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SINGULAR MEASURES AND THE LITTLE BLOCH SPACE

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Abstract

Aleksandrov, Anderson and Nicolau have found examples of inner functions that are in the little Bloch space with a specific rate of convergence to zero. As a corollary they obtain positive singular measures defined in the boundary of the unit disc that are simoultaneously symmetric and Kahane. Nevertheless their construction is very indirect. We give an explicit example of such measures by means of a martingale argument.

1. Introduction and some known results

A function f holomorphic in the unit disk $\mathbb{D}$ is said to be in the little Bloch space $\mathcal{B}_0$, if

$$\lim_{|z|\to 1} |f'(z)|(1-|z|^2) = 0.$$

Inner functions I are bounded holomorphic functions on $\mathbb{D} = \{|z| < 1\}$ such that

$$\lim_{r\to 1} |I(re^{i\theta})| = 1, \text{ for almost every } \theta \in [0, 2\pi].$$

Obviously, finite Blaschke products are in $\mathcal{B}_0$. Sarason [**S**] constructed an infinite Blaschke product in $\mathcal{B}_0$, from the singular inner function associated to a measure μ whose indefinite integral is in the little Zygmund class. Bishop [**B**] gave a characterization of inner functions in the little Bloch space in terms of a certain associated measure, which for infinite Blaschke products turns out to be a characterization in terms of the distribution of its zeroes.

Aleksandrov, Anderson and Nicolau [**AAN**] have found examples of inner functions I with specified rates of convergence to 0 of $(1-|z|^2)|I'(z)|$. Precisely,

Theorem. *There exists an inner function I such that*

$$(*)\qquad \lim_{|z|\to 1} \frac{|I'(z)|(1-|z|^2)}{1-|I(z)|^2} = 0.$$

As a corollary they obtain that for every continuous function $a(t)$, defined in $[0,1)$ with $a(0)=0$ and $a(t)>0$ if $t>0$, there exists an inner function I such that

$$\lim_{|z|\to 1} \frac{|I'(z)|(1-|z|^2)}{a(1-|I(z)|^2)} = 0.$$

A (singular) inner function like I can be obtained by first constructing a positive singular measure μ on $\partial\mathbb{D}$ for which

$$|\mu(J)-\mu(J')| = o(\mu(J)), \text{ as } |J|\to 0,$$

whenever J, J' are contiguous intervals of the same size and then considering the associated inner function

$$I(z) = \exp\left\{-\int \frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\mu(\theta)\right\}.$$

Their construction of the function I satisfying $(*)$ is indirect (see [**AAN**] for details). Anderson has asked for a direct construction. In this note we produce an explicit example.

After this paper was written I learnt about some overlapping work of Wayne Smith. His paper is titled: *Inner functions in the hyperbolic little Bloch class.*

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2. The construction

From now on we will use the following notation: $J \backsim J'$ means that J, J' are contiguous intervals of the same length; *i.e.*, $\operatorname{clos}(J)\cap\operatorname{clos}(J')$ is only one point and $|J|=|J'|$, where $|J|$ denotes the Lebesgue measure of J.

Theorem. *There exists a positive measure* μ *on* $\mathbb{R}$ *such that:*

(1) $\mu(A) > 0$, *for every open set* A.

(2) *For every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that if* $|J| < \delta$ *then*

$$\left|\frac{\mu(J)}{|J|} - \frac{\mu(J')}{|J'|}\right| < \varepsilon$$

for all $J \backsim J'$.

(3) *For all* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that if* $|J| < \delta$ *then*

$$\left|\frac{\mu(J) - \mu(J')}{\mu(J)}\right| < \varepsilon,$$

for all $J \backsim J'$.

(4) μ *is singular with respect to the Lebesgue measure.*

Proof: For the sake of clearness we are going to divide the proof into four stages labelled from A to D.

A. We are going to construct a 4-adic martingale $\{f_n\}$ on $[0,1]$. That is, the sequence of functions $\{f_n\}$ will be adapted to the standard 4-adic filtration of $[0,1]$. Precisely, we consider the sequence of partitions of $[0,1]$ into 4-adic intervals:

$$\mathcal{F}_n = \left\{ \left[\frac{j}{4^n}, \frac{j+1}{4^n}\right) : j = 0, \dots, 4^n - 1 \right\}, \quad n = 0, 1, 2, \dots .$$

Each f_n will be measurable with respect to the algebra generated by $\mathcal{F}_n$ and so each f_n will be constant on each $J \in \mathcal{F}_n$ therefore, it makes sense to refer to the value of f_n on $J \in \mathcal{F}_n$ as $f_n(J)$.

The martingale will have the following properties:

(a) $f_n(J) > 0$, for each J.

(b) For every $\varepsilon > 0$ there exists $\delta > 0$ such that if $|J| < \delta$ then

$$|f_n(J) - f_n(J')| < \varepsilon,$$

for each pair J, $J' \in \mathcal{F}_n$ with $J \backsim J'$.

(c) For all $\varepsilon > 0$ there exists $\delta > 0$ such that if $|J| < \delta$ then

$$\left|\frac{f_n(J) - f_n(J')}{f_n(J)}\right| < \varepsilon,$$

for each pair J, $J' \in \mathcal{F}_n$ with $J \backsim J'$,

(d) $f_n(x) \to 0$ *a.e.* x.

Once $\{f_n\}$ has been constructed then one defines a measure μ on $[0,1]$ as follows, for $J \in \mathcal{F}_n$,

$$\mu(J) = f_n(J)|J|;$$

this is a consistent definition of a measure on each $\sigma(\mathcal{F}_n)$ since $\{f_n\}$ is a martingale. Any other interval $\tilde{J} \subset [0,1]$ can be written as a disjoint union of 4-adic intervals, so $\mu(\tilde{J}) = \sum_{J \subset \tilde{J},\, J \text{ 4-adic}} \mu(J)$. We extend μ in the usual way to a Borel measure on $[0,1]$.

Notice that for 4-adic intervals properties (a) to (d) of the martingale imply trivially properties (1) to (4) of the measure.

B. Now we are going to develop the actual construction of the martingale. Let us consider three sequences of positive real numbers $\{\alpha_k\}$, $\{\varepsilon_k\}$, $\{M_k\}$ which have the following properties,

$$\alpha_k \searrow 0;\ \varepsilon_k \searrow 0,\ \varepsilon_0 = 1;\ M_k \nearrow \infty,\ M_1 > 2;$$

$$\frac{\alpha_{k-1}}{\varepsilon_k} < 1;\ \frac{\alpha_{k-1}}{\varepsilon_k} \searrow 0;\ \varepsilon_k < \varepsilon_{k-1} - \alpha_{k-1};$$

$$M_k \varepsilon_{k-1} \searrow 0.$$

Define $m_k = \min\{m \in \mathbb{N} : 0 < \alpha_{k-1} - m\alpha_k \leq \alpha_k\}$, and denote $\tilde{M}_k = \prod_{j=1}^{k} M_j$.

We also require ε_k and M_k to be multiples of α_k and α_k to divide α_{k-1}. These last conditions are only needed for technical reasons.

We will construct the martingale recursively: we start with a random walk with step α_1 and absorbing barriers at ε_1 and $M_1 = \tilde{M}_1$. At a certain time, say n_1, we change the step to a smaller one α_2 and consider barriers at ε_2, $M_2\varepsilon_1$ and $\tilde{M}_2$. At time n_2, we again change the step and the barriers as indicated above. We continue the construction indefinitely in this way. We will refer to ε_k as the *bottom barrier*, $M_k\varepsilon_{k-1}$ as the *middle barrier* and $\tilde{M}_k$ as the *top barrier*. We have chosen the ε_k's and M_k's to be possible positions of the random walk of step α_k, so it could land exactly at any of the barriers.

The main idea is to "trap" "many" walks in a narrow band near zero (between ε_k and $M_k\varepsilon_{k-1}$) by waiting long enough. Then we change the step to a smaller one and make the band narrower and closer to zero. The top barrier plays no crucial role but it is useful for technical reasons.

From now on the n_k's denote the time at which we change the step in the random walk. The criteria for choosing the n_k's will be given later.

To begin with, define $f_0(J) = 1$ for $J = [0, 1]$.

To start with the induction, let us suppose we have defined the martingale for $n \leq n_{k-1}$, that is $\{f_n\}_{n \leq n_{k-1}}$ and that we have chosen n_j's for $j \leq k - 1$.

From now on and in order to avoid endless repetitions, we will consider I, I', J, J' 4-adic intervals such that $J \subset I$, $J' \subset I'$ and $J \backsim J'$. Also we will write $f_n(J)$ only for $J \in \mathcal{F}_n$ and I_n will always denote an interval $I_n \in \mathcal{F}_n$.

For $n > n_{k-1}$, we will construct the martingale considering four different cases that depend on what has been happening up to step $n - 1$.

(i) If the random walk at time $n - 1$ is between the bottom barrier and the middle barrier, or if it is between the middle barrier and the top barrier, then we let it run "freely"; *i.e.*,

> if $\varepsilon_k < f_{n-1}(I) < M_k \varepsilon_{k-1}$, or if $M_k \varepsilon_{k-1} < f_{n-1}(I) < \tilde{M}_k$, then
>
> $$f_n(J) = f_{n-1}(I) + \alpha_k \zeta_n(J),$$
>
> where $\zeta_n(J) \in \{1, -1\}$, $\sum_{J \subset I, J \in \mathcal{F}_n} \zeta_n(J) = 0$, and the $\zeta_n(J)$ are chosen so that if $f_{n-1}(I) - f_{n-1}(I') > 0$ then $\zeta_n(J) = -1$ and $\zeta_n(J') = 1$.

(ii) If the random walk reaches the bottom barrier or the top barrier then we stop it, *i.e.*,

> if $f_{n-1}(I) = \varepsilon_k$ or $f_{n-1}(I) = \tilde{M}_k$ then, $f_n(J) = f_{n-1}(I)$, for all $J \subset I$.

(iii) If the random walk is on the middle barrier and it is the first time it has reached that barrier and has never been above it, then we stop the random walk (so we don't let it go too high), *i.e.*,

> if $f_{n-1}(I) = M_k \varepsilon_{k-1}$ and if for every $I_j \supset I$, $n_{k-1} \leq j < n - 1$ $f_j(I_j) \leq M_k \varepsilon_{k-1}$, then $f_n(J) = f_{n-1}(I)$, for all $J \subset I$.

(iv) Finally, if the random walk is on the middle barrier and it has already been above it at some time before, we let it run "freely", (since we still have the chance it will reach the lower level at a later time), *i.e.*,

> if $f_{n-1}(I) = M_k \varepsilon_{k-1}$ and if there exists a j, $n_{k-1} \leq j < n - 1$, $I_j \supset I$, such that $f_j(I_j) > M_k \varepsilon_{k-1}$, then $f_n(J) = f_{n-1}(I) + \alpha_k \zeta_n(J)$, where $\zeta_n(J)$ is as in (i).

Notice that $f_n \geq \varepsilon_k$.

To choose n_k we need the following lemma, whose proof we postpone to Section 3.

Lemma 1. *Given $\alpha > 0$, let ε, δ, M, be integer multiples of α so that $0 < \varepsilon < \delta < M$ and let a be an integer greater than 1.*

Let F_0 be a function measurable with respect to the σ-algebra generated by $\mathcal{F}_N$ for some $N \in \mathbb{N}$, whose values are integer multiples of α and lie between δ and M; i.e., $\operatorname{range}(F_0) \in \alpha\mathbb{N}$ *and $\delta \leq F_0(x) \leq M$ for all $x \in [0,1]$.*

For $J \in \mathcal{F}_{N+n}$, $I \in \mathcal{F}_{N+n-1}$, and $I_0 \in \mathcal{F}_N$ with $J \subset I \subset I_0$, we define

$$F_n(J) = F_{n-1}(I) + \alpha\eta_n(J)$$

where η_n is chosen so that,

- *if $\delta \leq F_0(I_0) \leq a\delta$ then, $\{F_n\}$ is a random walk of step α that starts at $F_0(I_0)$ and has absorbing barriers at ε and $a\delta$,*
- *if $a\delta \leq F_0(I_0) \leq M$, then $\{F_n\}$ is a random walk with step α that starts at $F_0(I_0)$ and has absorbing barriers at ε and aM.*

Under these assumptions there exists an integer $\tilde{n} > 0$ such that,

$$|\{F_{\tilde{n}} = \varepsilon\}| > 1 - \frac{1}{a}.$$

Remark. The definition of η_n in Lemma 1 is a shorter way to write items (i)-(iv) in the definition of the martingale. In this case, the botton barrier is ε, the middle barrier is $a\delta$ and the top barrier is aM.

Take $F_0 = f_{n_{k-1}}$ (therefore $N = n_{k-1}$), $\alpha = \alpha_k$, $\delta = \varepsilon_{k-1}$, $\varepsilon = \varepsilon_k$, $M = \tilde{M}_{k-1}$ and $a = M_k$.

Notice that $F_n = f_{n+n_{k-1}}$. Let $\tilde{n}_k$ be the $\tilde{n}$ given by the lemma and set $n_k = \tilde{n}_k + n_{k-1}$. It follows that,

$$|\{f_{n_k} = \varepsilon_k\}| > 1 - \frac{1}{M_k}.$$

Observe also that

$$(**) \qquad n_k > m_k + n_{k-1},$$

where $m_k = \min\{m \in \mathbb{N} : 0 < \alpha_{k-1} - m\alpha_k \leq \alpha_k\}$. This requires a little argument: by the way the martingale has been constructed, we have for $n > n_{k-1}$, that

$$f_n(J) \geq f_{n_{k-1}}(I_{n_{k-1}}) - (n - n_{k-1})\alpha_k.$$

Since ε_{k-1} is the lowest position the martingale could have reached at time n_{k-1}, that is, $f_{n_{k-1}}(I_{n_{k-1}}) > \varepsilon_{k-1}$, we have that,

$$f_n(J) \geq \varepsilon_{k-1} - (n - n_{k-1})\alpha_k.$$

Recall that ε_k's have been chosen so that $\varepsilon_{k-1} > \varepsilon_k + \alpha_{k-1}$. Then,

$$f_n(J) \geq \varepsilon_k + \alpha_{k-1} - (n - n_{k-1})\alpha_k.$$

Therefore if $f_n(J)$ has reached the bottom barrier n, must be so that $\alpha_{k-1} - (n - n_{k-1})\alpha_k \leq 0$ and then $n > m_k + n_{k-1}$. Observe that if $\alpha_{k-1} - (n - n_{k-1})\alpha_k < 0$ then either the random walk could not have been at the lowest position at time n_{k-1}, that is, $f_{n_{k-1}}(I_{n_{k-1}}) > \varepsilon_{k-1}$ or the random walk is on the barrier ε_k after certain time.

In particular, at time n_k, "many" random walks could have reached the bottom barrier, and so $n_k > m_k + n_{k-1}$.

Finally, notice that a similar argument works with the top and middle barriers. That means that the random walk cannot reach a barrier before time $m_k + n_{k-1}$.

To construct the martingale for times $n \geq n_k$ we restart the process again with new step α_{k+1} and barriers at ε_{k+1}, $M_{k+1}\varepsilon_k$ and $\tilde{M}_{k+1}$.

C. Next we verify that the martingale we have just constructed satisfies the properties (a)-(d).

(a) $f_n > 0$ for all $n \in \mathbb{N}$. In particular, if $n \leq n_k$ then $f_n \geq \varepsilon_k > 0$.

(b) $|f_n(J) - f_n(J')| \to 0$, $|J| \to 0$, $J \backsim J'$. We have to consider several situations, depending on whether the intervals are children of the same parent or not.

b.1. *Same parent:* $J, J' \subset I$, then

$$|f_n(J) - f_n(J')| = \begin{cases} 0 \\ 2\alpha_k \end{cases}$$

for $n_{k-1} \leq n < n_k$.

b.2. *Different parents:* $J \subset I$, $J' \subset I'$. We have to distinguish between two different situations, that depend on whether the martingale has changed its step.

(i) With change in the step of the martingale, *i.e.*, $n_{k-1} = n - 1 < n < n_k$.

Assume that $0 \leq f_{n-1}(I) - f_{n-1}(I') \leq 2\alpha_{k-1}$. We will see in (ii.2) that it is, in fact, the only possibility since the time we change the step $n_{k-1} = n-1$, is bigger than $m_{k-1}+n_{k-2}$ (see (∗∗) after Lemma 1). So, in this case we have,

$$f_n(J) = f_{n-1}(I) - \alpha_k$$
$$f_n(J') = f_{n-1}(I') + \alpha_k,$$

and thus $-2\alpha_k \leq f_n(J) - f_n(J') \leq 2(\alpha_{k-1} - \alpha_k)$.

(ii) With no change on the step, *i.e.*, $n_{k-1} < n-1 < n \leq n_k$. We will use an induction argument, but again several cases come into consideration.

(ii.1) Suppose first that at time $n-1$, neither $f_{n-1}(I)$ nor $f_{n-1}(I')$ have reached a barrier, and that $0 \leq f_{n-1}(I) - f_{n-1}(I') \leq 2\alpha_k$ (that is, for example I and I' are children of the same parent or that $n \geq m_k + n_{k-1}$), then

$$f_n(J) = f_{n-1}(I) - \alpha_k$$
$$f_n(J') = f_{n-1}(I') + \alpha_k$$

and therefore, $0 \leq f_n(J') - f_n(J) \leq 2\alpha_k$.

(ii.2) Let us suppose again that at time $n-1$, $f_{n-1}(I)$ and $f_{n-1}(I')$ have reached no barrier, and assume also that $2\alpha_k \leq f_{n-1}(I) - f_{n-1}(I') \leq 2\alpha_{k-1} - 2m\alpha_k$ for $m < m_k$, (*i.e.* $n < m_k + n_{k-1}$ so it has not been long enough to avoid the effect of change of the step), then

$$f_n(J) = f_{n-1}(I) - \alpha_k$$
$$f_n(J') = f_{n-1}(I') + \alpha_k$$

and so, $0 \leq f_n(J) - f_n(J') \leq 2\alpha_{k-1} - 2(m+1)\alpha_k$.

Note that when time runs long enough, *i.e.*, $n \geq m_k + n_{k-1}$, we get $|f_n(J) - f_n(J')| \leq 2\alpha_k$, and we are ready again to continue with the induction argument. In particular for n_k, $|f_{n_k}(J) - f_{n_k}(J')| \leq 2\alpha_k$, and so, as mentioned, the assumption in case (i) of change of step turns out to be always true.

(ii.3) And the remaining case is when $f_{n-1}(I)$ has reached a barrier, and $f_{n-1}(I')$ has not (when both have reached a barrier, trivially we get $f_n(J) = f_n(J')$). Since the random walk has already reached a barrier, by the observation after Lemma 1, $n-1 > m_k + n_{k-1}$, and then, $|f_{n-1}(I) - f_{n-1}(I')| \leq 2\alpha_k$, so

$$f_n(J) = f_{n-1}(I)$$
$$f_n(J') = f_{n-1}(I') + \zeta_n(J')\alpha_k$$

therefore, $|f_n(J) - f_n(J')| \leq \alpha_k$.

In both cases b.1 and b.2 we get, $|f_n(J) - f_n(J')| \leq 2(\alpha_{k-1} - \alpha_k)$ and $\alpha_k \searrow 0$, therefore property (b) holds.

(c) Now we will prove that

$$\frac{|f_n(J) - f_n(J')|}{|f_n(J)|} \to 0$$

for intervals $J \backsim J'$, $n_{k-1} < n \leq n_k$, then

$$\left| \frac{f_n(J) - f_n(J')}{f_n(J)} \right| \leq 2 \frac{\alpha_{k-1}}{\varepsilon_k},$$

and α_k and ε_k have been chosen so that

$$\frac{\alpha_{k-1}}{\varepsilon_k} \searrow 0.$$

(d) To verify $f_n(x) \to 0$ a.e. x we just have to check that $\liminf_{n\to\infty} f_n(x) = 0$ a.e. x: Because of the Martingale Convergence Theorem and since the martingale is positive, we already know that $\lim_{n\to\infty} f_n(x)$ exists a.e. x. We need to define the following sets. Let

$$\begin{aligned} B_k &= \{x : \varepsilon_k \leq f_{n_k}(x) \leq \varepsilon_{k-1} M_k\} \\ A_k &= \{x : f_{n_k}(x) = \varepsilon_k\}. \end{aligned}$$

Clearly, because of the way the martingale was constructed $B_k \supset A_{k-1}$. So, by the observation made after Lemma 1, we obtain $|B_k| \geq |A_{k-1}| > 1 - \frac{1}{M_{k-1}}$, which together with $\varepsilon_k \searrow 0$, $\varepsilon_{k-1} M_k \searrow 0$ and $M_k \nearrow \infty$, yields $f_n(x) \to 0$ for *a.e.* x.

As we have mentioned above, properties (a) to (d) of the martingale imply trivially properties (1) to (4) of the measure for 4-adic intervals.

D. For general intervals one should use Kahane's argument in [**K**, p. 190], where essentially he compares the μ-length of a general interval $\tilde{J}$, with the μ-length of the smallest 4-adic interval J that intersects $\tilde{J}$ with $|J| > |\tilde{J}|$. In our case, extra care needs to be taken because of the change of step in the martingale, but this causes only minor changes.

3. Proof of Lemma 1

As a matter of fact we will prove a little more, that is, we will show that there exists a $\tilde{n} > 0$ such that

$$1 - \frac{1}{a} < |\{x : F_{\tilde{n}}(x) = \varepsilon\}| < 1 - \frac{1}{a} + \frac{\varepsilon}{a\delta}.$$

We will need a lemma about random walks, see [**F**, p. 344–345] for references.

Lemma 2. *Given $j, M, k \in \mathbb{N}$, $M > k > j$, let S_n be the random walk with step $\alpha > 0$ and absorbing barriers at αj and αM. Then*

$$P(\cup_{n=1}^{\infty}\{S_n = \alpha j\} \mid \{S_0 = \alpha k\}) = 1 - \frac{k-j}{M-j}.$$

Here $P(A \mid B)$ denotes the probability of A conditioned on B.

Proof of Lemma 1: By conditioning,

$$\begin{aligned} P(\cup_{n=1}^{\infty}\{F_n = \varepsilon\}) = {} & P(\cup_{n=1}^{\infty}\{F_n = \varepsilon\} \mid \{F_0 = \delta\}) \cdot P(\{F_0 = \delta\}) \\ & + P(\cup_{n=1}^{\infty}\{F_n = \varepsilon\} \mid \{F_0 \neq \delta\}) \cdot P(\{F_0 \neq \delta\}). \end{aligned}$$

Now, using the fact that it is easier to reach the bottom barrier if the random walk starts at δ (which is the lowest position) than if it starts at any other position; *i.e.*,

$$P(\cup_{n=1}^{\infty}\{F_n = \varepsilon\} \mid \{F_0 = \delta\}) > P(\cup_{n=1}^{\infty}\{F_n = \varepsilon\} \mid \{F_0 \neq \delta\})$$

and so by Lemma 2 above, we obtain,

$$\begin{aligned} P(\cup_{n=1}^{\infty}\{F_n = \varepsilon\}) &< P(\cup_{n=1}^{\infty}\{F_n = \varepsilon\} \mid \{F_0 = \delta\}) \\ &= 1 - \frac{\delta - \varepsilon}{a\delta - \varepsilon} \\ &< 1 - \frac{1}{a} + \frac{\varepsilon}{a\delta}. \end{aligned}$$

Also, since the random walk has less chances to reach the bottom barrier if it starts at the highest position; *i.e.*, $P(\cup_{n=1}^{\infty}\{F_n = \varepsilon\} \mid \{F_0 = M\}) \leq P(\cup_{n=1}^{\infty}\{F_n = \varepsilon\} \mid \{F_0 \neq \delta\})$, we get,

$$
\begin{aligned}
P(\cup_{n=1}^{\infty}\{F_n = \varepsilon\}) &\geq P(\cup_{n=1}^{\infty}\{F_n = \varepsilon\} \mid \{F_0 = \delta\}) \cdot P(\{F_0 = \delta\}) \\
&\quad + P(\cup_{n=1}^{\infty}\{F_n = \varepsilon\} \mid \{F_0 = M\}) \cdot P(\{F_0 \neq \delta\}) \\
&= \left(1 - \frac{\delta - \varepsilon}{a\delta - \varepsilon}\right) \cdot P(\{F_0 = \delta\}) \\
&\quad + \left(1 - \frac{M - \delta}{aM - \delta}\right) \cdot (1 - P(\{F_0 = \delta\})) \\
&> 1 - \frac{1}{a}.
\end{aligned}
$$

Therefore, there exists $\tilde{n} > 0$ such that,

$$1 - \frac{1}{a} < P(\cup_{n=1}^{\tilde{n}}\{F_n = \varepsilon\}) < 1 - \frac{1}{a} + \frac{\varepsilon}{a\delta}.$$

Because ε is an absorbing barrier, once the random walk reaches ε it never escapes. Consequently,

$$\cup_{n=1}^{\tilde{n}}\{F_n = \varepsilon\} = \{F_{\tilde{n}} = \varepsilon\},$$

which proves Lemma 1. ■

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