DTU Library

A generalization of Gale's lemma

Alishahi, Meysam; Hajiabolhassan, Hossein

WILEY

ARTICLE

A generalization of Gale's lemma

Meysam Alishahi[1] | Hossein Hajiabolhassan[2,3] (iD)

[1] School of Mathematical Sciences, Shahrood University of Technology, Shahrood, Iran

[2] Department of Applied Mathematics and Computer Science, Technical University of Denmark, DK-2800 Lyngby, Denmark

[3] Department of Mathematical Sciences, Shahid Beheshti University, G.C., Tehran, Iran

Correspondence
Hossein Hajiabolhassan, Department of Applied Mathematics and Computer Science, Technical University of Denmark, DK-2800 Lyngby, Denmark, and Department of Mathematical Sciences, Shahid Beheshti University, G.C., P.O. Box 19839-69411, Tehran, Iran.
Email: hhhaji@yahoo.com

Abstract

In this work, we present a generalization of Gale's lemma. Using this generalization, we introduce two sharp combinatorial lower bounds for $\mathrm{coind}(B_0(G)) + 1$ and $\mathrm{coind}(B(G)) + 2$, the two classic topological lower bounds for the chromatic number of a graph G.

KEYWORDS

box complex, chromatic number of graphs, Gale's lemma

1 | INTRODUCTION AND MAIN RESULTS

1.1 | Backgrounds and motivations

Throughout the article, for positive integers k and n, two symbols $[n]$ and $\binom{[n]}{k}$ stand for the set $\{1, \ldots, n\}$ and the family of all k-subsets of $[n]$, respectively. For a positive integer s, a subset A of $[n]$ is said to be an *s-stable subset* if $s \leq |i - j| \leq n - s$ for each $i \neq j \in A$. Throughout the article, the family of all s-stable k-subsets of $[n]$ is denoted by $\binom{[n]}{k}_s$. For $n \geq 2k$, *the Kneser graph* $\mathrm{KG}(n, k)$ is a graph whose vertex set consists of all k-subsets of $[n]$ and two vertices are adjacent if their corresponding k-sets are disjoint. Kneser 1955 [11] proved that $\mathrm{KG}(n, k)$ can be properly colored with $n - 2k + 2$ colors. He also conjectured that this is the best possible, i.e. $\chi(\mathrm{KG}(n, k)) \geq n - 2k + 2$. In 1978, Lovász in a fascinating article [12], using the Borsuk–Ulam theorem, gave an affirmative answer to this conjecture.

For an integer $d \geq -1$, by the symbol S^d, we mean the d-dimensional sphere. For an $x \in S^d$, $H(x)$ is *the open hemisphere centered at x*, i.e. $H(x) = \{y \in S^d : \langle x, y \rangle > 0\}$. There is a well-known lemma due to Gale [8] that asserts that for every $k \geq 1$ and every $n > 2k$, there is an n-set $Z \subset S^{n-2k}$ such that for any $x \in S^{n-2k}$, the open hemisphere $H(x)$ contains at least k points of Z. In particular, if we identify the set Z with $[n]$, then for any $x \in S^{n-2k}$, the open hemisphere $H(x)$ contains some vertex of $\mathrm{KG}(n, k)$. Soon after the announcement of the Lovász breakthrough [12], Bárány [5] presented a short proof of the Lovász–Kneser theorem based on Gale's lemma. Next, Schrijver [16] generalized Gale's lemma by proving that there is an n-set $Z \subset S^{n-2k}$ and a suitable identification of Z with $[n]$ such that for any $x \in S^{n-2k}$, the open hemisphere $H(x)$ contains a 2-stable subset of size at least k, i.e. the

open hemisphere $H(x)$ contains some member of $\binom{[n]}{k}_2$. Using this generalization, Schrijver [16] found a specific vertex-critical subgraph of $KG(n, k)$ the same chromatic number as $KG(n, k)$. This subgraph, known as the Schrijver graph $SG(n, k)$, is an induced subgraph of $KG(n, k)$ whose vertex set consists of all 2-stable k-subsets of $[n]$. It is worth mentioning that Greene [9] showed that Bárány's idea can be used in a more tricky way to avoid the use of Gale's lemma while Greene's trick does not seem to work for the case of Schrijver graphs, that is, Gale's lemma is still essential for a Bárány type proof of Schrijver's theorem.

1.2 | Main results

For an $X = (x_1, \ldots, x_n) \in \{+, -, 0\}^n$, an *alternating subsequence of X* is a subsequence of nonzero terms of X such that each of its two consecutive members have different signs. In other words, $x_{j_1}, \ldots, x_{j_m}$ $(1 \leq j_1 < \cdots < j_m \leq n)$ is an alternating subsequence of X if $x_{j_i} \neq 0$ for each $i \in [m]$ and $x_{j_i} \neq x_{j_{i+1}}$ for $i = 1, \ldots, m - 1$. The length of the longest alternating subsequence of X is denoted by $\mathrm{alt}(X)$. We also set $\mathrm{alt}(0, \ldots, 0) = 0$. Moreover, define

$$X^+ = \{j : x_j = +\} \quad \text{and} \quad X^- = \{j : x_j = -\}.$$

Note that, by abuse of notation, we can write $X = (X^+, X^-)$.

Let V be a nonempty finite set of size n. *The signed-power set of V*, denoted by $P_s(V)$, is defined as follows;

$$P_s(V) = \{(A, B) : A, B \subseteq V, \ A \cap B = \varnothing\}.$$

For two pairs (A, B) and (C, D) in $P_s(V)$, by $(A, B) \subseteq (C, D)$, we mean $A \subseteq C$ and $B \subseteq D$. Note that $(P_s(V), \subseteq)$ is a partially order set (poset). *A signed-increasing property $\mathcal{P}$*, is a superset-closed family $\mathcal{P} \subseteq P_s(V)$, i.e. for any $F_1 \in \mathcal{P}$, if $F_1 \subseteq F_2 \in P_s(V)$, then $F_2 \in \mathcal{P}$. Clearly, for any bijection $\sigma : [n] \to V$, the map $X \mapsto X_\sigma = (\sigma(X^+), \sigma(X^-))$ is an identification between $\{+, -, 0\}^n$ and $P_s(V)$. Let $\sigma : [n] \to V$ be a bijection and $\mathcal{P} \subseteq P_s(V)$ be a signed-increasing property. Define

$$\mathrm{alt}(\mathcal{P}, \sigma) = \max \left\{ \mathrm{alt}(X) : X \in \{+, -, 0\}^n \text{ with } X_\sigma \notin \mathcal{P} \right\}.$$

Also, define *the alternation number of $\mathcal{P}$* to be the following quantity;

$$\mathrm{alt}(\mathcal{P}) = \min\{\mathrm{alt}(\mathcal{P}, \sigma) : \sigma : [n] \to V \text{ is a bijection}\}.$$

Let $d \geq 0$ be an integer, S^d be the d-dimensional sphere, and $Z \subset S^d$ be a finite set. For an $x \in S^d$, define $Z_x = (Z_x^+, Z_x^-) \in P_s(Z)$ where $Z_x^+ = H(x) \cap Z$ and $Z_x^- = H(-x) \cap Z$. Now, we are in a position to state the first main result of this article (Lemma 1), which is a generalization of Gale's lemma. The fact that the following lemma generalizes Gale's lemma will be more clarified later, see Corollary 1 and the discussion after it.

Lemma 1. *Let n be a positive integer, V be an n-set, and $\sigma : [n] \to V$ be a bijection. Also, let $\mathcal{P} \subseteq P_s(V)$ be a signed-increasing property and set $d = n - \mathrm{alt}(\mathcal{P}, \sigma) - 1$. If $d \neq -1$, then there are a multiset $Z \subset S^d$ of size n and a suitable identification of Z with V such that for any $x \in S^d$, $Z_x \in \mathcal{P}$. In particular, for $d \geq 1$, Z can be a set.*

A hypergraph $\mathcal{H}$ is a pair $(V(\mathcal{H}), E(\mathcal{H}))$ where $V(\mathcal{H})$ is a finite nonempty set, called the vertex set of $\mathcal{H}$, and $E(\mathcal{H})$ is a family containing some nonempty distinct subsets of $V(\mathcal{H})$, called the edge set of $\mathcal{H}$. For a set $U \subseteq V(\mathcal{H})$, the induced subhypergraph $\mathcal{H}[U]$ of $\mathcal{H}$, is a hypergraph with vertex set U and

edge set $\{e \in E(\mathcal{H}) \ : \ e \subseteq U\}$. A graph G is a hypergraph such that each of its edges has cardinality two. A t-coloring of a hypergraph $\mathcal{H}$ is a map $c \ : \ V(\mathcal{H}) \rightarrow [t]$ such that for no edge $e \in E(\mathcal{H})$, we have $|c(e)| = 1$. The minimum possible t for which $\mathcal{H}$ admits a t-coloring, denoted by $\chi(\mathcal{H})$, is called the chromatic number of $\mathcal{H}$. For a hypergraph $\mathcal{H}$, *the Kneser graph* $\mathrm{KG}(\mathcal{H})$ is a graph whose vertex set is $E(\mathcal{H})$ and two vertices are adjacent if their corresponding edges are vertex disjoint. It is known that for any graph G, there are infinitely many hypergraphs $\mathcal{H}$ for which G and $\mathrm{KG}(\mathcal{H})$ are isomorphic. Each of those hypergraphs $\mathcal{H}$ is called a *Kneser representation of G*. Note that if we set $K_n^k = ([n], \binom{[n]}{k})$ and $\widetilde{K_n^k} = ([n], \binom{[n]}{k}_2)$, then $\mathrm{KG}(K_n^k) = \mathrm{KG}(n, k)$ and $\mathrm{KG}(\widetilde{K_n^k}) = \mathrm{SG}(n, k)$. The *colorability defect* of a hypergraph $\mathcal{H}$, denoted by $\mathrm{cd}(\mathcal{H})$, is the minimum number of vertices that should be excluded so that the induced subhypergraph on the remaining vertices is 2-colorable. Dol'nikov [7] improved Lovász's result [12] by proving that for any hypergraph $\mathcal{H}$, we have $\chi(\mathrm{KG}(\mathcal{H})) \geq \mathrm{cd}(\mathcal{H})$.

Let $\mathcal{H} = (V, E)$ be a hypergraph and $\sigma \ : \ [n] \rightarrow V(\mathcal{H})$ be a bijection. Define

$$\mathrm{alt}(\mathcal{H}, \sigma) = \max \left\{ \mathrm{alt}(X) \ : \ X \in \{+, -, 0\}^n \text{ s.t. } \max \left(|E(\mathcal{H}[\sigma(X^+)])|, |E(\mathcal{H}[\sigma(X^-)])| \right) = 0 \right\}$$

and

$$\mathrm{salt}(\mathcal{H}, \sigma) = \max \left\{ \mathrm{alt}(X) \ : \ X \in \{+, -, 0\}^n \text{ s.t. } \min \left(\left| E(\mathcal{H}[\sigma(X^+)]) \right|, \left| E(\mathcal{H}[\sigma(X^-)]) \right| \right) = 0 \right\}.$$

In other words, $\mathrm{alt}(\mathcal{H}, \sigma)$ (resp. $\mathrm{salt}(\mathcal{H}, \sigma)$) is the maximum possible $\mathrm{alt}(X)$, where $X \in \{+, -, 0\}^n$, such that each of (resp. at least one of) $\sigma(X^+)$ and $\sigma(X^+)$ contains no edge of $\mathcal{H}$. We also define

$$\mathrm{alt}(\mathcal{H}) = \min_{\sigma} \mathrm{alt}(\mathcal{H}, \sigma) \quad \text{and} \quad \mathrm{salt}(\mathcal{H}) = \min_{\sigma} \mathrm{salt}(\mathcal{H}, \sigma),$$

where the minimum is taken over all bijections $\sigma \ : \ [n] \rightarrow V(\mathcal{H})$. The present authors, using Tucker's lemma [21], introduced two tight combinatorial lower bounds for the chromatic number of $\mathrm{KG}(\mathcal{H})$ improving Dol'nikov's lower bound.

Theorem A. *[1] For any hypergraph $\mathcal{H}$, we have*

$$\chi(KG(\mathcal{H})) \geq \max \left(|V(\mathcal{H})| - alt(\mathcal{H}), |V(\mathcal{H})| - salt(\mathcal{H}) + 1 \right).$$

Let $\mathcal{H} = (V, E)$ be a hypergraph and $\sigma \ : \ [n] \rightarrow V(\mathcal{H})$ be a bijection. Note that if we set

$$\mathcal{P}_1 = \left\{ (A, B) \in P_s(V) \ : \ \text{at least one of } A \text{ and } B \text{ contains some edge of } \mathcal{H} \right\}$$

and

$$\mathcal{P}_2 = \left\{ (A, B) \in P_s(V) \ : \ \text{both of } A \text{ and } B \text{ contain some edges of } \mathcal{H} \right\},$$

then $\mathrm{alt}(\mathcal{P}_1, \sigma) = \mathrm{alt}(\mathcal{H}, \sigma)$ and $\mathrm{alt}(\mathcal{P}_2, \sigma) = \mathrm{salt}(\mathcal{H}, \sigma)$. Therefore, in view of Lemma 1, we have the next result.

Corollary 1. *For a hypergraph $\mathcal{H} = (V, E)$ and a bijection $\sigma \ : \ [n] \rightarrow V(\mathcal{H})$, we have the following assertions.*

a) *If $d = |V| - alt(\mathcal{H}, \sigma) - 1$ and $alt(\mathcal{H}, \sigma) \neq |V|$, then there are a multiset $Z \subset S^d$ of size $|V|$ and a suitable identification of Z with V such that for any $x \in S^d$, $H(x)$ or $H(-x)$ contains some edge of $\mathcal{H}$. In particular, for $d \geq 1$, Z can be a set.*

b) *If $d = |V| - salt(\mathcal{H}, \sigma) - 1$ and $salt(\mathcal{H}, \sigma) \neq |V|$, then there are a multiset $Z \subset S^d$ of size $|V|$ and a suitable identification of Z with V such that for any $x \in S^d$, the open hemisphere $H(x)$ contains some edge of $\mathcal{H}$. In particular, for $d \geq 1$, Z can be a set.*

For two positive integers n and k, where $n > 2k$, and for the identity bijection $I : [n] \longrightarrow [n]$, one can see that $salt(\widetilde{K_n^k}, I) = 2k - 1$. Therefore, by the second part of Corollary 1, we have the strengthening of Gale's lemma given by Schrijver [16]: *there is an n-subset Z of S^{n-2k} and a suitable identification of Z with $[n]$ such that for any $x \in S^{n-2k}$, the open hemisphere $H(x)$ contains at least a 2-stable subset of $[n]$ with size k.*

Note that for any graph G, there are several hypergraphs $\mathcal{H}$ such that $KG(\mathcal{H})$ and G are isomorphic. By the help of Lemma 1 and as the second main result of this article, we provide two combinatorial approximations for two important topological lower bounds for the chromatic number of a graph G, namely, $\mathrm{coind}(B_0(G)) + 1$ and $\mathrm{coind}(B(G)) + 2$, see [18]. The quantities $\mathrm{coind}(B_0(G))$ and $\mathrm{coind}(B(G))$ are respectively the coindices of two box complexes $B_0(G)$ and $B(G)$ that will be defined in Section 2. It should be mentioned that the two inequalities $\chi(G) \geq \mathrm{coind}(B_0(G)) + 1$ $\chi(G) \geq \mathrm{coind}(B(G)) + 2$ are already proved [18] and we restate them in the following theorem just to emphasize that this theorem is an improvement of Theorem A.

Theorem 1. *Let G be a graph and $\mathcal{H} = (V, E)$ be a hypergraph such that $KG(\mathcal{H})$ and G are isomorphic. Then the following inequalities hold;*

a) $\chi(G) \geq \mathrm{coind}(B_0(G)) + 1 \geq |V(\mathcal{H})| - alt(\mathcal{H})$,

b) $\chi(G) \geq \mathrm{coind}(B(G)) + 2 \geq |V(\mathcal{H})| - salt(\mathcal{H}) + 1$.

Note that, in addition to presenting another proof for Theorem A, Theorem 1 also reveals a new way to estimate $\mathrm{coind}(B_0(G))$ and $\mathrm{coind}(B(G))$ for a graph G as well. Also, it is worth noting that the results in [1–3] confirm that we can evaluate the chromatic number of some family of graphs by computing $alt(-)$ or $salt(-)$ for an appropriate choice of a hypergraph, while it seems that it is not easy to directly determine the values of $\mathrm{coind}(B_0(-)) + 1$ and $\mathrm{coind}(B(-)) + 2$ for these graphs by topological methods.

Remark. Motivated by Corollary 1, we can assign to any hypergraph $\mathcal{H}$ two topological parameters; the *dimension of $\mathcal{H}$* and the *strong dimension of $\mathcal{H}$*, denoted respectively by $\dim(\mathcal{H})$ and $\mathrm{sdim}(\mathcal{H})$. The dimension of $\mathcal{H}$ (resp. strong dimension of $\mathcal{H}$) is the maximum integer $d \geq -1$ such that the vertices of $\mathcal{H}$ can be identified with a multiset $Z \subset S^d$ so that for any $x \in S^d$, at least one of (resp. both of) open hemispheres $H(x)$ and $H(-x)$ contains some edge of $\mathcal{H}$. In view of Corollary 1 and with a proof similar to the one of Theorem 1, one can prove the following corollary.

Corollary 2. *For a hypergraph $\mathcal{H}$, we have the following inequalities;*

a) $\mathrm{coind}(B_0(KG(\mathcal{H}))) \geq \dim(\mathcal{H}) \geq |V(\mathcal{H})| - alt(\mathcal{H}) - 1$,

b) $\mathrm{coind}(B(KG(\mathcal{H}))) \geq \mathrm{sdim}(\mathcal{H}) \geq |V(\mathcal{H})| - salt(\mathcal{H}) - 1$.

This article is organized as follows. Section 2 contains a brief review of elementary but essential preliminaries and definitions that will be needed throughout the article. Section 3 is devoted to the proof of Lemma 1 and Theorem 1. Also, in this section, as an application of the generalization of Gale's lemma (Lemma 1), we reprove a result by Chen [6] concerning the multichromatic number of stable Kneser graphs.

2 | TOPOLOGICAL PRELIMINARIES

This section is devoted to a brief review of some basic tools that we will need elsewhere in the article. However, we assume basic knowledge in topological combinatorics. More details on that topic can be found for instance in the book by Matoušek [13] or in the article by Simonyi and Tardos [17]. A $\mathbb{Z}_2$-*space* is a pair (T, ν), where T is a topological space and ν is an *involution*, i.e. $\nu : T \rightarrow T$ is a continuous map such that ν^2 is the identity map. For an $x \in T$, two points x and $\nu(x)$ are called *antipodal*. The $\mathbb{Z}_2$-space (T, ν) is called *free* if there is no $x \in T$ such that $\nu(x) = x$. For instance, one can see that the unit sphere $S^d \subset \mathbb{R}^{d+1}$ with the involution given by the antipodal map $- : x \mapsto -x$ is free. For two $\mathbb{Z}_2$-spaces (T_1, ν_1) and (T_2, ν_2), a continuous map $f : T_1 \rightarrow T_2$ is called a $\mathbb{Z}_2$-*map* if $f \circ \nu_1 = \nu_2 \circ f$. The existence of such a map is denoted by $(T_1, \nu_1) \xrightarrow{\mathbb{Z}_2} (T_2, \nu_2)$. For a $\mathbb{Z}_2$-space (T, ν), we define the $\mathbb{Z}_2$-index and $\mathbb{Z}_2$-coindex of (T, ν), respectively, as

$$\mathrm{ind}(T, \nu) = \min\{d \geq 0 \;\; (T, \nu) \xrightarrow{\mathbb{Z}_2} (S^d, -)\}$$

and

$$\mathrm{coind}(T, \nu) = \max\{d \geq 0 \;\; (S^d, -) \xrightarrow{\mathbb{Z}_2} (T, \nu)\}.$$

If for any $d \geq 0$, there is no $(T, \nu) \xrightarrow{\mathbb{Z}_2} (S^d, -)$, then we set $\mathrm{ind}(T, \nu) = \infty$. Also, if (T, ν) is not free, then $\mathrm{ind}(T, \nu) = \mathrm{coind}(T, \nu) = \infty$.

For simplicity of notation, when the involution is understood from the context, we speak about T rather than the pair (T, ν); also, we set $\mathrm{ind}(T, \nu) = \mathrm{ind}(T)$ and $\mathrm{coind}(T, \nu) = \mathrm{coind}(T)$. Throughout the article, we endow the unit sphere $S^d \subset \mathbb{R}^{d+1}$ with the involution given by the antipodal map. Note that if $T_1 \xrightarrow{\mathbb{Z}_2} T_2$, then $\mathrm{ind}(T_1) \leq \mathrm{ind}(T_2)$ and $\mathrm{coind}(T_1) \leq \mathrm{coind}(T_2)$. Two $\mathbb{Z}_2$-spaces T_1 and T_2 are $\mathbb{Z}_2$-equivalent, denoted by $T_1 \xleftrightarrow{\mathbb{Z}_2} T_2$, if $T_1 \xrightarrow{\mathbb{Z}_2} T_2$ and $T_2 \xrightarrow{\mathbb{Z}_2} T_1$. In particular, $\mathbb{Z}_2$-equivalent spaces have the same index and also same coindex.

In the following, we introduce the concept of simplicial complex that provides a bridge between combinatorics and topology. A simplicial complex can be viewed as a combinatorial object, called abstract simplicial complex, or as a topological space, called geometric simplicial complex. Here, we just review the definition of an abstract simplicial complex. However, it should be mentioned that we can assign a geometric simplicial complex to an abstract simplicial complex, called its geometric realization, and vice versa. An *abstract simplicial complex* is a pair $L = (V, K)$, where V (the vertex set of L) is a set and $K \subseteq 2^V$ (the set of simplices of L) is a hereditary collection of subsets of V, i.e. if $A \in K$ and $B \subseteq A$, then $B \in K$. Any set $A \in K$ is called a *simplex of L*. The geometric realization of an abstract simplicial complex L is denoted by $||L||$. For two abstract simplicial complexes $L_1 = (V_1, K_1)$ and $L_2 = (V_2, K_2)$, a simplicial map $f : L_1 \rightarrow L_2$ is a map from V_1 to V_2 preserving the simplices, i.e. if $A \in K_1$, then $f(A) \in K_2$. A *simplicial involution* is a simplicial map $\nu : L \rightarrow L$ such that ν^2 is the identity map. A *simplicial $\mathbb{Z}_2$-complex* is a pair (L, ν) where L is a simplicial complex and $\nu : L \rightarrow L$ is a simplicial involution. A simplicial complex (L, ν) is called *free* if there is no simplex A of L such that $\nu(A) = A$. For two simplicial $\mathbb{Z}_2$-complexes (L_1, ν_1) and (L_2, ν_2), the map $f : L_1 \rightarrow L_2$ is called a *simplicial $\mathbb{Z}_2$-map* if f is a simplicial map and $f \circ \nu_1 = \nu_2 \circ f$. The existence of a simplicial $\mathbb{Z}_2$-map $f : L_1 \rightarrow L_2$ implies the existence of a continuous $\mathbb{Z}_2$-map $||f|| : ||L_1|| \xrightarrow{\mathbb{Z}_2} ||L_2||$ that is called the geometric realization of f. We respectively use $\mathrm{ind}(L)$ and $\mathrm{coind}(L)$ to refer to $\mathrm{ind}(||L||)$ and $\mathrm{coind}(||L||)$.

The existence of a homomorphism between two graphs is an important and generally challenging problem in graph theory. In particular, in general, it is a hard task to determine the chromatic number of a graph G. In the following, we assign some free simplicial $\mathbb{Z}_2$-complexes to graphs in such a way that graph homomorphisms give rise to $\mathbb{Z}_2$-maps of the corresponding complexes. For a graph $G = (V(G), E(G))$ and two disjoint subsets $A, B \subseteq V(G)$, define $G[A, B]$ to be the induced bipartite subgraph of G whose parts are A and B, that is, a subgraph of G whose vertex set is $A \cup B$ and edge set is $\{uv \in E(G) : u \in A \text{ and } v \in B\}$.

Box Complex. For a graph $G = (V(G), E(G))$ and a subset $A \subseteq V(G)$, set

$$CN(A) = \{v \in V(G) : \ av \in E(G) \text{ for all } a \in A \ \} \subseteq V(G) \setminus A.$$

The *box complex* of a graph G, denoted by $B(G)$, is a free simplicial $\mathbb{Z}_2$-complex with vertex set $V(G) \uplus V(G) = V(G) \times [2]$ and the following set of simplices

$$\{A \uplus B : \ A, B \subseteq V(G), \ A \cap B = \emptyset, \ G[A, B] \text{ is complete bipartite and } CN(A) \neq \emptyset \neq CN(B)\}.$$

Also, one can consider another box complex $B_0(G)$ with vertex set $V(G) \uplus V(G) = V(G) \times [2]$ and the following set of simplices

$$\{A \uplus B : \ A, B \subseteq V(G), \ A \cap B = \emptyset, \ G[A, B] \text{ is complete bipartite}\}.$$

The involution on $B(G)$ (resp. $B_0(G)$) is given by interchanging the two copies of $V(G)$; that is, $(v, 1) \mapsto (v, 2)$ and $(v, 2) \mapsto (v, 1)$ for any $v \in V(G)$. In view of these involutions, one can consider $||B(G)||$ and $||B_0(G)||$ as free $\mathbb{Z}_2$-spaces. One can check that any graph homomorphism $G \to H$ implies that there are two simplicial $\mathbb{Z}_2$-maps $B(G) \overset{\mathbb{Z}_2}{\to} B(H)$ and $B_0(G) \overset{\mathbb{Z}_2}{\to} B_0(H)$; and consequently, $\operatorname{ind}(B(G)) \leq \operatorname{ind}(B(H))$, $\operatorname{coind}(B(G)) \leq \operatorname{coind}(B(H))$, $\operatorname{ind}(B_0(G)) \leq \operatorname{ind}(B_0(H))$, and $\operatorname{coind}(B_0(G)) \leq \operatorname{coind}(B_0(H))$. It is known that $B(K_n)$ and $B_0(K_n)$ are $\mathbb{Z}_2$-equivalent to S^{n-2} and S^{n-1}, respectively. Hence, $\chi(G) \geq \operatorname{ind}(B(G)) + 2 \geq \operatorname{coind}(B(G)) + 2$ and $\chi(G) \geq \operatorname{ind}(B_0(G)) + 1 \geq \operatorname{coind}(B_0(G)) + 1$. Indeed, it is known that (see [13,17,18])

$$\chi(G) \geq \operatorname{ind}(B(G)) + 2 \geq \operatorname{ind}(B_0(G)) + 1 \geq \operatorname{coind}(B_0(G)) + 1 \geq \operatorname{coind}(B(G)) + 2. \qquad (1)$$

For more about different simplicial complexes assigned to graphs and their relationships, one can see [14].

3 | PROOF OF MAIN RESULTS

This section is devoted to the proofs of main results. It should be mentioned that the following proof is based on an idea similar to that used in an interesting proof of Ziegler for Gale's lemma (see page 67 in [13]).

Proof of Lemma 1. For simplicity of notation, assume that $V = \{v_1, \ldots, v_n\}$ where $\sigma(i) = v_i$. Consider the following curve

$$\gamma = \{(1, t, t^2, \ldots, t^d) \in \mathbb{R}^{d+1} : t \in \mathbb{R}\}$$

and set $W = \{w_1, w_2, \ldots, w_n\}$, where $w_i = \gamma(i)$ for $i = 1, 2, \ldots, n$. Now, let $Z = \{z_1, z_2, \ldots, z_n\} \subseteq S^d$ be a set such that $z_i = (-1)^i \frac{w_i}{||w_i||}$ for any $1 \leq i \leq n$. Note that if $d \geq 1$, then Z is a set. Consider

the identification between V and Z such that $v_i \in V$ is identified with z_i for any $1 \le i \le n$. It can be checked that every hyperplane of $\mathbb{R}^{d+1}$ passing trough the origin intersects γ in no more than d points. Moreover, if a hyperplane intersects the curve in exactly d points, then the hyperplane cannot be tangent to the curve; and consequently, at each intersection point, the curve passes from one side of the hyperplane to the other side.

In what follows, for any $y \in S^d$, we will show that $Z_y \in \mathcal{P}$ completing the proof. On the contrary, suppose that there is a $y \in S^d$ such that $Z_y \notin \mathcal{P}$. Let h be the hyperplane passing trough the origin that contains the boundary of $H(y)$. We can move this hyperplane continuously to a position such that it still contains the origin and has exactly d points of $W = \{w_1, w_2, \ldots, w_n\}$ while during this movement no points of W crosses from one side of h to the other side. Consequently, during the aforementioned movement, no points of $Z = \{z_1, z_2, \ldots, z_n\}$ crosses from one side of h to the other side. Hence, at each of these intersections, γ passes from one side of h to the other side. Let h^+ and h^- be two open half-spaces determined by the hyperplane h. Now consider $X = (x_1, x_2, \ldots, x_n) \in \{+, -, 0\}^n \setminus \{\mathbf{0}\}$ such that

$$
x_i = \begin{cases}
0 & \text{if } w_i \text{ is on } h \\
+ & \text{if } w_i \text{ is in } h^+ \text{ and } i \text{ is even} \\
+ & \text{if } w_i \text{ is in } h^- \text{ and } i \text{ is odd} \\
- & \text{otherwise}
\end{cases}.
$$

Assume that $x_{i_1}, x_{i_2}, \ldots, x_{i_{n-d}}$ are nonzero entries of X, where $i_1 < i_2 < \cdots < i_{n-d}$. It is easy to check that any two consecutive terms of x_{i_j}'s have different signs. Since X has $n - d = \text{alt}(\mathcal{P}, \sigma) + 1$ nonzero entries, we have $\text{alt}(X) = \text{alt}(-X) = \text{alt}(\mathcal{P}, \sigma) + 1$; and therefore, both X_σ and $(-X)_\sigma$ are in $\mathcal{P}$. Also, one can see that either $X_\sigma \subseteq Z_y$ or $(-X)_\sigma \subseteq Z_y$. Therefore, since $\mathcal{P}$ is a signed-increasing property, we have $Z_y \in \mathcal{P}$ that is a contradiction. ∎

Multichromatic Number of Stable Kneser Graphs. For positive integers n, k, and s, *the s-stable Kneser graph* $\text{KG}(n, k)_s$ is an induced subgraph of $\text{KG}(n, k)$ whose vertex set is $\binom{[n]}{k}_s$. In other words, $\text{KG}(n, k)_s = \text{KG}([n], \binom{[n]}{k}_s)$. The chromatic number of stable Kneser graphs has been studied in several articles, for instance see [4,10,15]. Meunier [15] posed a conjecture about the chromatic number of stable Kneser hypergraphs that is a generalization of a conjecture of Alon et al. [4]. Restricting to Kneser graphs rather than Kneser hypergraphs, Meunier's conjecture asserts that the chromatic number of $\text{KG}(n, k)_s$ is $n - s(k - 1)$ for $n \ge sk$ and $s \ge 2$. Clearly, in view of Schrijver's result [16], this conjecture is true for $s = 2$. Moreover, for $s \ge 4$ and n sufficiently large, Jonsson [10] gave an affirmative answer to the graph case of Meunier's conjecture.

For two positive integers m and n with $n \ge m$, an m-fold n-coloring of a graph G is a homomorphism from G to $\text{KG}(n, m)$. The m-th multichromatic number of a graph G, denoted by $\chi_m(G)$, is defined as follows

$$
\chi_m(G) = \min \{n : G \to \text{KG}(n, m)\}.
$$

Note that $\chi_1(G) = \chi(G)$. An m-fold n-coloring of a graph G is called a multicoloring of G with color set $[n]$. The following conjecture of Stahl [19] has received a considerable attention in the literature.

Conjecture A. ([19]) For positive integers m, n, and k with $n \ge 2k$, we have

$$
\chi_m(\text{KG}(n, k)) = \left\lceil \frac{m}{k} \right\rceil (n - 2k) + 2m.
$$

Stahl [20] proved the accuracy of this conjecture for $k = 2, 3$ and arbitrary values of m. Chen [6] studied the multichromatic number of s-stable Kneser graphs and generalized Schrijver's result. In what follows, as an application of Lemma 1, we present another proof of Chen's result. Note that, as a special case of Chen's result, we have the chromatic number of s-stable Kneser graphs provided that s is even.

Theorem B. *[6] For positive integers n, k, and s with $n \geq sk$, if s is an even integer and $k \geq m$, then* $\chi_m(\mathrm{KG}(n,k)_s) = n - sk + sm$.

Proof. It is straightforward to check that $\chi_m(\mathrm{KG}(n,k)_s) \leq n - sk + sm$. For a proof of this observation, we refer the reader to [6]. Therefore, it is enough to show $\chi_m(\mathrm{KG}(n,k)_s) \geq n - sk + sm$. For the set $[n]$, let $\mathcal{P} = \mathcal{P}(n,k,s) \subseteq P_s([n])$ be a signed-increasing property such that $(A, B) \in \mathcal{P}$ if both A and B contain at least $\frac{s}{2}$ pairwise disjoint s-stable k-subsets of $[n]$. One can see that $\mathrm{alt}(\mathcal{P}, I) = sk - 1$ where $I : [n] \rightarrow [n]$ is the identity bijection. Thus, by Lemma 1, for $d = n - sk$, there exists a multiset $Z \subset S^d$ of size n such that under a suitable identification of Z with V, for any $x \in S^d$, we have $Z_x \in \mathcal{P}$. In other words, for any $x \in S^d$, the open hemisphere $H(x)$ contains at least $\frac{s}{2}$ pairwise disjoint vertices of $\mathrm{KG}(n,k)_s$. Now let $c : V(\mathrm{KG}(n,k)_s) \rightarrow \binom{C}{m}$ be an m-fold C-coloring of $\mathrm{KG}(n,k)_s$. For each $i \in \{1, 2, \ldots, C - sm + 1\}$, define A_i to be a set consisting of all $x \in S^d$ such that $H(x)$ contains some vertex v of $\mathrm{KG}(n,k)_s$ having the color i in its assigned color set, i.e. $i \in c(v)$. Furthermore, define $A_{C-sm+2} = S^d \setminus \bigcup_{i=1}^{C-sm+1} A_i$. One can check that each A_i contains no pair of antipodal points, i.e. $A_i \cap (-A_i) = \varnothing$; and also, for each $i \in \{1, 2, \ldots, C - sm + 1\}$, A_i is an open subset of S^d and A_{C-sm+2} is thus closed. Note that the Borsuk–Ulam theorem asserts that for any covering of S^d by $d + 1$ sets $B_1, \ldots, B_{d+1}$, each B_i open or closed, there exists an i such that B_i contains a pair of antipodal points. Accordingly, we must have $C - sm + 2 \geq d + 2 = n - sk + 2$ that completes the proof. ∎

In what follows, in view of Lemma 1 and with a similar approach as in proof of Proposition 8 of [17], we prove Theorem 1.

Proof of Theorem 1. Let $\sigma : [n] \rightarrow V(\mathcal{H})$ be an arbitrary bijection. To prove the first part, set $d = |V| - \mathrm{alt}(\mathcal{H}, \sigma) - 1$. In view of inequalities of (1) and the definition of $\mathrm{alt}(\mathcal{H})$, it is sufficient to prove that $\mathrm{coind}(B_0(G)) + 1 \geq |V| - \mathrm{alt}(\mathcal{H}, \sigma)$. If $d \leq 0$, then the assertion follows from the definition of coindex. Hence, suppose $d \geq 1$. Now in view of Corollary 1, there exists an n-set $Z \subset S^d$ and an identification of Z with V such that for any $x \in S^d$, at least one of open hemispheres $H(x)$ and $H(-x)$ contains some edge of $\mathcal{H}$.

For any vertex A of $\mathrm{KG}(\mathcal{H})$ and any $x \in S^d$, define $D_A(x)$ to be the smallest distance of a point in $A \subset S^d$ from the set $S^d \setminus H(x)$. Note that $D_A(x) > 0$ if and only if $H(x)$ contains A. Define

$$D(x) = \sum_{A \in E}(D_A(x) + D_A(-x)).$$

Since, for any $x \in S^d$, at least one of $H(x)$ and $H(-x)$ contains some edge of $\mathcal{H}$, we have $D(x) > 0$. Thus, the map

$$f(x) = \frac{1}{D(x)}\left(\sum_{A \in E} D_A(x)||(A,1)|| + \sum_{A \in E} D_A(-x)||(A,2)||\right)$$

is a $\mathbb{Z}_2$-map from S^d to $||B_0(\mathrm{KG}(\mathcal{H}))||$. It implies $\mathrm{coind}(B_0(G)) \geq d$.

b) To prove the second part, set $d = n - \mathrm{salt}(\mathcal{H}, \sigma) - 1$. In view of the inequalities of (1) and the definition of $\mathrm{salt}(\mathcal{H})$, it is sufficient to prove that $\mathrm{coind}(B(G)) + 2 \geq |V| - \mathrm{salt}(\mathcal{H}, \sigma) + 1$. If $d \leq 0$, then

the assertion follows from the definition of coindex. Hence, suppose $d \geq 1$. Now in view of Corollary 1, there is an n-set $Z \subset S^d$ and an identification of Z with V such that for any $x \in S^d$, the open hemisphere $H(x)$ contains some edge of $\mathcal{H}$. Define $D(x) = \sum_{A \in E} D_A(x)$. Since the open hemisphere $H(x)$ contains some edge of $\mathcal{H}$, we have $D(x) > 0$ for each $x \in S^d$. Thus, the map

$$f(x) = \frac{1}{2D(x)} \sum_{A \in E} D_A(x)||(A, 1)|| + \frac{1}{2D(-x)} \sum_{A \in E} D_A(-x)||(A, 2)||$$

is a $\mathbb{Z}_2$-map from S^d to $||B(\mathrm{KG}(\mathcal{H}))||$. This implies $\mathrm{coind}(B(G)) \geq d$. ∎

ACKNOWLEDGMENT

ORCID

Hossein Hajiabolhassan (iD) http://orcid.org/0000-0001-8146-9764

REFERENCES

[1] M. Alishahi and H. Hajiabolhassan, *On the chromatic number of general Kneser hypergraphs*, J. Combin. Theory, Ser. B **115** (2015), 186–209.

[2] M. Alishahi and H. Hajiabolhassan, On the chromatic number of matching graphs, ArXiv e-prints, 2015.

[3] M. Alishahi and H. Hajiabolhassan, *Chromatic number via Turán number*, Discrete Math. **340**(10) (2017), 2366–2377.

[4] N. Alon, L. Drewnowski, and T. Łuczak, *Stable Kneser hypergraphs and ideals in ℕ with the Nikodým property*, Proc. Amer. Math. Soc. **137**(2) (2009), 467–471.

[5] I. Bárány, *A short proof of Kneser's conjecture*, J. Combin. Theory Ser. A **25**(3) (1978), 325–326.

[6] P.-A. Chen, *On the multichromatic number of s-stable Kneser graphs*, J. Graph Theory **79**(3) (2015), 233–248.

[7] V. L. Dol'nikov, *A combinatorial inequality*, Sibirsk. Mat. Zh. **29**(3) (1988), 53–58, 219.

[8] D. Gale, Neighboring vertices on a convex polyhedron. In *Linear inequalities and related system*, Annals of Mathematics Studies, no. 38, . Princeton University Press, Princeton, NJ, 1956, pp. 255–263.

[9] J. E. Greene, *A new short proof of Kneser's conjecture*, Am. Math. Monthly **109**(10) (2002), 918–920.

[10] J. Jonsson, On the chromatic number of generalized stable Kneser graphs. *manuscript*, 2012. https://people.kth.se/jakobj/doc/submitted/stablekneser.pdf

[11] M. Kneser, *Ein Satz über abelsche Gruppen mit Anwendungen auf die Geometrie der Zahlen*, Math. Z. **61** (1955), 429–434.

[12] L. Lovász, *Kneser's conjecture, chromatic number, and homotopy*, J. Combin. Theory Ser. A **25**(3) (1978), 319–324.

[13] J. Matoušek, *Using the Borsuk-Ulam theorem*. Universitext. Springer-Verlag, Berlin, 2003. Lectures on topological methods in combinatorics and geometry, Written in cooperation with Anders Björner and Günter M. Ziegler.

[14] J. Matoušek and G. M. Ziegler, *Topological lower bounds for the chromatic number: A hierarchy*, Jahresber. Deutsch. Math.-Verein. **106**(2) (2004), 71–90.

[15] F. Meunier, *The chromatic number of almost stable Kneser hypergraphs*, J. Combin. Theory Ser. A **118**(6) (2011), 1820–1828.

[16] A. Schrijver, *Vertex-critical subgraphs of Kneser graphs*, Nieuw Arch. Wisk. (3) **26**(3) (1978), 454–461.

[17] G. Simonyi and G. Tardos, *Local chromatic number, Ky Fan's theorem and circular colorings*, Combinatorica **26**(5) (2006), 587–626.

[18] G. Simonyi, G. Tardos, and S. T. Vrećica, *Local chromatic number and distinguishing the strength of topological obstructions*, Trans. Amer. Math. Soc. **361**(2) (2009), 889–908.

[19] S. Stahl, *n-tuple colorings and associated graphs*, J. Combin. Theory Ser. B **20**(2) (1976), 185–203.

[20] S. Stahl, *The multichromatic numbers of some Kneser graphs*, Discrete Mathematics **185** (1998), 287–291.

[21] A. W. Tucker, Some topological properties of disk and sphere. In *Proc. First Canadian Math. Congress*, Montreal, 1945, pp. 285–309. University of Toronto Press, Toronto, 1946.