z[B/J]$ defined earlier on B/J. We first assume that J is an invariant subspace of B with index one. What we really need is to decompose T into summands with small spectra, which can be accomplished in the framework of classical holomorphic functional calculus. Recall that $\mathcal{O}(\sigma(T))$ denotes the algebra of all analytic functions on some neighbourhood of the compact set $\sigma(T)$ and let $U \supset \sigma(T)$ be an open (not neccessarily connected) set whose boundary ∂U consists of a finite number of rectifiable Jordan curves oriented in the positive direction as customary in the theory of complex variables. Suppose $U \cup \partial U$ is contained in the domain of analyticity of a given $f \in \mathcal{O}(\sigma(T))$. Associate to f the following operator on B/J:

$$f(T) = \frac{1}{2\pi i} \int_{\partial U} f(\lambda)(\lambda - T)^{-1}\, d\lambda.$$

Since J has index one, it follows from ([**27**, Theorem 4.5]) that the spectrum of the operator $M_z[B/J]$ is discrete in Ω. Consider now the isolated portions $\sigma_1 = \sigma(T) \cap E$ and $\sigma_2 = \sigma(T) \setminus E$ of the compact set $\sigma(T)$. Let e_1 be a function in $\mathcal{O}(\sigma(T))$ which is equal to 1 near σ_1 and equal to 0 near the other portion σ_2 and define $Q_1 = e_1(T)$. Analogously one can consider the function e_2, which is equal to 1 near σ_2 and equal to 0 near σ_1, to obtain $Q_2 = e_2(T)$. It is well-known that Q_1 and Q_2 are projections, simply because of $e_1^2 = e_1$, and $e_2^2 = e_2$. Moreover, since in a neighbourhood of $\sigma(T)$ we have $e_1 + e_2 = 1$ and $e_1 \cdot e_2 = e_2 \cdot e_1 = 0$, it follows that

$$Q_1 + Q_2 = \text{identity}, \quad \text{and} \quad Q_1 Q_2 = Q_2 Q_1 = 0.$$

These equalities imply that

$$B/J = \operatorname{im} Q_1 \oplus \operatorname{im} Q_2, \quad \operatorname{im} Q_1 = \ker Q_2, \quad \text{and} \quad \operatorname{im} Q_2 = \ker Q_1,$$

where $\operatorname{im} Q_j$ stands for the image of B/J under Q_j for $j = 1, 2$. For the direct sum, we note that if $x \in \operatorname{im} Q_1 \cap \operatorname{im} Q_2$, then we have $Q_1(y) = x = Q_2(z)$ for some y and z in B/J. Thus $y = Q_1^2(y) = Q_1 Q_2(z) = 0$, and hence $x = Q_1(y) = Q_1(0) = 0$. Moreover, $\operatorname{im} Q_1$ and $\operatorname{im} Q_2$ are two closed invariant subspaces of B/J with the property $\sigma(T|_{\operatorname{im} Q_j}) = \sigma_j$, for $j = 1, 2$ [**26**, p. 84]. Put

$$J_2 = \{f \in B : f + J \in \operatorname{im} Q_1\} \supset J.$$

It is obvious that J_2 is a closed invariant subspace of B, and that $\operatorname{im} Q_1 = J_2/J$. The closed invariant subspace J_1 of B can be found in an analogous manner:

$$J_1 = \{f \in B : f + J \in \operatorname{im} Q_2\}.$$

From the above the inclusion $J \subset J_1 \cap J_2$ is obvious. For the reverse inclusion, suppose that f is an element of the intersection of J_1 and J_2. Then $f + J \in \operatorname{im} Q_1 \cap \operatorname{im} Q_2 = \{0\}$, or equivalently $f \in J$. So far we have proved (a). To prove the other two statements of the lemma we first notice that B/J_1 is isomorphic to $\frac{B/J}{J_1/J}$, and secondly that because of $B/J = J_1/J \oplus J_2/J$ we also have $\frac{B/J}{J_1/J} \cong J_2/J$. Hence we have $B/J_1 \cong J_2/J$. Now define $T_1 : B/J_1 \to B/J_1$ by $f + J_1 \mapsto zf + J_1$, and $T_2 : J_2/J \to J_2/J$ sending $f + J$ to $zf + J$. Since $T_2 = T|_{J_2/J}$ and $J_2/J = \operatorname{im} Q_1$, we get $\sigma(T_2) = \sigma_1$. But, we also have $\sigma(T_1) = \sigma(T_2)$, because of the isomorphism

$$\Psi : J_2/J \to B/J_1$$

which takes $f + J$ to $f + J_1$, for $f \in B$. Hence

$$\Psi(T_2(f + J)) = \Psi(zf + J) = zf + J_1 = T_1(f + J_1)$$
$$= T_1(\Psi(f + J)).$$

Therefore $\Psi T_2 = T_1 \Psi$ or equivalently $T_2 = \Psi^{-1} T_1 \Psi$. Similarly, we obtain $\lambda - T_2 = \Psi^{-1}(\lambda - T_1)\Psi$, for every $\lambda \in \mathbb{C}$. This means that $\lambda - T_1$ is invertible if and only if $\lambda - T_2$ is invertible, therefore $\sigma(T_1) = \sigma(T_2)$ and finally

$$\sigma(M_z[B/J_1]) = \sigma(M_z[J_2/J]) = \sigma_1,$$

which proves part (b). Similarly one can prove part (c). So far, we have proved the lemma with the words "multiplier invariant" replaced by "invariant". Now let J be a multiplier invariant subspace of B of index one. We want to show that J_1 and J_2 are also multiplier invariant subspaces. Suppose that $f \in J_1$, then $f + J \in \operatorname{im} Q_2 = \ker Q_1$. Assume that $\varphi \in M(B)$, and that γ_2 is a curve around σ_2 oriented in the positive direction. We can regard the operator $M_\varphi \in \mathcal{L}(B/J)$ both as left multiplication by φ, and as right multiplication by φ. Since the operator $M_\varphi = M_\varphi[B/J]$ commutes with $(\lambda - T)^{-1}$, it follows that

$$\begin{aligned} Q_1 M_\varphi &= \left(\frac{1}{2\pi i} \int_{\gamma_2} (\lambda - T)^{-1} \, d\lambda \right) M_\varphi \\ &= M_\varphi \left(\frac{1}{2\pi i} \int_{\gamma_2} (\lambda - T)^{-1} \, d\lambda \right) \\ &= M_\varphi Q_1. \end{aligned}$$

Hence

$$Q_1(\varphi f + J) = Q_1 M_\varphi[B/J](f + J) = M_\varphi[B/J] Q_1(f + J) = M_\varphi[B/J](0) = 0,$$

since $Q_1(f + J) = 0$. This shows that $\varphi f \in J_1$, that is, J_1 is a multiplier invariant subspace. Similar reasoning shows that J_2 is also multiplier invariant. ■

Lemma 3.9. *Let B be a Banach space of analytic functions on Ω satisfying the axioms (0-1)-(0-4), and let J be an invariant subspace of B. If $g \in \mathcal{O}(\overline{\Omega})$, then*

$$g(M_z[B/J]) = M_g[B/J],$$

where the right hand side is the operator of multiplication by g on the quotient space B/J.

Proof: We first note that by Proposition 1.5(b), $g \in M(B)$ and hence the operator $M_g[B/J]$ is well-defined. Now, let U be a finitely connected smoothly bordered domain containing $\overline{\Omega}$ and such that $\overline{U}$ lies in the domain of analyticity of g. Assuming that ∂U is oriented in the positive direction, and that $h \in B$, we have

$$\begin{aligned} g\big(M_z[B/J]\big)\big(h(z)+J\big) &= \frac{1}{2\pi i}\int_{\partial U} g(\lambda)\big(\lambda - M_z[B/J]\big)^{-1}\big(h(z)+J\big)\,d\lambda \\ &= \frac{1}{2\pi i}\int_{\partial U}\left(\frac{g(\lambda)h(z)}{\lambda - z} + J\right)d\lambda \\ &= h(z)\left(\frac{1}{2\pi i}\int_{\partial U}\frac{g(\lambda)}{\lambda - z}\,d\lambda + J\right) \\ &= h(z)\big(g(z)+J\big) \\ &= M_g[B/J]\big(h(z)+J\big). \end{aligned}$$

Note that for $\lambda \in \partial U$, we have $h(z)/(\lambda - z) \in B$, by the axiom (0-3). ■

Proposition 3.10. *Assume that B satisfies the shrinking domain condition. Let J be a multiplier invariant subspace of B having index one, and let J_1 and J_2 be as in Lemma 3.8. If $\mathcal{O}(\overline{\Omega}_1)$ and $\mathcal{O}(\overline{\Omega}_2)$ are dense in B_1 and B_2, respectively, then each $f \in J$ has a factorization $f = f_1 \cdot f_2$, where $f_1 \in J_1 \cap B_1$ and $f_2 \in J_2 \cap B_2$.*

Proof: Choose the functions f_1 and f_2 as in Lemma 3.7. What remains to be proved is $f_1 \in J_1$ and $f_2 \in J_2$. Since the proofs are similar, we shall just verify that $f_1 \in J_1$. By assumption, for $f_2 \in B_2$ we can find a sequence of functions $g_k \in \mathcal{O}(\overline{\Omega}_2)$ converging to f_2 in B_2. We recall that B satisfies the shrinking domain condition, so we have $B_2 = B^{(2)} \cap \mathcal{O}(\Omega_2)$. Consequently, g_k tends to f_2 in $\mathcal{O}(\Omega_2) \subset \mathcal{O}(\sigma_1)$, where σ_1 is the spectrum of $M_z[B/J_1]$ as in Lemma 3.8. We know that if the space $\mathcal{O}(\Omega_2)$ is

equipped with the uniform convergence topology, then the holomorphic functional calculus is continuous [**31**, p. 244]. Therefore

$$g_k(M_z[B/J_1]) \to f_2(M_z[B/J_1]), \quad \text{as} \quad k \to \infty.$$

Considering the fact proved in Lemma 3.9 that $g_k(M_z[B/J_1]) = M_{g_k}[B/J_1]$, we get

$$M_{g_k}[B/J_1](f_1 + J_1) \to f_2(M_z[B/J_1])(f_1 + J_1), \quad \text{as} \quad k \to \infty.$$

We claim that $g_k f_1 \to f$ in the norm of B. Putting off the proof of this claim to the end, we get

$$f_2(M_z[B/J_1])(f_1 + J_1) = (\lim_{k\to\infty} g_k f_1) + J_1 = f + J_1 = 0 + J_1, \tag{3-7}$$

because $f \in J \subset J_1$. From the construction of f_2 we know that $Z_{\Omega_2}(f_2) = Z_\Omega(f) \setminus E$, (see Lemma 3.7); thus f_2 has no zeros in a neighbourhood of σ_1. This means that the operator $f_2(M_z[B/J_1])$ is invertible and furthermore, its inverse in $\mathcal{L}(B/J_1)$ is $(1/f_2)(M_z[B/J_1])$ [**31**, p. 244]. Therefore from the equality (3-7) we have

$$f_1 + J_1 = (1/f_2)(M_z[B/J_1])(0 + J_1) = 0 + J_1,$$

or equivalently $f_1 \in J_1$. We now turn to prove our postponed claim. We want to show that $\|g_k f_1 - f\|_B = \|f_1(g_k - f_2)\|_B \to 0$, as $k \to \infty$. To see this, we first notice that

$$\|f\|_B \asymp \|f\|_{B^{(1)}} + \|f\|_{B^{(2)}},$$

where $B^{(1)}$ and $B^{(2)}$ are as in Definition 3.5. Here, the notation $\asymp$ is used to indicate that the two norms are equivalent.This estimate comes from the direct sums involved in B, $B^{(1)}$ and $B^{(2)}$ (see Definition 3.5) together with the closed graph theorem. It is also immediate from the definition of norm in the space of multipliers that

$$\|f\|_{B^{(1)}} \le \|f_2\|_{M(B^{(1)})} \cdot \|f_1\|_{B^{(1)}},$$

and

$$\|f\|_{B^{(2)}} \le \|f_1\|_{M(B^{(2)})} \cdot \|f_2\|_{B(2)}.$$

Since

$$\|f_1(g_k - f_2)\|_B \asymp \|f_1(g_k - f_2)\|_{B^{(1)}} + \|f_1(g_k - f_2)\|_{B^{(2)}},$$

we obtain

$$\|f_1(g_k - f_2)\|_B \leq C\Big(\|f_1\|_{B^{(1)}} \cdot \|g_k - f_2\|_{M(B^{(1)})} + \|g_k - f_2\|_{B^{(2)}} \cdot \|f_1\|_{M(B^{(2)})}\Big),$$

where C is some constant. We know that $g_k \to f_2$ in the space $\mathcal{O}(\Omega_2)$, so $\|g_k - f_2\|_{M(B^{(1)})} \to 0$, since $\mathcal{O}(\Omega_2) \subset M(B^{(1)})$ by the holomorphic functional calculus. Also, $g_k \to f_2$ in the norm of B_2, and so that we get $\|g_k - f_2\|_{B^{(2)}} \to 0$. As a result, $\|f_1(g_k - f_2)\|_B \to 0$ as $k \to \infty$, which completes the proof of the proposition. ■

For an $M(B_1)$-invariant subspace I, we write

$$\Lambda(I) = \operatorname{clos}(I \cdot M(B))$$

for the smallest multiplier invariant subspace of B containing I.

Theorem 3.11. *Assume that $B = B_1 \oplus B_2^0$ is a Banach space of analytic functions on Ω which contains 1, satisfies the axioms (0-1)-(0-4), and the shrinking domain condition. Assume further that $\mathcal{O}(\overline{\Omega}_1)$ and $\mathcal{O}(\overline{\Omega}_2)$ are dense in B_1 and B_2, respectively. Then*

(a) *The mapping $I \mapsto \Lambda(I)$ is a bijection from the set of all $M(B_1)$-invariant subspaces I with $\sigma(M_z[B_1/I]) \subset \Omega_2$ onto the set of all $M(B)$-invariant subspaces J with $\sigma(M_z[B/J]) \subset \Omega_2$. Also*

$$\sigma(M_z[B/\Lambda(I)]) = \sigma(M_z[B_1/I]),$$

and the inverse mapping is given by $J \mapsto J \cap B_1$.

(b) *The mapping L_I is a surjective module-homomorphism $B \to B_1/I$ which is canonical on B_1 and has kernel $\Lambda(I)$, for all $M(B_1)$-invariant subspaces I with*

$$\sigma(M_z[B_1/I]) \subset \Omega_2.$$

(c) *The quotient Banach spaces B_1/I and $B/\Lambda(I)$ are canonically module-isomorphic for all $M(B_1)$-invariant subspaces I with*

$$\sigma(M_z[B_1/I]) \subset \Omega_2.$$

Proof: We start by proving (b); that L_I is a surjective module-homomorphism which is also canonical on B_1 follows from Proposition 3.3. What therefore has to be proved is that $\ker L_I = \Lambda(I)$. By Proposition 3.4(a), we have already proved that $\Lambda(I) \subset \ker L_I$. Writing $\ker L_I = J$, we have from Proposition 3.4(b)

$$J \cap B_1 = (\ker L_I) \cap B_1 = I.$$

This equality makes the mapping $I \mapsto \ker L_I$ injective. Now, we want to show that $J \subset \Lambda(I)$. We intend to make use of our assumption that $\mathcal{O}(\overline{\Omega}_2)$ is dense in B_2. By Proposition 3.4(c), we know that

$$\sigma(M_z[B/J]) = \sigma(M_z[B_1/I]) \subset \Omega_2,$$

thus Corollary 2.4 implies that the invariant subspace J has index one. This allows us to apply Proposition 3.10 to J. According to Proposition 3.10, we know that each element f of the multiplier invariant subspace J has a product decomposition

$$f = f_1 \cdot f_2, \quad f_1 \in J_1 \cap B_1, \quad f_2 \in J_2 \cap B_2,$$

where J_1, J_2 are multiplier invariant subspaces of B, $J_1 \cap J_2 = J$, $\sigma_{J_1} \cap \sigma_{J_2} = \emptyset$, $\sigma_{J_1} \cup \sigma_{J_2} = \sigma_J$, and $\sigma_{J_1} \subset \Omega_2$, $\sigma_{J_2} \subset \Omega_1$. We may assume that $J_1 = J$ and $J_2 = B_2$, because we have made the assumption that the spectrum of the operator $M_z[B/J]$ lies in $\Omega \cup \partial\Omega_1$ so that the portion σ_2 of this spectrum in the situation of Proposition 3.10 consists of finitely many points, hence Proposition 2.5 implies that $J_2 = pB$ for some polynomial p with zeros in σ_{J_2}. Write $f_2 = pg$ for some $g \in B$, and note that p devides any element of J_2 (see Proposition 2.5), so that in fact we have $g \in B_2$. Therefore $f = (f_1 \cdot p) \cdot g$, where $f_1 \cdot p \in J_1 \cap J_2 = J$, and $g \in B_2$. But $f_1 \cdot p \in B_1$ so that $f_1 \cdot p \in J \cap B_1 = I$, by Proposition 3.4(b), proving that

$$J = (J \cap B_1) \cdot B_2 = I \cdot B_2.$$

We now turn to prove that $J \subset \Lambda(I)$. For $f \in J$, we have $f = f_1 \cdot f_2$ where $f_1 \in J \cap B_1 = I$, and $f_2 \in B_2$. Since, by assumption, $\mathcal{O}(\overline{\Omega}_2)$ is dense in B_2, it follows that f_2 can be approximated in B_2 by a sequence of elements of this space, say $f_{2,j}$. To show that $J \subset \Lambda(I)$, it suffices to verify that

$$\|f - f_1 f_{2,j}\|_B = \|f_1(f_2 - f_{2,j})\|_B \to 0, \quad \text{as} \quad j \to \infty,$$

because

$$f_1 f_{2,j} \in (J \cap B_1) \cdot M(B_2) = I \cdot M(B_2) \subset I \cdot M(B).$$

As in the proof of Proposition 3.10, we have

$$\|f_1(f_2 - f_{2,j})\|_B \asymp \|f_1(f_2 - f_{2,j})\|_{B^{(1)}} + \|f_1(f_2 - f_{2,j})\|_{B^{(2)}},$$

and therefore

$$\begin{aligned}&\|f_1(f_2 - f_{2,j})\|_B \\ &\quad \le C\Big(\|f_1\|_{B^{(1)}} \cdot \|f_2 - f_{2,j}\|_{M(B^{(1)})} + \|f_2 - f_{2,j}\|_{B^{(2)}} \cdot \|f_1\|_{M(B^{(2)})}\Big),\end{aligned}$$

where C is some constant. Since $f_{2,j} \to f_2$ in the space $\mathcal{O}(\Omega_2)$, we obtain $\|f_2 - f_{2,j}\|_{M(B^{(1)})} \to 0$, because $\mathcal{O}(\Omega_2) \subset M(B^{(1)})$ by the holomorphic functional calculus, and finally $\|f_2 - f_{2,j}\|_{B^{(2)}} \to 0$ by our assumption. Therefore we obtain $\|f_1(f_2 - f_{2,j})\|_B \to 0$, as $j \to \infty$ completing the proof of part (b).

To prove part (c), we note that for an $M(B_1)$-invariant subspace I with condition $\sigma(M_z[B_1/I]) \subset \Omega_2$, the surjective module-homomorphism $L_I : B \to B_1/I$ induces an isomorphism

$$\bar{L_I} : B/\ker L_I = B/\Lambda(I) \longrightarrow B_1/I.$$

Since $L_I(f) = f + I$ for $f \in B_1$, it follows that $(\bar{L_I})^{-1} : B_1/I \to B/\Lambda(I)$ has to coincide with the canonical homomorphism $B_1/I \to B/\Lambda(I)$. Note that for $f \in B_1$ we have $\bar{L_I}(f + \ker L_I) = L_I(f)$ if and only if $(\bar{L_I})^{-1}(L_I(f)) = f + \ker L_I$, or

$$(\bar{L_I})^{-1}(f + I) = f + \ker L_I = f + \Lambda(I), \quad f \in B_1.$$

Concerning the isomorphism $\bar{L_I}$, it should be noted that for every $f \in B$ and every $\varphi \in M(B)$ we have

$$\begin{aligned}\bar{L_I}(\varphi f + J) = L_I(\varphi f) &= L_{I,M}(\varphi) L_I(f) \\ &= L_{I,M}(\varphi) \bar{L_I}(f + J).\end{aligned}$$

This together with the fact that $M(B)$ acts on B/J according to

$$\varphi \cdot (f + J) = \varphi f + J, \quad \varphi \in M(B) \text{ and } f \in B,$$

suggest that we might call $\bar{L_I}$ a multiplier module-isomorphism, that is, a surjective and injective multiplier module-homomorphism.

Regarding part (a), note that we have already shown, in part (b) above, that for any $M(B_1)$-invariant subspace I with $\sigma(M_z[B_1/I]) \subset \Omega_2$ we have

$$\ker L_I = \Lambda(I).$$

This together with Proposition 3.4(b) gives

$$\Lambda(I) \cap B_1 = (\ker L_I) \cap B_1 = I,$$

which in turn proves that the mapping $I \mapsto \Lambda(I)$ is injective. To prove the surjectivity of this mapping, let J be an arbitrary $M(B)$-invariant subspace with $\sigma(M_z[B/J]) \subset \Omega_2$. As above, we may again assume that $J = J_1$ and $J_2 = B_2$ where J_1 and J_2 are as in Proposition 3.10. Therefore we have $J = (J \cap B_1) \cdot B_2$. Let $f \in J$ be written as $f = f_1 \cdot f_2$ where $f_1 \in J \cap B_1$ and $f_2 \in B_2$. By our assumption, we can approximate $f_2 \in B_2$ by a sequence of functions in $\mathcal{O}(\overline{\Omega}_2)$. As in part (b), it turns out that

$$f \in \operatorname{clos}\{(J \cap B_1) \cdot M(B_2)\},$$

and hence

$$J \subset \operatorname{clos}\{(J \cap B_1) \cdot M(B_2)\} \subset \operatorname{clos}\{(J \cap B_1) \cdot M(B)\}.$$

This together with the obvious fact that $J \supset \operatorname{clos}\{(J \cap B_1) \cdot M(B)\}$ prove that we have in fact the equality

$$J = \operatorname{clos}\{(J \cap B_1) \cdot M(B)\} = \Lambda(J \cap B_1),$$

which in turn proves the surjectivity of Λ. Finally, we recall that the equality of the two spectra was shown in Proposition 3.4(c). Now, the proof of our theorem is complete. ■

Remark 3.12. Theorem 3.11 supplies a one-to-one correspondence between the multiplier invariant subspaces J of B with $\sigma(M_z[B/J]) \subset \Omega_2$ and the multiplier invariant subspaces I of B_1 with $\sigma(M_z[B_1/I]) \subset \Omega_2$. What can we do when we wish to describe the structure of all multiplier invariant subspaces of index one in B in terms of corresponding lattices of multiplier invariant subspaces in B_1 and B_2, respectively? We note first that by a suitable Möbius transformation, we may interchange the roles of B_1 and B_2. This means that under appropriate assumptions on B, B_1, and B_2, Theorem 3.11 also provides a one-to-one correspondence between the multiplier invariant subspaces J of B with $\sigma(M_z[B/J]) \subset \Omega_1$ and the multiplier invariant subspaces I of B_2 with $\sigma(M_z[B_2/I]) \subset \Omega_1$. A general multiplier invariant subspace J of B of index one splits into $J = J_1 \cap J_2$, where J_1 and J_2 are two other multiplier invariant subspaces of B such that the spectra associated to the operators $M_z[B/J_1]$ and $M_z[B/J_2]$ are disjoint parts of the spectrum $\sigma(M_z[B/J])$ in accordance with Lemma 3.8. By Theorem 3.11 (applied to J_1 and J_2, respectively), J_1 corresponds to $I_1 = J_1 \cap B_1$ and J_2 corresponds to $I_2 = J_2 \cap B_2$.

So, to describe J, we essentially just need to understand I_1 and I_2. Moreover, by Proposition 3.10, each element $f \in J$ has a factorization $f = f_1 \cdot f_2$, where $f_1 \in I_1$ and $f_2 \in I_2$. It follows that I_1 takes care of the boundary behavior of the functions in J near $\partial\Omega_1$, and that I_2 takes care of the boundary behavior of the functions in J near $\partial\Omega_2$. It should be pointed out that all known examples of multiplier invariant subspaces of index one involve boundary behavior (if we omit the rather trivial ones generated by polynomials, corresponding to finitely many zeros).

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