

Publicacions Matemàtiques, Vol 41 (1997), 481–487.

## ***P*-NILPOTENT COMPLETION IS NOT IDEMPOTENT**

GEOK CHOO TAN

### *Abstract*

Let  $P$  be an arbitrary set of primes. The  $P$ -nilpotent completion of a group  $G$  is defined by the group homomorphism  $\eta : G \rightarrow G_{\widehat{P}}$  where  $G_{\widehat{P}} = \operatorname{invlim}(G/\Gamma_i G)_P$ . Here  $\Gamma_2 G$  is the commutator subgroup  $[G, G]$  and  $\Gamma_i G$  the subgroup  $[G, \Gamma_{i-1} G]$  when  $i > 2$ . In this paper, we prove that  $P$ -nilpotent completion of an infinitely generated free group  $F$  does not induce an isomorphism on the first homology group with  $\mathbf{Z}_P$  coefficients. Hence,  $P$ -nilpotent completion is not idempotent. Another important consequence of the result in homotopy theory (as in [4]) is that any infinite wedge of circles is  $R$ -bad, where  $R$  is any subring of rationals.

### 1. Introduction

For a group  $G$ , we denote by  $\Gamma_2 G$  the commutator subgroup  $[G, G]$  and  $\Gamma_i G$  the subgroup  $[G, \Gamma_{i-1} G]$  when  $i > 2$ . A group  $G$  is *nilpotent* if  $\Gamma_i(G)$  is trivial for some  $i$ . The *nilpotency class*  $\operatorname{nil}(G)$  of  $G$  is the least  $c$  such that  $\Gamma_c(G)$  is trivial. Let  $P$  be a set of prime numbers. There is a well-known  $P$ -localization in the category of nilpotent groups, [7]. We denote this localization on a nilpotent group  $N$  by  $e : N \rightarrow N_P$ .

The  $P$ -nilpotent completion or  $\mathbf{Z}_P$ -completion of a group  $G$  is defined to be the group homomorphism  $\eta : G \rightarrow G_{\widehat{P}}$  where  $G_{\widehat{P}} = \operatorname{invlim}(G/\Gamma_i G)_P$ , with  $i$  running through all finite ordinals. For each  $i$ , the group homomorphism  $G \rightarrow (G/\Gamma_i G)_P$  defines a localization on the category  $\mathcal{G}$  of groups. Its universal property gives rise to a natural map  $(G_{\widehat{P}}/\Gamma_i G_{\widehat{P}})_P \rightarrow (G/\Gamma_i G)_P$ . Passing to inverse limit, we obtain a natural transformation  $\chi : (G_{\widehat{P}})_{\widehat{P}} \rightarrow G_{\widehat{P}}$  so that  $((\ )_{\widehat{P}}, \eta, \chi)$  is a monad on  $\mathcal{G}$ .

Let  $F$  be a free group on an infinitely countable set of generators. In [4, Proposition IV.5.4], it is proved that the abelianization of  $\eta : F \rightarrow F_{\widehat{\mathbf{Z}}} = \operatorname{invlim}(F/\Gamma_i F)$  is not an isomorphism. This result is used to verify that  $\mathbf{Z}$ -completion (which is  $P$ -nilpotent completion when  $P$  is the set of all primes) is not idempotent in [3].

We study these proofs closely and obtain a similar proof of the non-idempotence of  $P$ -nilpotent completion for any set  $P$  of primes. We use results from orthogonal pairs, idempotent monads and the  $P$ -localization on the category of nilpotent groups.

Although the  $P$ -nilpotent completion is not idempotent on the category of groups, a procedure to obtain an idempotent monad from it is described in [5]. This turns out to be the minimal  $P$ -localization, which is also obtained in [2]. It is the “smallest” (in the sense that it provides the least local objects) idempotent monad which extends  $P$ -localization on the category of nilpotent groups to the category of groups. This minimal  $P$ -localization coincides with the  $P$ -nilpotent completion on groups which have finitely generated abelianization [3] and groups with stable lower central series [2].

## 2. The $P$ -nilpotent completion is not idempotent

Let  $\mathcal{C}$  be a category,  $X$  be an object of  $\mathcal{C}$  and  $f : A \rightarrow B$  be a morphism of  $\mathcal{C}$ . Then  $X$  and  $f$  are said to be *orthogonal* to each other, denoted by  $X \perp f$  or  $f \perp X$ , if  $f^* : \mathcal{C}(B, X) \cong \mathcal{C}(A, X)$ . For a class  $D$  of objects in  $\mathcal{C}$ , the *orthogonal complement* of  $D$  in  $\mathcal{C}$ , denoted by  $D^\perp$ , is the class of morphisms orthogonal to every object in  $D$ . Dually, the orthogonal complement of  $S$  can be defined for a class  $S$  of morphisms. An *orthogonal pair*  $(S, D)$  in  $\mathcal{C}$  comprises a collection  $S$  of morphisms in  $\mathcal{C}$  and a collection  $D$  of objects in  $\mathcal{C}$  satisfying  $S = D^\perp$  and  $D = S^\perp$ . Every idempotent monad (see [8, p. 133]) (also known as localization in [6]) is associated with a unique orthogonal pair.

Let  $a_1, a_2, \dots$  be elements of a group  $G$ . We define  $[a_1, a_2] = a_1^{-1}a_2^{-1}a_1a_2$  and  $[a_1, a_2, \dots, a_k] = [[a_1, \dots, a_{k-1}], a_k]$  recursively for  $k \geq 3$ .

**Proposition 1.** *Let  $F$  be the free group on  $a_1, \dots, a_k$ . For every positive integer  $n$ ,  $[a_1, \dots, a_k]^n$  does not belong to the subgroup of  $\Gamma_2 F$  that is generated by  $\Gamma_{k+1} F$  and  $\Gamma_2 \Gamma_2 F$ .*

*Proof:* Replacing  $F$  by the quotient  $F/\langle \Gamma_{k+1} F, \Gamma_2 \Gamma_2 F \rangle$ , the proposition becomes: for each  $n$ , there exists a group  $G$  with the following properties: (i) The commutator subgroup  $\Gamma_2 G$  is abelian, (ii)  $G$  is nilpotent of class  $k + 1$ , and (iii) there exists  $x \in \Gamma_k G$  such that  $x^n \neq 1$ .

However, it is enough to pick a prime  $p$  that does not divide  $n$  and find a  $p$ -group  $G$  such that  $\Gamma_2 G$  is abelian and  $G$  is nilpotent of class  $k + 1$ . For any positive integer  $m$ , consider the  $\mathbf{Z}/p$  vector space  $V$  on a

basis  $\{v_1, v_2, \dots, v_{p^m}\}$  and let  $\sigma \in GL(V)$  where

$$\sigma(v_i) = \begin{cases} v_i + v_{i+1} & \text{if } i \leq p^m - 1 \\ v_{p^m} & \text{if } i = p^m. \end{cases}$$

For each positive integer  $j \leq p^m$ ,

$$\sigma^j(v_i) = \begin{cases} \sum_{l=0}^j \binom{j}{l} v_{i+l} & \text{if } i \leq p^m - j \\ \sum_{l=0}^r \binom{j}{l} v_{i+l} & \text{if } i > p^m - j, \text{ where } r = p^m - i \end{cases}$$

so that the order of  $\sigma$  is  $p^m$ . The semi-direct product group of  $V$  and  $\langle \sigma \rangle$  is a  $p$ -group whose commutator subgroup is abelian and has nilpotency class  $p^m$  (see [1] and [9]). By choosing  $m \geq k + 1$  and factoring this semi-direct product group by the  $k + 1$ -th lower central term we obtain a group  $G$  with the required properties. ■

Let  $P$  be a fixed set of prime numbers. We use the notation  $n \in P$  to mean all prime divisors of  $n$  are in  $P$  and  $P'$  to denote the complement of  $P$  in the set of all primes. A group  $G$  is said to be  $P$ -local if the map  $g \mapsto g^n$  is a bijection for all  $n \in P'$ . A group homomorphism  $f : G \rightarrow K$  is said to be (i)  $P$ -injective if for any two elements  $g_1, g_2 \in G$  such that  $f(g_1) = f(g_2)$ , there exists an integer  $n \in P'$  such that  $g_1^n = g_2^n$ ; (ii)  $P$ -surjective if for every  $k \in K$ , there exists an integer  $n \in P'$  such that  $k^n \in \text{Im}f$ ; and (iii)  $P$ -bijective if  $f$  is both  $P$ -injective and  $P$ -surjective.

On the category of nilpotent groups, there is a well-known  $P$ -localization [7], which is denoted by  $e : N \rightarrow N_P$  for each nilpotent group  $N$ , where  $N_P$  is  $P$ -local nilpotent and  $e$  is a  $P$ -bijection.

The  $P$ -nilpotent completion or  $\mathbf{Z}_P$ -completion of a group  $G$  is defined to be the group homomorphism  $\eta : G \rightarrow G_{\widehat{P}}$  induced by the group homomorphisms  $G \rightarrow G/\Gamma_i G \xrightarrow{e} (G/\Gamma_i G)_P$ , where  $G_{\widehat{P}} = \text{invlim}(G/\Gamma_i G)_P$ , with  $i$  running through all finite ordinals. For each  $i$ , the above group homomorphism  $G \rightarrow (G/\Gamma_i G)_P$  defines an idempotent monad on  $\mathcal{G}$  whose universal property enables us to complete the following diagram

$$\begin{array}{ccc} G_{\widehat{P}} & \longrightarrow & (G_{\widehat{P}}/\Gamma_i G_{\widehat{P}})_P \\ \downarrow & & \\ (G/\Gamma_i G)_P & & \end{array}$$

by a unique map  $(G_{\widehat{P}}/\Gamma_i G_{\widehat{P}})_P \rightarrow (G/\Gamma_i G)_P$ . Passing to inverse limits, we obtain a natural transformation  $\chi : (G_{\widehat{P}})_{\widehat{P}} \rightarrow G_{\widehat{P}}$  so that  $((\ )_{\widehat{P}}, \eta, \chi)$  is a monad on  $\mathcal{G}$ .

Let  $\mathcal{G}$  be the category of groups and  $\mathcal{G}'$  be the full subcategory of groups  $G$  such that the natural homomorphism  $G_{\widehat{P}} \rightarrow (G_{\widehat{P}})_{\widehat{P}}$  is an isomorphism. Then  $((\ )_{\widehat{P}})$  restricts to an idempotent monad on  $\mathcal{G}'$ . Let  $(S', D')$  be the associated orthogonal pair. Since every abelian group  $A$  satisfies  $A_{\widehat{P}} \cong A_P$ , all abelian groups are objects of  $\mathcal{G}'$ ; moreover, all  $P$ -local abelian groups are in  $D'$ .

For any group  $G$  in  $\mathcal{G}'$ , the completion homomorphism  $\eta : G \rightarrow G_{\widehat{P}}$  is in  $S'$  and hence it is orthogonal to all  $P$ -local abelian groups. From this fact it follows that, for all groups  $G$  in  $\mathcal{G}'$ , the natural map

$$(G/\Gamma_2 G)_P \rightarrow (G_{\widehat{P}}/\Gamma_2 G_{\widehat{P}})_P$$

induced by  $\eta$  is an isomorphism. Thus, if  $G$  is in  $\mathcal{G}'$ , then  $H_1(G; \mathbf{Z}_P) \cong H_1(G_{\widehat{P}}; \mathbf{Z}_P)$ .

For any group  $G$ , we denote by  $\gamma_i$  the projection of  $G$  onto  $G/\Gamma_i G$ , by  $\theta_i$  the natural epimorphism from  $G_{\widehat{P}}$  onto  $(G/\Gamma_i G)_P$ , by  $\bar{\eta}$  the abelianization of  $\eta : G \rightarrow G_{\widehat{P}}$ , and by  $e$  the  $\bar{P}$ -localization homomorphism. Since  $(G/\Gamma_2 G)_P$  is abelian, there is a unique homomorphism  $\bar{\theta}_2 : G_{\widehat{P}}/\Gamma_2 G_{\widehat{P}} \rightarrow (G/\Gamma_2 G)_P$  such that  $\bar{\theta}_2 \gamma_2 = \theta_2$ . Now we have

$$\bar{\theta}_2 \bar{\eta} \gamma_2 = \bar{\theta}_2 \gamma_2 \eta = \theta_2 \eta = e \gamma_2.$$

Since  $\gamma_2$  is surjective, we infer that  $\bar{\theta}_2 \bar{\eta} = e$ . Under the assumption that the group  $G$  is in the subcategory  $\mathcal{G}'$ , both  $\bar{\eta}$  and  $e$  are  $P$ -bijections. It follows that  $\bar{\theta}_2$  is a  $P$ -bijection as well. Hence, we have proved the following result.

**Proposition 2.** *For a group  $G$ , if the natural homomorphism  $G_{\widehat{P}} \rightarrow (G_{\widehat{P}})_{\widehat{P}}$  is an isomorphism, then the homomorphism  $H_1(G; \mathbf{Z}_P) \rightarrow H_1(G_{\widehat{P}}; \mathbf{Z}_P)$  induced by the  $P$ -completion map  $G \rightarrow G_{\widehat{P}}$  and the homomorphism  $H_1(G_{\widehat{P}}; \mathbf{Z}_P) \rightarrow H_1(G; \mathbf{Z}_P)$  induced by the projection  $G_{\widehat{P}} \rightarrow (G/\Gamma_2 G)_P$  are isomorphisms, and they are inverse to each other.*

We next prove that if  $F$  is a free group on an infinite set of generators, then  $\bar{\theta}_2$  is not  $P$ -injective. This implies that  $F$  is not in  $\mathcal{G}'$ , as desired.

Thus, we shall assume that  $\bar{\theta}_2$  is  $P$ -injective and arrive at a contradiction. Pick a countable subset of free generators of  $F$  and label them as  $\{a_{ij}\}$ , where  $1 \leq j \leq i$ . Denote by  $F_m$  the free group generated by  $a_{m1}, \dots, a_{mm}$ . Let  $\pi_m$  be the projection of  $F$  onto  $F_m$  sending all other

generators to 1. Likewise, we denote by  $\hat{\pi}_m$ , the induced homomorphism  $F_{\hat{P}} \rightarrow (F_m)_{\hat{P}}$  and by  $\eta_m$ , the completion map  $F_m \rightarrow (F_m)_{\hat{P}}$ .

Consider the element  $b = (b_2, b_3, \dots) \in F_{\hat{Z}}$ , where  $b_2 = 1$  and, for  $m \geq 2$ ,  $b_{m+1}$  is the class of

$$[a_{21}, a_{22}][a_{31}, a_{32}, a_{33}] \cdots [a_{m1}, \dots, a_{mm}]$$

in  $F/\Gamma_{m+1}F$ . Since the natural map  $F_{\hat{Z}} \rightarrow F_{\hat{P}}$  is injective, we may view  $b$  as an element of  $F_{\hat{P}}$  as well. In fact we have

$$\hat{\pi}_m(b) = \eta_m([a_{m1}, \dots, a_{mm}]).$$

Since  $1 = \theta_2(b) = \bar{\theta}_2(\gamma_2(b))$  and  $\bar{\theta}_2$  is assumed to be  $P$ -injective, it follows that  $\gamma_2(b)^n = 1$  for some  $n \in P'$ . Hence,  $b^n \in \Gamma_2 F_{\hat{P}}$ . Therefore we may write

$$b^n = [u_1, u_2] \cdots [u_{2k-1}, u_{2k}],$$

with  $u_i \in F_{\hat{P}}$  for all  $i$ . Now, for each  $i$ , we have  $\theta_2(u_i)^{t_i} = e(\gamma_2(z_i))$ , for some  $t_i \in P'$  and  $z_i \in F$  because  $e$  is  $P$ -surjective and  $\gamma_2$  is surjective. Since only a finite number of generators of  $F$  are involved in  $z_i$ , we have  $\pi_m(z_i) = 1$  for all  $i$  and all  $m$  except for a finite number of indices  $m_1, \dots, m_r$ . Choose any  $m \neq m_1, \dots, m_r$ , which will remain fixed in the rest of the argument.

Let  $\psi_m$  be the unique homomorphism that renders the following diagram commutative:

$$\begin{array}{ccccccc} F_{\hat{P}} & \xrightarrow{\theta_2} & (F/\Gamma_2 F)_P & \xleftarrow{e} & (F/\Gamma_2 F) & \xleftarrow{\gamma_2} & F \\ \downarrow \hat{\pi}_m & & \downarrow \psi_m & & \downarrow & & \downarrow \pi_m \\ (F_m)_{\hat{P}} & \xrightarrow{\theta_2} & (F_m/\Gamma_2 F_m)_P & \xleftarrow{e} & (F_m/\Gamma_2 F_m) & \xleftarrow{\gamma_2} & F_m \end{array}$$

For each  $i$ , we have

$$\psi_m(\theta_2(u_i))^{t_i} = \psi_m(e(\gamma_2(z_i))) = e(\gamma_2(\pi_m(z_i))) = 1.$$

Since the target of  $\psi_m$  is a  $P$ -local group, we infer that  $\psi_m(\theta_2(u_i)) = 1$ , and hence  $\theta_2(\hat{\pi}_m(u_i)) = 1$ . Therefore,  $\theta_{m+1}(\hat{\pi}_m(u_i))$  belongs to the kernel of the reduction map  $(F_m/\Gamma_{m+1}F_m)_P \rightarrow (F_m/\Gamma_2F_m)_P$ , that is,

$$\theta_{m+1}(\hat{\pi}_m(u_i)) \in (\Gamma_2 F_m/\Gamma_{m+1}F_m)_P$$

for all  $i$ . Now observe that

$$\begin{aligned} \theta_{m+1}(\eta_m([a_{m1}, \dots, a_{mm}]^n)) &= \theta_{m+1}(\hat{\pi}_m(b^n)) \\ &= [\theta_{m+1}(\hat{\pi}_m(u_1)), \theta_{m+1}(\hat{\pi}_m(u_2))] \cdots [\theta_{m+1}(\hat{\pi}_m(u_{2k-1})), \theta_{m+1}(\hat{\pi}_m(u_{2k}))], \end{aligned}$$

which is an element of  $(\Gamma_2\Gamma_2F_m/\Gamma_{m+1}F_m)_P$ . Hence, there is an integer  $q \in P'$  and an element  $x \in \Gamma_2\Gamma_2F_m/\Gamma_{m+1}F_m$  such that

$$\theta_{m+1}(\eta_m([a_{m1}, \dots, a_{mm}]^n))^q = e(x).$$

Since we have the commutative diagram

$$\begin{array}{ccc} F_m & \xrightarrow{\eta_m} & (F_m)_{\hat{P}} \\ \gamma_{m+1} \downarrow & & \downarrow \theta_{m+1} \\ F_m/\Gamma_{m+1}F_m & \xrightarrow{e} & (F_m/\Gamma_{m+1}F_m)_P \end{array}$$

where  $F_m/\Gamma_{m+1}F_m$  is torsion-free and hence the localization map  $e$  is injective, we infer that

$$\gamma_{m+1}([a_{m1}, \dots, a_{mm}]^{nq}) = x.$$

It follows that  $[a_{m1}, \dots, a_{mm}]^{nq}$  belongs to the subgroup of  $F_m$  generated by  $\Gamma_2\Gamma_2F_m$  and  $\Gamma_{m+1}F_m$ . This contradicts Proposition 1. We have thus shown

**Theorem 3.** *Let  $F$  be a free group on an infinite set of generators. Then, for any set of primes  $P$ , the natural homomorphisms  $H_1(F; \mathbf{Z}_P) \rightarrow H_1(F_{\hat{P}}; \mathbf{Z}_P)$  and  $\eta : F_{\hat{P}} \rightarrow (F_{\hat{P}})_{\hat{P}}$  both fail to be isomorphisms.*

We thus conclude that  $P$ -nilpotent completion is not idempotent on the category of groups. As in [4, Proposition IV.5.4], it follows from our theorem and [4, Proposition IV.5.3] that

**Corollary 4.** *Any infinite wedge of circles is  $R$ -bad, where  $R$  is any subring of the rationals.*

**Acknowledgements.** The author wishes to thank Jon Berrick, Carles Casacuberta and Garth Warner for many helpful comments and the referee for his/her suggestions on the presentation of this paper.

### References

1. R. BAER, Nilpotent groups and their generalizations, *Trans. Amer. Math. Soc.* **47** (1940), 393–434.
2. A. J. BERRICK AND G. C. TAN, The minimal extension of  $P$ -localization on groups, *Math. Proc. Cambridge Philos. Soc.* **118** (1996), 243–255.
3. A. K. BOUSFIELD, Homological localization towers for groups and  $\pi$ -modules, *Mem. Amer. Math. Soc.* **10(186)** (1977), ?.
4. A. K. BOUSFIELD AND D. M. KAN, “*Homotopy Limits, Completions and Localizations*,” Lecture Notes in Math. **304**, Springer-Verlag, Berlin, 1972.
5. C. CASACUBERTA, A. FREI AND G. C. TAN, Extending localization functors, *J. Pure Appl. Algebra* **103** (1995), 149–165.
6. C. CASACUBERTA, G. PESCHKE AND M. PFENNIGER, Orthogonal pairs in categories and localization, in “*Proc. Adams Memorial Symposium*,” London Math. Soc. Lecture Note Ser. **175**, Camb. Univ. Press, Cambridge, 1992, pp. 211–223.
7. P. HILTON, G. MISLIN AND J. ROITBERG, “*Localization of Nilpotent Groups and Spaces*,” North-Holland Math. Studies **15**, North-Holland, Amsterdam, 1975.
8. S. MACLANE, “*Categories for the Working Mathematician*,” Graduate Texts in Math. **5**, Springer-Verlag, Berlin, 1972.
9. M. SUZUKI, “*Group Theory II*,” Grundlehren der mathematischen Wissenschaften **248**, A series of Comprehensive Studies in Mathematics, Springer Verlag, 1986.

Department of Mathematics  
Faculty of Science  
National University of Singapore  
10 Kent Ridge Crescent  
SINGAPORE 119260

Primera versió rebuda el 3 de Maig de 1996,  
darrera versió rebuda el 20 de Juny de 1997