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ON RADIAL LIMIT FUNCTIONS FOR ENTIRE SOLUTIONS OF SECOND ORDER ELLIPTIC EQUATIONS IN $\mathbf{R}^2$

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Abstract

Given a homogeneous elliptic partial differential operator L of order two with constant complex coefficients in $\mathbf{R}^2$, we consider entire solutions of the equation $Lu = 0$ for which

$$\lim_{r\to\infty} u(re^{i\varphi}) =: U(e^{i\varphi})$$

exists for all $\varphi \in [0, 2\pi)$ as a finite limit in $\mathbf{C}$. We characterize the possible "radial limit functions" U. This is an analog of the work of A. Roth for entire holomorphic functions. The results seem new even for harmonic functions.

1. Introduction and Main Results

Let

$$Lv = c_{11}v_{x_1x_1} + 2c_{12}v_{x_1x_2} + c_{22}v_{x_2x_2}$$

be an homogeneous partial differential operator of order two with constant complex coefficients in $\mathbf{R}^2$ satisfying the ellipticity condition

$$c_{11}\xi_1^2 + 2c_{12}\xi_1\xi_2 + c_{22}\xi_2^2 \neq 0$$

for all $(\xi_1, \xi_2) \neq (0, 0)$, ξ_1, $\xi_2 \in \mathbf{R}$.

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Let λ_1, λ_2 be the (complex) roots of the characteristic equation $c_{11}\lambda^2 + 2c_{12}\lambda + c_{22} = 0$. It follows from the ellipticity condition that $\lambda_1, \lambda_2 \notin \mathbf{R}$. We define

$$\partial_1 = \frac{\partial}{\partial x_1} - \lambda_1 \frac{\partial}{\partial x_2}, \quad \partial_2 = \frac{\partial}{\partial x_1} - \lambda_2 \frac{\partial}{\partial x_2} \quad \text{if } \lambda_1 \neq \lambda_2,$$

or

$$\partial_1 = \frac{\partial}{\partial x_1} - \lambda_1 \frac{\partial}{\partial x_2}, \quad \partial_2 = \frac{\partial}{\partial x_1} + \lambda_1 \frac{\partial}{\partial x_2} \quad \text{if } \lambda_1 = \lambda_2.$$

We then have the following decomposition of L:

$$Lv = \begin{cases} c_{11}\partial_1(\partial_2(v)), & \text{if } \lambda_1 \neq \lambda_2; \\ c_{11}\partial_1^2(v), & \text{if } \lambda_1 = \lambda_2. \end{cases}$$

We also introduce the following new coordinates:

$$z_1 = \frac{\lambda_2}{\lambda_2 - \lambda_1}\left(x_1 + \frac{1}{\lambda_2}x_2\right), \quad z_2 = \frac{\lambda_1}{\lambda_1 - \lambda_2}\left(x_1 + \frac{1}{\lambda_1}x_2\right) \quad \text{if } \lambda_1 \neq \lambda_2;$$

or

$$z_1 = \frac{1}{2}\left(x_1 - \frac{1}{\lambda_1}x_2\right), \qquad z_2 = \frac{1}{2}\left(x_1 + \frac{1}{\lambda_1}x_2\right) \qquad \text{if } \lambda_1 = \lambda_2.$$

The following "orthogonality" relations then are easily obtained:

$$\begin{aligned} \partial_1 z_1 = 1 \quad \partial_1 z_2 = 0 \\ \partial_2 z_1 = 0 \quad \partial_2 z_2 = 1. \end{aligned} \tag{1}$$

Finally, we identify $z = x_1 + ix_2$ in $\mathbf{C}$ and $x = (x_1, x_2)$ in $\mathbf{R}^2$ and, for $s = 1$ and 2, we define $T_s(z) = z_s$ (which are linear nondegenerate transformations of $\mathbf{R}^2$).

For any set E in $\mathbf{R}^2$, denote by $L(E)$ the family of all functions v, each defined on its own neighbourhood Ω_v of E, such that $Lv = 0$ in Ω_v in the classical sense. We note that for E open, one can take $\Omega_v = E$ for all v. Functions in $L(E)$ and $L(\mathbf{R}^2)$ are called L-*analytic* on E and L-*entire* respectively.

It is well known that (for E open) each function $v \in L(E)$ is real-analytic on E, and that each continuous function v satisfying $Lv = 0$ on E in the distributional sense is in $L(E)$. From these facts, using (1), one can prove the following well known result [**1**, Chapter IV, §6, (4.77)] (see also [**5**] for a simple direct proof).

Proposition 1. *Let D be any domain in $\mathbf{C}$ and L be as above.*

1. *If D is simply connected and if $\lambda_1 \neq \lambda_2$, then*
   1a) $v \in L(D)$ *if and only if there exist f_1 holomorphic in $T_1(D)$ and f_2 holomorphic in $T_2(D)$ such that*
   $$v(z) = f_1(T_1(z)) + f_2(T_2(z)) = f_1(z_1) + f_2(z_2)$$
   for all $z \in D$. In particular, L-entire functions u are of the form $u(z) = f_1(z_1) + f_2(z_2)$ where f_1, f_2 are entire holomorphic functions.

   1b) There exist in $\mathbf{C} \setminus \{0\}$ a fixed analytic branch $\log(z_1 z_2^{\nu})$ of the multivalued function $\mathrm{Log}(z_1 z_2^{\nu})$ and a nonzero complex constant C_L depending only on L such that
   $$\Phi_L(z) = C_L \log(z_1 z_2^{\nu})$$
   is a fundamental solution of L, where $\nu = 1$ if $\mathrm{sgn}(\mathrm{Im}\,\lambda_1) \neq \mathrm{sgn}(\mathrm{Im}\,\lambda_2)$, and $\nu = -1$ otherwise.

2. *If $\lambda_1 = \lambda_2$, then*
   2a) $v \in L(D)$ if and only if there exist g_1 and g_2 holomorphic in $T_2(D)$ such that
   $$v(z) = T_1(z) g_1(T_2(z)) + g_2(T_2(z)) = z_1 g_1(z_2) + g_2(z_2)$$
   for all $z \in D$. In particular, L-entire functions u are of the form $u(z) = z_1 g_1(z_2) + g_2(z_2)$ where g_1, g_2 are entire holomorphic functions.

   2b) $\Phi_L(z) = C_L \frac{z_1}{z_2}$ is a fundamental solution of L, where C_L is a nonzero complex constant depending only on L.

3. *If $\{v_n\} \subset L(D)$ and $\{v_n\}$ converges uniformly to v on compact subsets of D as $n \longrightarrow \infty$, then $v \in L(D)$.*

We just note that 1b) and 2b) follow from 1a) and 2a) respectively, and from the definition of fundamental solution. It is not difficult to check that if $\mathrm{sgn}(\mathrm{Im}\,\lambda_1) \neq \mathrm{sgn}(\mathrm{Im}\,\lambda_2)$ (respectively $\mathrm{sgn}(\mathrm{Im}\,\lambda_1) = \mathrm{sgn}(\mathrm{Im}\,\lambda_2)$), then the increment of the polar argument of $(z_1 z_2)$ (respectively (z_1/z_2)) around the origin is zero, and thus some analytic branch of the function $\log(z_1 z_2)$ (respectively $\log(z_1/z_2)$) exists in $\mathbf{R}^2 \setminus \{(0,0)\}$.

Example 1. For the Laplacian $L = \Delta$, one has $\lambda_1 = i$, $\lambda_2 = -i$, $z_1 = z/2$, $z_2 = \bar{z}/2$ and

$$\partial_1 = \frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2} =: 2\frac{\partial}{\partial z},$$

$$\partial_2 = \frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2} =: 2\frac{\partial}{\partial \bar{z}},$$

$$\Phi_\Delta(z) = \frac{1}{4\pi}\log\left(\frac{z\bar{z}}{4}\right).$$

For the Bitsadze operator

$$L = \frac{\partial^2}{\partial \bar{z}^2} = \frac{1}{4}\left(\frac{\partial^2}{\partial x_1^2} + 2i\frac{\partial^2}{\partial x_1 \partial x_2} - \frac{\partial^2}{\partial x_2^2}\right),$$

one gets $\lambda_1 = \lambda_2 = -i$, $z_1 = \bar{z}/2$, $z_2 = z/2$ and

$$\partial_1 = 2\frac{\partial}{\partial \bar{z}}, \quad \partial_2 = 2\frac{\partial}{\partial z}, \quad \Phi_L(z) = \frac{1}{\pi}\frac{\bar{z}}{z}.$$

In order to formulate our main results (Theorems 1 and 2), we need the following characterization of radially constant solutions of the equation $Lv = 0$.

Proposition 2. *Let $J = \{z \in \mathbf{C} : \varphi_1 < \arg z < \varphi_2\}$, $\varphi_1 < \varphi_2 \leq \varphi_1 + 2\pi$ denote an (infinite) open sector with vertex at 0. Let $v \in L(J)$ and assume that $v(z) = v(re^{i\varphi}) = v(e^{i\varphi})$ does not depend on r.*

1. *If $\lambda_1 \neq \lambda_2$, then there exist α, $\beta \in \mathbf{C}$ and a fixed analytic branch $\log(z_1/z_2)$ of $\operatorname{Log}(z_1/z_2)$ in J such that, for $z \in J$,*

$$v(z) = \alpha \log\frac{z_1}{z_2} + \beta$$
$$= \alpha\log\left(\frac{\cos\varphi + \frac{1}{\lambda_2}\sin\varphi}{\cos\varphi + \frac{1}{\lambda_1}\sin\varphi}\right) + \beta =: v_{12}^*(e^{i\varphi}). \tag{2}$$

2. *If $\lambda_1 = \lambda_2$, then there exist α, $\beta \in \mathbf{C}$ such that, for $z \in J$,*

$$v(z) = \alpha\frac{z_1}{z_2} + \beta$$
$$= \alpha\left(\frac{\cos\varphi - \frac{1}{\lambda_1}\sin\varphi}{\cos\varphi + \frac{1}{\lambda_1}\sin\varphi}\right) + \beta =: v_1^*(e^{i\varphi}). \tag{3}$$

   (For this case, $J = \mathbf{C}\setminus\{0\}$ is also allowed.)

Example 2. For $L=\Delta$, one has $v_{12}^*(e^{i\varphi})=\alpha\varphi+\beta$, $\varphi_1<\varphi<\varphi_2$, and for $L=\partial^2/\partial\bar{z}^2$, $v_1(e^{i\varphi})=\alpha e^{-2i\varphi}+\beta$, where α and β are any complex constants.

Theorem 1. *Let u be an entire solution of the equation $Lu=0$ such that*

$$\lim_{r\to+\infty} u(re^{i\varphi}) =: U(e^{i\varphi}) \tag{4}$$

exists and is finite for all $\varphi\in[0,2\pi)$. Then

A) *U is of Baire class 1 on $S=\{e^{i\varphi}:\varphi\in[0,2\pi)\}$; that is, U is a pointwise limit on S of a sequence of continuous functions on S.*

B) *There is an open set $I=\cup_{j=1}^{\infty}I_j$, where the I_j are disjoint open arcs on S (and $I_j=\emptyset$ is possible for some j, but $I_j\neq S$) with the following properties:*

B1) I is everywhere dense on S;

B2) On each I_j, $U(e^{i\varphi})$ is of the form $v_{12}^*(e^{i\varphi})$ if $\lambda_1\neq\lambda_2$ (respectively of the form $v_1^*(e^{i\varphi})$, if $\lambda_1=\lambda_2$), (see (2) and (3));

B3) The limit (4) is uniform on each compact subset of each I_j.

Conversely, let U be a function defined on S and I be an open subset of S with $I=\cup_{j=1}^{\infty}I_j$, where the I_j are disjoint open arcs. If (A), (B1) and (B2) above are satisfied, then there exists an L-entire function u with the properties:

a) *$\lim_{r\to\infty} u(re^{i\varphi})=U(e^{i\varphi})$ for each φ;*

b) *The limit in (a) holds uniformly on each compact subset of I_j for each j.*

Moreover, if U_1 is of Baire class 1 on S and $U_1(e^{i\varphi})=\partial U(e^{i\varphi})/\partial\varphi$ on I, then the function u can be chosen such that (a) and (b) are satisfied and

$$\lim_{r\to+\infty}\frac{\partial u(re^{i\varphi})}{\partial r}=0,\qquad \lim_{r\to+\infty}\frac{\partial u(re^{i\varphi})}{\partial\varphi}=U_1(e^{i\varphi})$$

for all $\varphi\in[0,2\pi)$.

Let K be a compact set in S. Let $RP(K)$ (respectively $RU(K)$) denote the set of all functions g on K for which there exists $u=u_g\in L(\mathbf{R}^2)$ such that $u(re^{i\varphi})\longrightarrow g(e^{i\varphi})$ for each $\varphi\in K$ (respectively $u(re^{i\varphi})\longrightarrow g(e^{i\varphi})$ uniformly on K) as $r\to\infty$.

Theorem 2.

a) *For each compact set K in S, $g \in RP(K)$ if and only if g is of Baire class* 1 *on K and there exists a countable family of disjoint open arcs $\{I_j\}_{j=1}^{\infty}$ in K such that $K \setminus \cup_{j=1}^{\infty} I_j$ is nowhere dense in S and on each I_j, g is of the form $v_{12}^*(e^{i\varphi})$ (when $\lambda_1 \neq \lambda_2$) or $v_1^*(e^{i\varphi})$ (when $\lambda_1 = \lambda_2$) (see Proposition 2). In particular, $RP(K)$ consists of all Baire class* 1 *functions on K if and only if K has an empty interior on S.*

b) *Let K be a compact set in S, $K \neq S$. Then $g \in RU(K)$ if and only if $g \in C(K)$ and g is of the form $v_{12}^*(e^{i\varphi})$ (when $\lambda_1 \neq \lambda_2$) or $v_1^*(e^{i\varphi})$ (when $\lambda_1 = \lambda_2$) in each connected component of the interior of K in S. In particular, $RU(K) = C(K)$ if and only if K is nowhere dense in S. If $K = S$, then $RU(K)$ contains only constant functions.*

2. Proofs

We first establish the following uniqueness theorem for L-analytic functions.

Lemma 1. *Let D be any domain in $\mathbf{C}$ and $v \in L(D)$. If the set $G_v = \{z = x_1 + ix_2 \in D \mid \nabla v(z) := (\partial v(z)/\partial x_1, \partial v(z)/\partial x_2) = (0,0)\}$ has at least one accumulation point inside D, then v is constant in D.*

Proof: From Proposition 1 and equations (1), one has $\partial_1 v = f_1'(z_1)$ for $\lambda_1 \neq \lambda_2$ and $\partial_1 v = g_1(z_2)$ for $\lambda_1 = \lambda_2$, where f_1' and g_1 are holomorphic on $T_1(D)$ and $T_2(D)$ respectively. By assumption, $f_1' = 0$ on $T_1(G_v)$ (respectively $g_1 = 0$ on $T_2(G_v)$). It thus follows from the uniqueness theorem for holomorphic functions that $f_1 \equiv \text{const}$ in $T_1(D)$ (respectively $g_1 \equiv 0$ in $T_2(D)$). An analogous study of $\partial_2 v$ completes the proof of Lemma 1. ■

Proof of Proposition 2: We shall consider only the case $\lambda_1 \neq \lambda_2$, the proof for the case $\lambda_1 = \lambda_2$ being similar. Let $v \in L(J)$, $v = v(e^{i\varphi})$. Let $v_0(z) = \log(z_1/z_2)$ be some fixed analytic branch of $\mathrm{Log}(z_1/z_2)$ in J. Simple calculations show that $\partial v_0(z)/\partial\varphi \neq 0$ and $\partial v_0/\partial r \equiv 0$ in J. Fixing some $\varphi_0 \in (\varphi_1, \varphi_2)$, we can thus find α and β in $\mathbf{C}$ such that $v - \alpha v_0 - \beta = 0$ and $\partial(v - \alpha v_0 - \beta)/\partial\varphi = 0$ on the ray $\{\arg z = \varphi_0\}$. It thus follows that $\nabla(v - \alpha v_0 - \beta) = 0$ on the ray $\{\arg z = \varphi_0\}$. Lemma 1 now gives the desired result. ■

Proof of Theorem 1: The scheme of the proof is analogous to that of A. Roth [**7**] (see also [**3**, Chapter IV, § 5A]). The main new tools are some recent results in approximation theory ([**6**] and [**2**]).

Let $u\in L(\mathbf{R}^2)$ satisfy (4), then A) is a consequence of $\lim_{n\to\infty}u(ne^{i\varphi})=U(e^{i\varphi})$. Using a decreasing sequence of nested intervals and condition (4), one can prove that for each nonempty sector J'' with vertex at the origin, there exists a nonempty sector $J'=\{\varphi_1'<\arg z<\varphi_2'\}\subset J''$ with $\varphi_1'<\varphi_2'\le\varphi_1'+2\pi$ such that u is bounded on J' (see [**3**, p. 164]). Fix any φ_1 and φ_2 with $\varphi_1<\varphi_2$ and $[\varphi_1,\varphi_2]\subset(\varphi_1',\varphi_2')$. Let $u_n(z)=u(2^nz)$. We claim that the sequence $\{u_n(z)\}_{n=1}^{\infty}$ converges uniformly on compact subsets of the "closed" sector $J=\{\varphi_1\le\arg z\le\varphi_2\}$. From (4), it will follow that the limit function v does not depend on r. Since $v\in L(J)$ (see 3 of Proposition 1), Proposition 2 will give us B) in our theorem (see [**3**, p. 166] for more details). To prove the claim, it suffices to establish that $\{u_n\}$ converges uniformly on the compact set $K=\{\varphi_1\le\arg z\le\varphi_2,\ 1\le|z|\le 2\}$. In order to prove this last assertion, it is enough to check that $|\nabla u_n|$ is uniformly bounded on K and to use Ascoli-Arzela's theorem. Notice that $\sup\{|u_n(z)|\mid z\in J',\ n\ge 1\}<+\infty$, and $d:=\operatorname{dist}(K,\partial J')>0$ (here and in the sequel, ∂E is the boundary of a set E). Denote by Φ the fundamental solution of L, which is found in Proposition 1, and set $B(a,\delta)=\{z\in\mathbf{C}\mid |z-a|<\delta\}$, where $a\in\mathbf{C}$ and $\delta>0$. Fix $\psi\in C_0^\infty(B(0,d))$ such that $\psi=1$ in $B(0,d/2)$. Now fix $z_0\in K$ and put $\psi_0(z)=\psi(z-z_0)$. Then $\psi_0=0$ outside the ball $B(z_0,d)\subset J'$ and $\psi=1$ on $B(z_0,d/2)$. One has ([**6**, p. 255]) $u_n\psi=\Phi*L(u_n\psi)$, so that in $B(z_0,d/2)$, we can write (in the case $\lambda_1\ne\lambda_2$)

$$u_n(z)=\Phi*(Lu_n\,\psi+a_{11}\partial_1u_n\partial_2\psi+a_{11}\partial_2u_n\partial_1\psi+u_nL\psi)(z).$$

Since $\psi Lu_n\equiv 0$ and $a_{11}\partial_su_n\partial_{3-s}\psi=a_{11}\partial_s(u_n\partial_{3-s}\psi)-u_nL\psi$ ($s=1$ and 2), we obtain that, in $B(z_0,d/2)$,

$$\begin{aligned}u_n&=\Phi*(a_{11}\partial_1(u_n\partial_2\psi)+a_{11}\partial_2(u_n\partial_1\psi)-u_nL\psi)\\&=a_{11}(\partial_1\Phi)*(u_n\partial_2\psi)+a_{11}(\partial_2\Phi)*(u_n\partial_1\psi)-\Phi*(u_nL\psi).\end{aligned}$$

Now the desired uniform estimate for $|\nabla u_n(z_0)|$ can be obtained by making trivial estimates in the formula

$$\nabla u_n(z_0)=a_{11}\big[(\nabla\partial_1\Phi)*(u_n\partial_2\psi)+(\nabla\partial_2\psi)*(u_n\partial_1\psi)\big] \\ -(\nabla\Phi)*(u_nL\psi))\Big|_{z=z_0}.$$

The proof for the case $\lambda_1=\lambda_2$ is similar.

Let us now prove the second part of Theorem 1. Let $I = \cup_{j=1}^{\infty} I_j$, U, U_1 be as in (the second part of) Theorem 1. Put $I_0 = S \setminus I$, and for $j = 0, 1, \dots$ let $J_j = \{z \in \mathbf{C} \setminus \{0\} \mid e^{i \arg(z)} \in I_j\}$. Finally set $F_0 = \{z \in J_0 \mid |z| \geq 1\}$, $F_j = \{z \in J_j \mid \operatorname{dist}(z, \partial J_j) \geq 1\}$, $j = 1, 2, \dots$, and $F = \cup_{j=0}^{\infty} F_j$. Notice that each F_j and F are closed subsets of $\mathbf{C}$ and that the F_j $(j \geq 0)$ are pairwise disjoint. We note that if they are infinitely many F_j, they are pushed to ∞ (i.e. they are eventually outside any fixed compact set). It follows that there exist pairwise disjoint neighbourhoods Ω_j of F_j, $j = 0, 1, \dots$, with $\Omega_j \subset J_j$ for $j \geq 1$.

We first want to show that there exists a neighbourhood Ω_0' of F_0, $\Omega_0' \subset \Omega_0$, and a function $f \in C^1_{\mathrm{loc}}(\Omega_0')$ such that

$$\begin{aligned} &\lim_{r \to \infty} f(re^{i\varphi}) = U(e^{i\varphi}), \\ &\lim_{r \to \infty} \frac{\partial f(re^{i\varphi})}{\partial \varphi} = U_1(e^{i\varphi}), \qquad (5) \\ &\lim_{r \to \infty} \frac{\partial f(re^{i\varphi})}{\partial r} = 0, \end{aligned}$$

for each $e^{i\varphi} \in I_0$. The proof of this elementary fact is included for completeness.

Let $A_0 = \{|z| < 2\}$, $A_s = \{2^{s-1} < |z| < 2^{s+1}\})$, $s = 1, 2, \dots$, and let $\{\chi_s\}_{s=0}^{\infty}$ be a partition of unity on $\mathbf{C}$ subordinate to $\{A_s\}_{s=0}^{\infty}$ such that $\chi_s(z) = \chi_s(|z|)$ and $|\nabla \chi_s| \leq c/2^s$, where c is a constant independent of s. Since U and U_1 are of Baire class 1 on S, there exist sequences of continuous functions $\{V_s\}$, $\{W_s\}$ on S such that $V_s(e^{i\varphi}) \longrightarrow U(e^{i\varphi})$ and $W_s(e^{i\varphi}) \longrightarrow U_1(e^{i\varphi})$, for all $e^{i\varphi} \in S$ (and thus in particular for all $e^{i\varphi} \in I_0$). In addition we can choose the continuous functions V_s and W_s so that they are bounded by $2^{s/2}$.

Since V_s and W_s are uniformly continuous on S, there exists δ_s, $0 < \delta_s < 2^{-s}$, such that $|e^{i\varphi} - e^{i\varphi_0}| < \delta_s$ implies $|V_s(e^{i\varphi}) - V_s(e^{i\varphi_0})| < 1/2^s$ and $|W_s(e^{i\varphi}) - W_s(e^{i\varphi_0})| < 1/2^s$.

Since by assumption I_0 is nowhere dense in S, there exist open neighbourhoods N_s of I_0, $s = 0, 1, \dots$, such that $N_s = \cup_{k \geq 1} I_{sk}$ is the union of finitely many open arcs I_{sk} whose closures are disjoint and each I_{sk} is of length less than δ_s.

Now for each $s \geq 0$, define $\Omega_0^s = N_s^{(\varphi)} \times (2^{s-1}, 2^{s+1})^{(r)}$ and $\Omega_0^{sk} = I_{sk}^{(\varphi)} \times (2^{s-1}, 2^{s+1})^{(r)}$ in the (φ, r)-plane. We further require that the N_s $(s \geq 0)$ be chosen such that $\Omega_0^s \subset \Omega_0$.

We note that, by construction, V_s and W_s are almost constant on each of the sets I_{sk}. Fix $\varphi_{sk} \in I_0 \cap I_{sk}$. For $z = re^{i\varphi} \in \Omega_0^{sk}$, let $f_{sk}(z) :=$

$\alpha_{sk}\varphi + \beta_{sk}$, where α_{sk}, $\beta_{sk} \in \mathbf{C}$, are chosen such that $f_{sk}(e^{i\varphi_{sk}}) = V_s(e^{i\varphi_{sk}})$ and $\partial f_{sk}/\partial\varphi = \alpha_{sk} = W_s(e^{i\varphi_{sk}})$, so that $|\alpha_{sk}| \le 2^{s/2}$.

Let f_s be the function defined on Ω_0^s which is equal to f_{sk} on Ω_0^{sk}. And let $f = \sum_{s=0}^{\infty} f_s\chi_s$. Then f is well-defined on some neighbourhood Ω_0' of F_0. It is not too difficult to see that f satisfies (5). In the sequel, we identify Ω_0 and Ω_0'.

Using the localization scheme of Vitushkin (similarly to [**4**, Lemma 2.2(8), Corollary 6.3]), one can prove that for each $R > 0$, there exists $\{f_n^R\} \subset L(F_0^R)$, where $F_0^R = F_0 \cap \{|z| \le R\}$, such that $f_n^R \longrightarrow f$ in $C^1_{\text{jet}}(F_0^R)$ as $n \to +\infty$ (see [**4**] and [**2**, section 2.1]; in our particular case, since the interior of F_0 is empty and the union of all the lines in $\mathbf{C} \setminus F_0$ is everywhere dense, we only need a very simple part of the localization scheme).

Let us now consider the Banach space

$$V = \Big\{g \in C^1(\mathbf{R}^2) \;\Big|\; \|g\| := \sup_{z\in\mathbf{R}^2}\big\{\max\{|g(z)|, |\nabla g(z)|\}(1+|z|^2)\big\} < \infty\Big\}$$

with norm $\|\cdot\|$. This space satisfies the conditions (1)-(4) of [**2**]. From the fact that V is locally equivalent to the space $C^1(\mathbf{R}^2)$ and from the approximation properties of f on F_0^R mentioned above, it follows also that there exists a locally finite family of balls covering F_0 such that for each ball B in this family and for each $\varepsilon > 0$, there exists g such that $Lg = 0$ on some neighbourhood of $F_0 \cap \overline{B}$ and $\|f - g\|_{F_0\cap\overline{B}} < \varepsilon$ i.e. f is approximable locally on F_0 in the norm of V by (local) L-analytic functions. Theorem 2 in [**2**] now states that this is equivalent to global approximation, that is, for each $\varepsilon > 0$, there exists an L-analytic function g on (all of) F_0 such that $\|f - g\|_{F_0} < \varepsilon$.

Denote by $\mathbf{R}^2_\infty = \mathbf{R}^2 \cup \{\infty\}$ the one-point compactification of $\mathbf{R}^2$. Since $\mathbf{R}^2_\infty \setminus F_0$ is connected and locally connected (that is, F_0 is a RKL-set in the terminology of [**2**] (the letters stand for Roth-Keldysh-Lavrentieff)), we can use an analog of Runge's theorem obtained in [**2**, Theorem 1] to approximate in the norm of V L-analytic functions on F_0 by L-entire functions. We thus conclude that we can find an L-entire function h such that $\|f - h\|_{F_0} \le 1$. Using the estimate

$$|\partial\psi(z)/\partial\varphi| < |\nabla\psi(z)||z|, \tag{6}$$

this gives that (5) is satisfied when h is substituted for f.

Now define $v(z) = h(z)$ in Ω_0 and $v(z) = U(e^{i\arg(z)})$ in $\cup_{j=1}^{\infty}\Omega_j$. Then $v \in L(\Omega)$, where $\Omega = \cup_{j=0}^{\infty}\Omega_j$ is a neighbourhood of F, and F is a RKL-set. Thus again by [**2**, Theorem 1], we can find $u \in L(\mathbf{R}^2)$ with $\|v - u\|_F \le 1$. It suffices to notice, using (6) with $\psi = u - v$, that u is the desired L-entire function. Theorem 1 is proved. ■

Proof of Theorem 2: Part (a) of Theorem 2 trivially follows from Theorem 1, since it suffices to extend g from K to S by setting $g = 0$ on $S \setminus K$.

Suppose that $K \neq S$. The necessity in (b) is also a simple consequence of the proof of Theorem 1. To obtain the sufficiency in (b), we consider the closed set $F = \{z = re^{i\varphi} \in \mathbf{C} \mid e^{i\varphi} \in K,\, r \geq 1\}$ and the function $f(z) = f(re^{i\varphi}) := g(e^{i\varphi})$ on the RKL-set F.

An elementary proof (using only well known facts from one-dimensional real analysis) shows that for each $\varepsilon > 0$, there exists a finite number of disjoint open arcs I_j, whose union $I = \cup I_j$ contains K, and a function h_ε on I such that h_ε has the form v_{12}^* (or v_1^*) (see Proposition 2) on each I_j, and

$$\sup\big\{|g(e^{i\varphi}) - h_\varepsilon(e^{i\varphi})| \;\big|\; e^{i\varphi} \in K\big\} < \varepsilon.$$

Thus $f(z)$ is approximable uniformly on F by functions $h_\varepsilon(z) = h_\varepsilon(e^{i\arg(z)}) \in L(F)$.

The end of the proof is now similar to that of Theorem 1. We just need to take the following new approximation space:

$$V = \big\{\psi \in C(\mathbf{R}^2) \;\big|\; \|\psi\| = \sup_{z \in \mathbf{C}}(|\psi(z)|(1 + |z|)) < \infty\big\}.$$

Finally, if $K = S$, then $u = u_g$ must be bounded in $\mathbf{R}^2$, and hence $|\nabla u|$ is also bounded (see the beginning of the proof of Theorem 1). Then, considering $\partial_1 u$ and $\partial_2 u$ and using Proposition 1, we reduce the proof to an application of Liouville's Theorem for holomorphic functions. ■

References

1. A. V. Bitsadze, "*Boundary-value problems for second order elliptic equations*," North-Holland Series in Applied Mathematics and Mechanics **5**, North-Holland, Amsterdam, 1968.
2. A. Boivin and P. V. Paramonov, Approximation by meromorphic and entire solutions of elliptic equations in Banach spaces of distributions, *Sb. Math.* **189(4)** (1998), 481–502.
3. D. Gaier, "*Lectures on Complex Approximation*," Birkhäuser, Boston Basel Stuttgart, 1987.
4. P. V. Paramonov, On harmonic approximation in the C^1-norm, *Math. USSR-Sb.* **71(1)** (1992), 183–207.
5. P. V. Paramonov and K. Yu. Fedorovski, On C^1-approximation of functions by polynomial solutions of homogeneous elliptic

equations of second order on compact sets in $\mathbf{R}^2$, Dep. in VINITI 2965-B96 (1996), 1–15. (In Russian).

6. P. V. Paramonov and J. Verdera, Approximation by solutions of elliptic equations on closed subsets of Euclidean space, *Math. Scand.* **74** (1994), 249–259.
7. A. Roth, Approximationseigenschaften und strahlengrenzwerte meromorpher und ganzer funktionen, *Comment. Math. Helv.* **11** (1938), 77–125.

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