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HOMOGENIZATION OF A CAPILLARY PHENOMENA

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Abstract

We study the height of a liquid in a tube when it contains a great number of thin vertical bars and when its border is finely strained. For this, one uses an epi-convergence method.

1. Introduction

Let T be a cylindrical tube of height infinity, containing a compressible liquid and a great number of thin cyclindrical bars of radius r which are distributed periodically with a period ϵ such that $r < \epsilon$ (see Figure 1). One supposes the bars are of the same material which is specified by the coefficient λ (see (1) below). In section 2 we show that (see Proposition 2 and Theorem 1) when $(\epsilon, r\lambda)$ tends to $(0,0,0)$ the height of the liquid grows indefinitely, does not grow or take a certain height according to the rate $\frac{r\lambda}{\epsilon^2}$ being $+\infty$, 0 or finite (see Proposition 2 and Theorem 1).

Suppose now that the border of the tube is made of two alternate periodic bands with a period ϵ. The band 1 of length r is encrusted in the other of length $\epsilon - r$ $(r < \epsilon)$ (see Figure 3). The different materials that constitute the bands are specified by the coefficients k_1 and k_2.

Keywords. Homogenization, epi-convergence, capillarity.

One shows that (see Theorem 2) when (ϵ, r) tends to $(0,0)$ the tube behaves like an intermediate and homogeneous one of coefficient $k_1 a + k_2(1-a)$, where $a = \lim_{(\epsilon,r)\to(0,0)} \frac{r}{\epsilon}$.

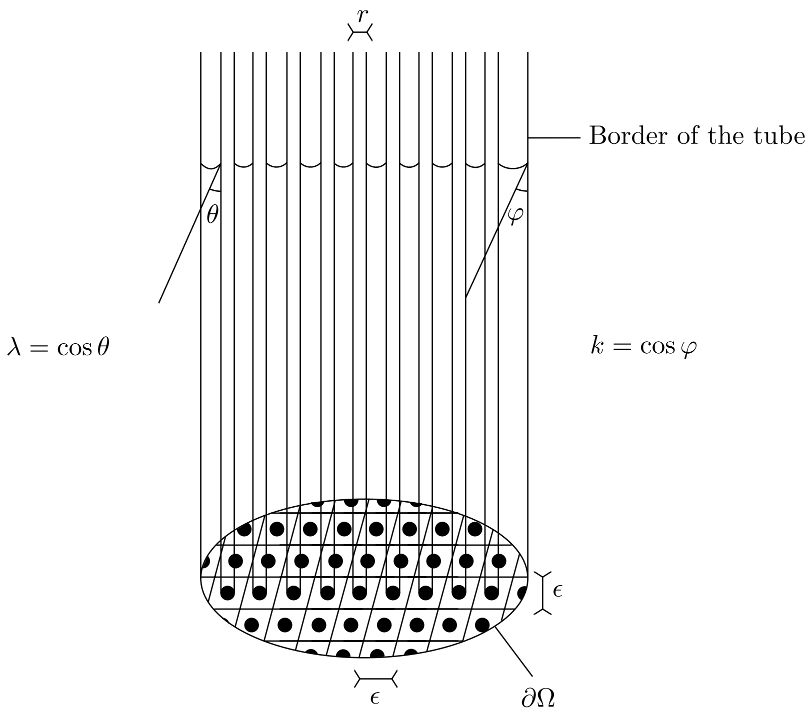

Figure 1. The tube with the bars.

2. Tube containing a great number of thin bars

2.1. Problem.

Let T be a cylindrical tube of base Ω with radius R and of height infinity, containing a compressible liquid of a finite volume. Let us divide Ω by a net of side ϵ and let us set in the center $x_{\epsilon,j}$ of the net $Y_{\epsilon,j}$ ($j \in I$ the integer part of the real $\frac{|\Omega|}{\epsilon^2}$) an homogeneous vertical cylindrical bar $T_{h,j}$ of radius r ($r < \epsilon$) where $h = (\epsilon, r, \lambda)$ (see Figures 1 and 2).

Let us denote the union of the bars $T_{h,j}$ by T_h and let $\Omega_h = \Omega \backslash T_h$.

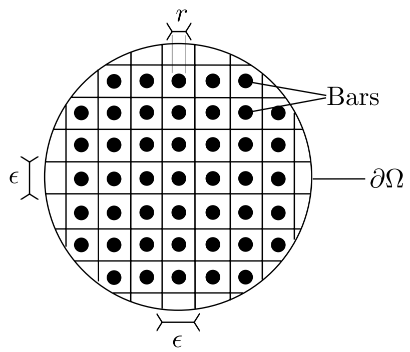

Figure 2

The height of the liquid is stabilizing when it satisfies the following minimization problem (see [**3**], [**4**]):

$$\text{Min}\{J_h(v),\, v \in BV(\Omega_h)\}, \tag{1}$$

where

$$J_h(v) = \int_{\Omega_h} \sqrt{1+|Dv|^2} + c\int_{\Omega_h} v^2 - \lambda \int_{\partial T_h} v - k\int_{\partial\Omega} v$$

for $v \in BV(\Omega_h)$ the space of functions of bounded variation defined in Ω (see [**5**] for the definition and the properties of $BV(\Omega)$):

$$BV(\Omega) = \left\{ v \in L^1(\Omega),\, \int_\Omega |Dv| < +\infty \right\}$$

where

$$\int_\Omega |Dv| = \sup\left\{ \sum_{k=1}^n \int_\Omega v(x)\frac{\partial\phi_k}{\partial x_k}(x)\,dx,\, \sum_{k=1}^n \phi_k^2(x) \le 1,\, \phi_k \in C_0^1(\Omega)\,\forall\, k \right\}.$$

$C_0^1(\Omega)$ is the space of functions defined in Ω, of class C^1 with a compact support included in Ω,

$$\int_\Omega \sqrt{1+|Dv|^2} = \sup\left\{ \int_\Omega v(x)\phi_0(x)\,dx - \sum_{k=1}^n \int_\Omega v(x)\frac{\partial\phi_k}{\partial x_k}(x)\,dx,\, \sum_{k=0}^n \phi_k^2(x) \le 1,\, \phi_k \in C_0^1(\Omega)\,\forall\, k \right\}.$$

The capillary constant c is related to the pressure term of the liquid, λ and k are respectively the cosine of the contact angles (liquid, bar), (liquid, border) and which specify respectively the bars material and the tube border. The constants c, λ and k are supposed real positive numbers.

Problem. We look for the limit behavior of the height of the liquid when $h = (\epsilon, r, \lambda)$ tends to $(0, 0, 0)$.

2.2. Existence and extension of the solution.

Proposition 1. *(See [**3**], [**4**]). The problem (1) has a unique positive solution in $BV(\Omega_h)$. Furthermore $u_h \in C^2(\overline{\Omega_h})$ and is constant on the boundary $\partial T_{h,j}$ of the bar $T_{h,j}$ for all $j \in I$.*

Let us cite a trace lemma in the space $BV(\Omega)$ which will be useful:

Lemma 1. *(See [**4**]). Suppose that Ω is of class C^2 and satisfying an internal sphere condition of radius r (i.e. for any boundary point $x \in \partial\Omega$, there is a ball B of radius r such that $B \subset \Omega$ and $x \in \partial B$). Then there exists a positive constant ct such that for any $v \in BV(\Omega)$,*

$$\int_{\partial\Omega} |v| \leq \int_{\Omega} |Dv| + \frac{ct}{r} \int_{\Omega} |v|.$$

Let us now state some estimates of u_h:

Proposition 2. *Let $a = \lim_{h\to 0} \frac{r\lambda}{\epsilon^2}$. Then,*

i) *If $a = +\infty$, then $\lim_{h\to 0} \int_{\Omega_h} u_h = +\infty$ and if a is finite, then*

$$\lim_{h\to 0} \int_{\Omega_h} u_h = \frac{k|\partial\Omega|}{2c} + \frac{\pi a|\Omega|}{c}.$$

ii) *If a is finite, then $\int_{\Omega_h} u_h$ and $\int_{\Omega_h} |Du_h|$ are uniformly bounded for $\frac{\lambda}{\epsilon}$ bounded.*

Proof: i) The problem (1) is equivalent to the Euler equation (see [**4**])

$$-\operatorname{Div} Tu_h + 2cu_h = 0 \quad \text{in} \quad \Omega_h \tag{2}$$

$$Tu_h.n = \lambda \quad \text{on} \quad \partial T_h \tag{3}$$

$$Tu_h.n = k \quad \text{on} \quad \partial\Omega, \tag{4}$$

where $Tv = \frac{Dv}{\sqrt{1+|Dv|^2}}$ and n the outer normal.

Multiplying (2) by the characteristic function χ_{Ω_h} and using the Green formula, (3) and (4),

$$\int_{\Omega_h} u_h = \frac{k|\partial\Omega|}{2c} + \frac{\pi|\Omega|}{c}\frac{\lambda r}{\epsilon^2}, \tag{5}$$

hence passing to the limit in (5), one obtains i).

ii) According to Lemma 1

$$\int_{\partial T_h} u_h \leq \int_{\Omega_h} |Du_h| + \frac{ct}{\epsilon}\int_{\Omega_h} u_h. \tag{6}$$

Let $\tilde{u}_h$ be the extension of u_h in Ω defined by

$$\tilde{u}_h = u_h \quad \text{in} \quad \Omega_h \tag{7}$$

$$\tilde{u}_h = u_{h,j} \quad \text{in} \quad T_{h,j};\ j \in I, \tag{8}$$

where $u_{h,j}$ is the value of u_h on $\partial T_{h,j}$.

Using again Lemma 1, one has

$$\begin{aligned}
\int_{\partial h} u_h &= \int_{\partial h} \tilde{u}_h \leq \int_\Omega |D\tilde{u}_h| + \frac{ct}{R}\int_\Omega \tilde{u}_h \\
&= \int_\Omega |Du_h| + \frac{ct}{R}\int_{\Omega_h} u_h + \frac{ct}{R}\int_{T_h} u_h \\
&= \int_\Omega |Du_h| + \frac{ct}{R}\int_{\Omega_h} u_h + ct\frac{r}{R}\int_{\partial T_h} u_h \\
&\leq (1 + ct.r)\int_{\Omega_h} |Du_h| + ct\int_{\Omega_h} u_h
\end{aligned} \tag{9}$$

where the last inequality of (9) is due to (6).

The inequality

$$J_h(u_h) \leq J_h(0) = |\Omega|,$$

(6) and (9) yield

$$(1 - k - \lambda - kr)\int_{\Omega_h} |Du_h| + c\int_\Omega u_h^2 \leq ct\left(1 + \frac{\lambda}{\epsilon}\right).$$

Hence the assertion ii) holds. ■

The following proposition yields an extension of u_h and of J_h in Ω which will be useful afterwards.

Proposition 3. *Suppose that* $\lim_{h\to 0} \frac{\lambda}{\epsilon}$ *is finite. Let* $\tilde{u}_h$ *be the extension of* u_h *in* Ω *which is defined by (7), (8) and* $\tilde{J}_h$ *the extension of* J_h *in* $BV(\Omega)$ *defined by*

$$\tilde{J}_h(v) = \int_\Omega \sqrt{1+|Dv|^2} + c\int_{\Omega_h} v^2 - \lambda \int_{\partial T_h} v_i - k\int_{\partial\Omega} v,$$

where v_i *is the trace of the restriction of* v *in* Ω_h*. Then*

i) $\tilde{J}_h$ *attains its minimum at the point* $\tilde{u}_h$*.*

ii) *The sequence* $\tilde{u}_h$ *is uniformly bounded in* $BV(\Omega)$*. In particular a subsequence of* $\tilde{u}_h$*, noted by* $\tilde{u}_h$*, converges to an element* u *(*$u \in BV(\Omega)$*) in* $L^1(\Omega)$*.*

Proof: i) For the existence and the regularity of the minimum of $\tilde{J}_h$ in $BV(\Omega)$ see [**3**], [**4**]. The minimum value of $\tilde{J}_h$ is attained at $\tilde{u}_h$ since $\tilde{J}_h(\tilde{u}_h) = J_h(u_h)$.

ii) From the definition of $\tilde{u}_h$,

$$\int_\Omega |D\tilde{u}_h| = \int_{\Omega_h} |Du_h|. \tag{10}$$

On the other hand, using (6), one has

$$\int_\Omega \tilde{u}_h = \int_{\Omega_h} u_h + \frac{r}{2}\int_{\partial T_h} u_h \le \Big(1 + ct\frac{r}{\epsilon}\Big)\int_{\Omega_h} u_h + \frac{r}{2}\int_{\Omega_h} |Du_h|. \tag{11}$$

Then, using Proposition 2, (10) and (11), one deduces that $\tilde{u}_h$ is uniformly bounded in $BV(\Omega)$ if $\lim_{(\lambda,\epsilon)\to(0,0)} \frac{\lambda}{\epsilon}$ is finite. ■

2.3. Epi-limit of the functional $\tilde{J}_h$.

The following proposition characterizes the epi-limit of the sequence $\tilde{J}_h$ (see [**1**], [**2**] for the definition and the properties of the epi-limit) when $h \to (0,0,0)$:

Proposition 4. *Suppose that* $a = \lim_{h\to 0} \frac{\lambda r}{\epsilon^2}$ *is finite. Then the sequence of the functionals* $\tilde{J}_h$ *epi-converges to the functional* J *in* $BV(\Omega)$ *endowed with the topology of* $L^1(\Omega)$ *where*

$$J(v) = \int_\Omega \sqrt{1+|Dv|^2} + c\int_\Omega v^2 - 2\pi a\int_\Omega v - k\int_{\partial\Omega} v$$

for $v \in BV(\Omega)$*.*

Proof: 1) Let $v \in BV(\Omega)$ and $v_h = v$. Since

$$\lambda \int_{\partial T_h} v = \frac{r\lambda}{\epsilon^2} \sum_{j\in I} \epsilon^2 \left(\frac{1}{|\partial T_{h,j}|} \int_{\partial T_{h,j}} v \right),$$

then

$$\lim_{h\to 0} \tilde{J}_h(v_h) = J(v).$$

2) Let v and $v_h \in BV(\Omega)$ such that v_h converges to v in $L^1(\Omega)$ when h tends to $(0,0,0)$. Let J'_h the functional defined in $BV(\Omega)$ by

$$J'_h(v) = \int_\Omega \sqrt{1+|Dv|^2} + c\int_{\Omega_h} v^2 - \lambda \int_{\partial T_h} v_i - k\int_{\partial\Omega} v,$$

where v_i is the trace, on ∂T_h, of the restriction of v to Ω_h.

Using the inequality (6),

$$\begin{aligned}
J'_h(v) - \tilde{J}_h(v_h) &\le \left(\int_\Omega \sqrt{1+|Dv|^2} - k\int_{\partial\Omega} v \right) \\
&\quad - \left(\int_\Omega \sqrt{1+|Dv_h|^2} - k\int_{\partial\Omega} v_h \right) \\
&\quad + \int_{\partial T_h} |v_h - v| + c\left(\int_\Omega v^2 - \int_{\Omega_h} v_h^2 \right) \\
&\le \left(\int_\Omega \sqrt{1+|Dv|^2} - k\int_{\partial\Omega} v \right) \\
&\quad - \left((1-\lambda)\int_\Omega \sqrt{1+|Dv_h|^2} - k\int_{\partial\Omega} v_h \right) \\
&\quad + ct\frac{\lambda}{\epsilon} \int_\Omega |v_h - h| + \lambda \int_\Omega |Dv| \\
&\quad + c\left(\int_\Omega v^2 - \int_{\Omega_h} v_h^2 \right).
\end{aligned} \tag{12}$$

Now, passing to the limitinf in (12), using the lower semi continuity of the functionsl $v \to \int_\Omega \sqrt{1+|Dv|^2} - k\int_{\partial\Omega} v$ in $BV(\Omega)$ according to the topology of $L^1(\Omega)$ (see [**6**]), Fatou Lemma, the fact that v_h converges to v in $L^1(\Omega)$ and that $\lim_{h\to 0} \frac{\lambda}{\epsilon}$ is finite, one deduces that

$$J(v) = \liminf_{h\to 0} J'_h(v) \le \liminf_{h\to 0} \tilde{J}_h(v_h). \quad \blacksquare$$

2.4. Limit behavior.

According to the properties of the epi-convergence (see [**1**], [**2**]), one deduces the limit behavior of the problem (1) from Propositions 2 and 3.

Theorem 1. *If* $\lim_{h\to 0}\frac{\lambda r}{\epsilon^2}$ *is finite, then the minimum* $\tilde{J}_h(\tilde{u}_h)$ *of* $\tilde{J}$ *converges to the minimum* $J(u)$ *and* $\tilde{u}_h$ *converges to* u *in* $L^1(\Omega)$ *when* h *tends to* $(0,0,0)$.

Physic interpretation. When the section, of radius r, of the bars $T_{h,j}$ satisfies $r \gg \frac{\epsilon^2}{\lambda}$, the liquid rises indefinitely; when $r \ll \frac{\epsilon^2}{\lambda}$, it does not rise and when $r \simeq a\frac{\epsilon^2}{\lambda}$, $a \in (0,+\infty)$, the capillary problem in the tube with the bars is approximated by a capillary problem in the tube without bars with the same capillary constant $2c$ but with $-2\pi a$ as Lagrange parameter.

3. Tube with a strained border

Let $n \in N^*$ and $\epsilon = \frac{|\Omega|}{n}$. Suppose now that the border of the tube is formed by two bands which are distributed alternately and periodically with the period ϵ (see Figures 3 and 4): the boundary $\partial\Omega$ is formed by n arcs $Y_{\epsilon,j}$, $j \in \{1,\dots,n\}$ of length ϵ. In the center of any arc $Y_{\epsilon,j}$ is encrusted an arc $T_{h,j}$ (band 1), $h = (\epsilon, r)$, of length r $(r < \epsilon)$. Suppose that the arcs $T_{h,j}$ are of the same material specified by the coefficient k_1 and the union of $Y_{\epsilon,j}\backslash T_{h,j}$ (band 2) is of another material specified by the coefficient k_2 (see (13)). Let us denote $T_h = \cup T_{h,j}$ and $(\partial\Omega)_h = \cup(Y_{\epsilon,j}\backslash T_{h,j})$.

The height of the liquid is stabilizing when it satisfies the minimization problem (see [**3**], [**4**]):

$$\text{Min}\{F_h(v),\, v \in BV(\Omega)\}, \tag{13}$$

where

$$F_h(v) = \int_{\Omega_h}\sqrt{1+|Dv|^2} + c\int_\Omega v^2 - k_1\int_{T_h} v - k_2\int_{(\partial\Omega)_h} v$$

for $v \in BV(\Omega)$; c is the capillary constant, k_1, k_2 are respectively the cosine of the contact angles (liquid, band 1) and (liquid, band 2). c, k_1, k_2 are supposed to be positive constants.

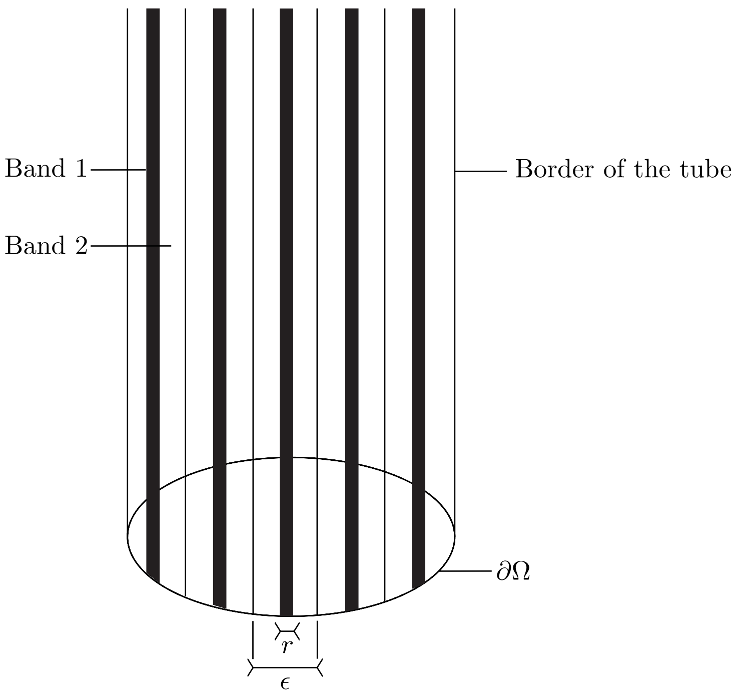

Figure 3. The tube with a strained border.

3.1. Existence.

Proposition 5. *The problem (13) has a unique positive solution in $BV(\Omega)$, $u_h \in C^2(\overline{\Omega_h})$ and u_h is bounded in $BV(\Omega)$. In particular a subsequence, noted u_h converges to an element u ($u \in BV(\Omega)$) in $L^1(\Omega)$.*

Proof: The proof of the existence is similar than the one of Proposition 1.

The problem (13) is equivalent to the equation

$$-\operatorname{Div} Tu_h + 2cu_h = 0 \quad \text{in} \quad \Omega \tag{14}$$

$$T_{u_h.n} = k_1 \quad \text{on} \quad T_h \tag{15}$$

$$Tu_h.n = k_2 \quad \text{on} \quad \partial\Omega \backslash T_h, \tag{16}$$

where $Tv = \frac{Dv}{\sqrt{1+|Dv|^2}}$ and n the outer normal.

Multiplying (14) by the characteristic function χ_{Ω_h} and using the Green formula, (15) and (16)

$$\int_\Omega u_h = \frac{1}{2c}(k_1|T_h| + k_2|\partial\Omega\backslash T_h|) = \frac{|\Omega|}{2c}\left(k_1\frac{r}{\epsilon} + k_2\frac{\epsilon - r}{\epsilon}\right). \tag{17}$$

The inequality

$$J_h(u_h) \le J_h(0) = |\Omega|$$

and (see Lemma 1)

$$\int_{\partial\Omega} u_h \le \int_\Omega |Du_h| + \frac{ct}{R}\int_\Omega u_h$$

yield

$$(1 - k_3)\int_\Omega |Du_h| + c\int_\Omega u_h^2 \le \frac{ct.k_3}{R}\int_\Omega u_h + |\Omega| \tag{18}$$

where $k_3 = \max(k_1, k_2)$. The boundedness of u_h in $BV(\Omega)$ is then a consequence of (17) and (18). ■

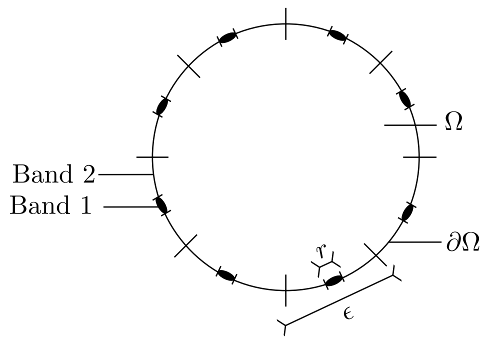

Figure 4

3.2. Epi-limit of the functional F_h.

The following proposition characterizes the epi-limit of the sequence F_h when $h \to (0, 0)$:

Proposition 6. *The sequence of the functionals F_h epi-converges to the functional F in $BV(\Omega)$ endowed within the topology of $L^1(\Omega)$ where*

$$F(v) = \int_\Omega \sqrt{1 + |Dv|^2} + c\int_\Omega v^2 - [k_1 a + k_2(1 - a)]\int_{\partial\Omega} v$$

for $v \in BV(\Omega)$; $a = \lim_{h\to(0,0)} \frac{r}{\epsilon} \in [0, 1]$.

Proof: 1) Let $v \in BV(\Omega)$ and $v_h = v$. Since

$$\int_{T_h} v = \frac{r}{\epsilon} \sum_{j \in I} \epsilon \left(\frac{1}{|T_{h,j}|} \right) \int_{\partial T_{h,j}} v)$$

and

$$\int_{\partial\Omega \setminus T_h} v = \frac{\epsilon - r}{\epsilon} \sum_{j \in I} \epsilon \left(\frac{1}{|Y_j \setminus T_{h,j}|} \right) \int_{Y_j \setminus T_{h,j}} v)$$

then

$$\lim_{h \to 0} F_h(v_h) = F(v).$$

2) Let v and $v_h \in BV(\Omega)$ such that v_h converges to v in $L^1(\Omega)$ when h tends to $(0,0)$.

Using Lemma 1

$$\begin{aligned} F_h(v) - F_h(v_h) &\leq \left(\int_\Omega \sqrt{1 + |Dv|^2} - \int_\Omega \sqrt{1 + |Dv_h|^2} \right) \\ &\quad + \int_{\partial\Omega} |v_h - v| + c \left(\int_\Omega v^2 - \int_{\Omega_h} v_h^2 \right) \\ &\leq \left(\int_\Omega \sqrt{1 + |Dv|^2} - \int_\Omega \sqrt{1 + |Dv_h|^2} \right) \\ &\quad + ct\frac{1}{R} \int_\Omega |v_h - v| + c \left(\int_\Omega v^2 - \int_\Omega v_h^2 \right). \end{aligned} \tag{19}$$

Now, passing to the limitinf in (19), using the lower semi continuity of the functional $v \to \int_\Omega \sqrt{1 + |Dv|^2}$ in $BV(\Omega)$ according to the topology of $L^1(\Omega)$ (see [**5**]), Fatou Lemma and the fact that v_h converges to v in $L^1(\Omega)$, one deduces that

$$F(v) = \liminf_{h \to 0} F_h(v) \leq \liminf_{h \to 0} F_h(v_h). \quad \blacksquare$$

3.3. Limit behavior.

From Proposition 6, one deduces the limit behavior of the problem (13).

Theorem 2. *i) The minimum of* F_h *converges, when* h *tends to* $(0,0)$, *to the minimum of* F *in* $BV(\Omega)$ *where*

$$F(v) \int_\Omega \sqrt{1 + |Dv|^2} + c \int_\Omega v^2 - [k_1 a + k_2(1-a)] \int_{\partial\Omega} v$$

for $v \in BV(\Omega)$ *and* $a = \lim_{h \to 0} \frac{r}{\epsilon} \in [0,1]$.

ii) The functional F *attains its minimum at the point* u *and* u_h *converges to* u *in* $L^1(\Omega)$ *when* h *tends to* $(0,0)$.

Physic interpretation. When the thickness r of the band 1 satisfies $r \ll \epsilon$ $(a = 0)$, the tube behaves like an homogeneous one whose border is made only of the band 2; when $r \simeq \epsilon$, it behaves like an homogeneous one whose border is made only of the band 1, and when $r \simeq a\epsilon$, $a \in (0,1)$, it behaves like an intermediate homogeneous one made only of a third material specified by its contact angle (liquid, border) whose cosine is $k_1 a + k_2(1-a)$.

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