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THE VORONOVSKAYA THEOREM FOR SOME LINEAR POSITIVE OPERATORS IN EXPONENTIAL WEIGHT SPACES

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Abstract

In this note we give the Voronovskaya theorem for some linear positive operators of the Szasz-Mirakjan type defined in the space of functions continuous on $[0, +\infty)$ and having the exponential growth at infinity.

Some approximation properties of these operators are given in [3], [4].

1. Preliminaries

1.1. Let $R_0 := [0, +\infty)$, $N := \{1, 2, \dots\}$, $N_0 := N \cup \{0\}$ and let $w_r(\cdot)$, $r > 0$, be the weight function defined on R_0 by the formula

$$(1) \quad w_r(x) := e^{-rx}.$$

Similarly as in [1] we denote by C_r , $r > 0$, the space of real-valued functions f defined on R_0 and such that $w_r f$ is a uniformly continuous and bounded function on R_0 . The norm in C_r is defined by

$$\|f\|_r := \sup_{x \in R_0} w_r(x) |f(x)|.$$

For a fixed $r > 0$ let

$$C_r^2 := \{f \in C_r : f', f'' \in C_r\}.$$

1.2. In [3] were introduced the following operators of the Szasz-Mirakjan type for functions $f \in C_r$, $r > 0$,

$$(2) \quad L_n^{(1)}(f; x) := \sum_{k=0}^{\infty} p_{n,k}(x) f\left(\frac{2k}{n}\right),$$

$$(3) \quad L_n^{(2)}(f; x) := \sum_{k=0}^{\infty} p_{n,k}(x) \frac{n}{2} \int_{I_{n,k}} f(t) dt,$$

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$x \in R_0$, $n \in N$, where

$$(4) \quad p_{n,k}(x) := \frac{1}{\cosh nx} \frac{(nx)^{2k}}{(2k)!}, \quad k \in N_0,$$

$\sinh x$, $\cosh x$, $\tanh x$ are the elementary hyperbolic functions and $I_{n,k} := [\frac{2k}{n}, \frac{2k+2}{n}]$, $k \in N_0$.

In [4] were introduced the operators

$$(5) \quad L_n^{(3)}(f; x) := \frac{f(0)}{1 + \sinh nx} + \sum_{k=0}^{\infty} q_{n,k}(x) f\left(\frac{2k+1}{n}\right),$$

$$(6) \quad L_n^{(4)}(f; x) := \frac{f(0)}{1 + \sinh nx} + \sum_{k=0}^{\infty} q_{n,k}(x) \frac{n}{2} \int_{I_{n,k}^*} f(t) dt,$$

$x \in R_0$, $n \in N$, where

$$(7) \quad q_{n,k}(x) := \frac{1}{1 + \sinh nx} \frac{(nx)^{2k+1}}{(2k+1)!},$$

and $I_{n,k}^* := [\frac{2k+1}{n}, \frac{2k+3}{n}]$ for $k \in N_0$.

We observe that the above operators are linear positive operators well-defined on every space C_r , $r > 0$, and

$$(8) \quad L_n^{(i)}(1; x) = 1, \quad 1 \leq i \leq 4,$$

for all $x \in R_0$ and $n \in N$.

In [3] and [4] it was proved that $L_n^{(i)}$, $1 \leq i \leq 4$, are operators from C_r into C_s for every fixed $s > r > 0$ provided n is large enough. Moreover in [3], [4] some approximation properties of these operators were given. In particular in [3], [4] we proved the following

Theorem A. *Suppose that r, s, n_0 are fixed numbers such that $s > r > 0$, $n_0 \in N$ and $n_0 > r (\ln \frac{s}{r})^{-1}$. If $f \in C_r$, then there exists a positive constant $M_1 \equiv M_1(n_0, r, s)$ depending only on n_0, r, s such that for all $x \in R_0$, $n_0 < n \in N$ and $1 \leq i \leq 4$*

$$w_s(x) \left| L_n^{(i)}(f; x) - f(x) \right| \leq M_1 \omega \left(f, C_r; \sqrt{\frac{x+1}{n}} \right),$$

where $\omega(f; C_r; \cdot)$ is the modulus of continuity of f , i.e.,

$$\omega(f; C_r; t) := \sup_{0 < h \leq t} \|f(\cdot + h) - f(\cdot)\|_r.$$

2. Auxiliary results

In this part we shall give some properties of the operators $L_n^{(i)}$. Let

$$\begin{aligned}
 (9) \quad S(nx) &:= \frac{\sinh nx}{1 + \sinh nx}, \\
 T(nx) &:= \frac{\cosh nx}{1 + \sinh nx}, \\
 V(nx) &:= 1 - \tanh nx,
 \end{aligned}$$

for $n \in N$ and $x \in R_0$. By elementary calculations from (2)-(8) and (9) we obtain

Lemma 1. For all $x \in R_0$ and $n \in N$ we have

$$\begin{aligned}
 L_n^{(1)}(t-x; x) &= -xV(nx), \\
 L_n^{(1)}((t-x)^2; x) &= \left(2x^2 - \frac{x}{n}\right) V(nx) + \frac{x}{n}, \\
 L_n^{(1)}((t-x)^4; x) &= \left(8x^4 - \frac{12x^3}{n} + \frac{4x^2}{n^2} - \frac{x}{n^3}\right) V(nx) + \frac{3x^2}{n^2} + \frac{x}{n^3}, \\
 L_n^{(2)}(t-x; x) &= -xV(nx) + \frac{1}{n}, \\
 L_n^{(2)}((t-x)^2; x) &= \left(2x^2 - \frac{3x}{n}\right) V(nx) + \frac{x}{n} + \frac{4}{3n^2}, \\
 L_n^{(2)}(t-x)^4; x) &= \left(8x^4 - \frac{28x^3}{n} + \frac{32x^2}{n^2} - \frac{21x}{n^3}\right) V(nx) + \frac{12x}{n^3} + \frac{16}{5n^4}, \\
 L_n^{(3)}(t-x; x) &= x(T(nx) - 1), \\
 L_n^{(3)}((t-x)^2; x) &= x^2(S(nx) - 2T(nx) + 1) + \frac{x}{n}V(nx), \\
 L_n^{(3)}((t-x)^4; x) &= x^4(7S(nx) - 8T(nx) + 1) + \frac{12x^3}{n}(T(nx) - S(nx)) \\
 &\quad + \frac{x^2}{n^2}(7S(nx) - 4T(nx)) + \frac{x}{n^3}T(nx), \\
 L_n^{(4)}(t-x; x) &= x(T(nx) - 1) + \frac{1}{n}S(nx), \\
 L_n^{(4)}((t-x)^2; x) &= x^2(S(nx) - 2T(nx) + 1) \\
 &\quad + \frac{2x}{n}(T(nx) - S(nx)) + \frac{4}{3n^2}S(nx), \\
 L_n^{(4)}((t-x)^4; x) &= x^4(7S(nx) - 8T(nx) + 1) + \frac{28x^3}{n}(T(nx) - S(nx)) \\
 &\quad + \frac{x^2}{n^2}(35S(nx) - 32T(nx)) + \frac{17x}{n^3}T(nx).
 \end{aligned}$$

Using Lemma 1, we shall prove two lemmas.

Lemma 2. *For every fixed $x_0 \in R_0$ one has*

$$(10) \quad \lim_{n \rightarrow \infty} nL_n^{(i)}(t - x_0; x_0) = \begin{cases} 0 & \text{if } i = 1, 3, \\ 1 & \text{if } i = 2, 4, \end{cases}$$

and

$$(11) \quad \lim_{n \rightarrow \infty} nL_n^{(i)}((t - x_0)^2; x_0) = x_0 \quad \text{for } 1 \leq i \leq 4.$$

Proof: We shall prove only (10) and (11) for $i = 3$, because the proof for $i = 1, 2, 4$ is analogous.

By Lemma 1 and (9) we have

$$nL_n^{(3)}(t - x; x) = \frac{nx}{e^{2nx}(1 + \sinh nx)} - \frac{nx}{(1 + \sinh nx)},$$

$$nL_n^{(3)}((t - x)^2; x) = \frac{nx^2}{1 + \sinh nx} - \frac{2nx^2}{e^{nx}(1 + \sinh nx)} + \frac{x \cosh nx}{1 + \sinh nx},$$

for every $x \in R_0$ and $n \in N$, which immediately yield (10) and (11). ■

Lemma 3. *For every fixed $x_0 \in R_0$ there exists a positive constant $M_2(x_0)$, depending only on x_0 , such that for all $n \in N$*

$$(12) \quad L_n^{(i)}((t - x_0)^4; x_0) \leq M_2(x_0)n^{-2}, \quad 1 \leq i \leq 4.$$

Proof: For example we shall prove (12) for $L_n^{(1)}$. By (9) we have for $n \in N$, $p \in N$ and $x \in R_0$

$$0 \leq x^p V(nx) = \frac{2x^p}{e^{2nx} + 1} \leq 2^{1-p} p! n^{-p}.$$

Applying the above inequality to the formula given in Lemma 1, we obtain

$$L_n^{(1)}((t - x_0)^4; x_0) \leq \frac{47}{n^4} + \frac{3x_0^2}{n^2} + \frac{x_0}{n^3} \leq M_2(x_0)n^{-2},$$

for every fixed $x_0 \geq 0$ and for all $n \in N$. ■

The proof of (12) for $i = 2, 3, 4$ is similar.

In the papers [3] (for $L_n^{(i)}$, $i = 1, 2$) and [4] (for $L_n^{(i)}$, $i = 3, 4$) we proved the following two lemmas.

Lemma 4. *Let $s > r > 0$ and let n_0 be a natural number such that*

$$(13) \quad n_0 > r \left(\ln \frac{s}{r} \right)^{-1}.$$

Then there exists a positive constant $M_3 \equiv M_3(r, s, n_0)$ depending only on r, s, n_0 such that for all $n > n_0$ and $i = 1, 2, 3, 4$

$$\left\| L_n^{(i)} \left(\frac{1}{w_r(t)}; \cdot \right) \right\|_s \leq M_3.$$

Lemma 5. *Suppose that r, s and n_0 are a numbers as in Lemma 4. Then there exists a positive constant $M_4 \equiv M_4(r, s, n_0)$ depending only on r, s, n_0 such that for all $x \geq 0, n > n_0$ and $i = 1, 2, 3, 4$*

$$(14) \quad w_s(x) L_n^{(i)} \left(\frac{(t-x)^2}{w_r(t)}; x \right) \leq M_3 \frac{x+1}{n}.$$

Applying the above lemmas, we shall prove

Lemma 6. *Suppose that x_0 is a fixed point on R_0 and $\varphi(\cdot; x_0)$ is a function belonging to a give space $C_r, r > 0$, such that $\lim_{t \rightarrow \infty} \varphi(t; x_0) = 0, (\lim_{t \rightarrow 0+} \varphi(t; 0) = 0)$. Then*

$$(15) \quad \lim_{n \rightarrow \infty} L_n^{(i)} (\varphi(t; x_0); x_0) = 0 \quad \text{for } 1 \leq i \leq 4.$$

Proof: We shall prove (15) for $i = 1$, because the proof of (15) for $i = 2, 3, 4$ is analogous.

Choose $\varepsilon > 0$ and M_3 as in Lemma 4. Then by the properties of $\varphi(\cdot; x_0)$ there exist positive constants $\delta \equiv \delta(\varepsilon, M_3)$ and M_5 such that

$$w_r(t) |\varphi(t; x_0)| < \frac{\varepsilon}{2 M_3} \quad \text{for } |t - x_0| < \delta,$$

$$w_r(t) |\varphi(t; x_0)| < M_5 \quad \text{for } t \geq 0.$$

Denoting by $Q_{n,1} := \{k \in N_0 : |\frac{2k}{n} - x_0| < \delta\}$ and $Q_{n,2} := \{k \in N_0 : |\frac{2k}{n} - x_0| \geq \delta\}$, we get for $s > r$ and $n > n_0$ by (1)-(4) and Lemma 4

$$\begin{aligned} w_s(x_0) \left| L_n^{(1)} (\varphi(t; x_0); x_0) \right| &\leq w_s(x_0) \sum_{k=0}^{\infty} p_{n,k}(x_0) \left| \varphi \left(\frac{2k}{n}; x_0 \right) \right| \\ &= w_s(x_0) \sum_{k \in Q_{n,1}} p_{n,k}(x_0) \left| \varphi \left(\frac{2k}{n}; x_0 \right) \right| \\ &\quad + w_s(x_0) \sum_{k \in Q_{n,2}} p_{n,k}(x_0) \left| \varphi \left(\frac{2k}{n}; x_0 \right) \right| \\ &:= \sum_1 + \sum_2 \end{aligned}$$

and

$$\begin{aligned} \sum_1 &< \frac{\varepsilon}{2M_3} w_s(x_0) \sum_{k=0}^{\infty} p_{n,k}(x_0) \left(w_r \left(\frac{2k}{n} \right) \right)^{-1} < \frac{\varepsilon}{2}, \\ \sum_2 &\leq M_5 w_s(x_0) \sum_{k \in Q_{n,2}} p_{n,k}(x_0) \left(w_r \left(\frac{2k}{n} \right) \right)^{-1}. \end{aligned}$$

Since $1 \leq \delta^{-2} \left(\frac{2k}{n} - x_0 \right)^2$ if $\left| \frac{2k}{n} - x_0 \right| \geq \delta$, we have

$$\begin{aligned} \sum_2 &\leq M_5 \delta^{-2} w_s(x_0) \sum_{k \in Q_{n,2}} p_{n,k}(x_0) \left(w_r \left(\frac{2k}{n} \right) \right)^{-1} \left(\frac{2k}{n} - x_0 \right)^2 \\ &\leq M_5 \delta^{-2} w_s(x_0) L_n^{(1)} \left(\frac{(t-x_0)^2}{w_r(t)}; x_0 \right), \end{aligned}$$

wich by (14) and (13) yields

$$\sum_2 \leq M_5 M_4 \frac{x_0 + 1}{n \delta^2} \quad \text{for all } n > n_0.$$

It is obvious that for fixed numbers $\varepsilon > 0$, $\delta > 0$, $M_3 > 0$, $M_4 > 0$, $n_0 \in N$ and $x_0 \geq 0$ there exist a natural number $n_1 > n_0$ depending on the above parameters such that for all $n_1 < n \in N$

$$M_4 M_5 \frac{x_0 + 1}{n \delta^2} < \frac{\varepsilon}{2}.$$

Hence we have

$$\sum_2 < \frac{\varepsilon}{2} \quad \text{for all } n > n_1.$$

Consequently,

$$w_s(x_0) |L_n^{(1)}(\varphi(t; x_0); x_0)| < \varepsilon \quad \text{for } n > n_1,$$

which proves that

$$\lim_{n \rightarrow \infty} w_s(x_0) L_n^{(1)}(\varphi(t; x_0); x_0) = 0.$$

From this and (1) assertion (15) follows for x_0 and $i = 1$. Thus the proof is completed. ■

3. Theorems of the Voronovskaya type

The Voronovskaya theorem for the Bernstein operators is given in [2].

We shall prove a similar theorem for the operators $L_n^{(i)}$.

Theorem 1. *Let $f \in C_r^2$ with some $r > 0$. Then*

$$(16) \quad \lim_{n \rightarrow \infty} n \left\{ L_n^{(i)}(f; x) - f(x) \right\} = \frac{x}{2} f''(x)$$

for every $x \in R_0$ and $i = 1, 3$.

Proof: Let $x_0 \geq 0$ be an arbitrary fixed point and $i = 1$. By the Taylor formula we have for $t \geq 0$

$$(17) \quad f(t) = f(x_0) + f'(x_0)(t - x_0) + \frac{1}{2} f''(x_0)(t - x_0)^2 + \psi(t; x_0)(t - x_0)^2,$$

where $\psi(\cdot; x_0)$ is a function belonging to the space C_r and $\lim_{t \rightarrow x_0} \psi(t; x_0) = 0$. By (2), (8) and (17) we get

$$(18) \quad L_n^{(1)}(f(t); x_0) = f(x_0) + f'(x_0) L_n^{(1)}(t - x_0; x_0) \\ + \frac{1}{2} f''(x_0) L_n^{(1)}((t - x_0)^2; x_0) + L_n^{(1)}(\psi(t; x_0)(t - x_0)^2; x_0)$$

for every $n \in N$. Using Lemma 2, we have

$$(19) \quad \lim_{n \rightarrow \infty} n L_n^{(1)}(t - x_0; x_0) = 0, \\ \lim_{n \rightarrow \infty} n L_n^{(1)}((t - x_0)^2; x_0) = x_0.$$

By (2) and the Hölder inequality we have for every $n \in N$

$$(20) \quad \left| L_n^{(1)}(\psi(t; x_0)(t - x_0)^2; x_0) \right| \\ \leq \left\{ L_n^{(1)}(\psi^2(t; x_0); x_0) \right\}^{\frac{1}{2}} \left\{ L_n^{(1)}((t - x_0)^4; x_0) \right\}^{\frac{1}{2}}.$$

Since for the function $\varphi(t; x_0) := \psi^2(t; x_0)$, $t \geq 0$, we have $\varphi(\cdot; x_0) \in C_{2r}$ and $\lim_{t \rightarrow x_0} \varphi(t; x_0) = 0$, we get by Lemma 6

$$(21) \quad \lim_{n \rightarrow \infty} L_n^{(1)}(\psi^2(t; x_0); x_0) \equiv \lim_{n \rightarrow \infty} L_n^{(1)}(\varphi(t; x_0); x_0) = 0.$$

Applying (21) and (12) to (20), we obtain

$$(22) \quad \lim_{n \rightarrow \infty} n L_n^{(1)}(\psi(t; x_0)(t - x_0)^2; x_0) = 0.$$

Now we immediately obtain (16) for a given x_0 and $i = 1$ from (18) by (19) and (22). This proves the desired assertion for $i = 1$. ■

Similarly we can prove the following

Theorem 2. *Suppose that $f \in C_r^2$ with some $r > 0$. Then*

$$\lim_{n \rightarrow \infty} n \left\{ L_n^{(i)}(f; x) - f(x) \right\} = f'(x) + \frac{x}{2} f''(x)$$

for every $x \in R_0$ and $i = 2, 4$.

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