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INTEGRABILITY OF A LINEAR CENTER PERTURBED BY A FIFTH DEGREE HOMOGENEOUS POLYNOMIAL*

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Abstract

In this work we study the integrability of two-dimensional autonomous system in the plane with linear part of center type and non-linear part given by homogeneous polynomials of fifth degree. We give a simple characterisation for the integrable cases in polar coordinates. Finally we formulate a conjecture about the independence of the two classes of parameters which appear on the system; if this conjecture is true the integrable cases found will be the only possible ones.

1. Introduction

We consider the system

$$\begin{aligned}\dot{x} &= -y + X_s(x,y),\\ \dot{y} &= x + Y_s(x,y),\end{aligned} \tag{1.1}$$

where $X_s(x,y)$ and $Y_s(x,y)$ are homogeneous polynomials of degree s, with $s \geq 2$.

The aim of this paper is to find the integrable cases of system (1.1) when $s = 5$ (see Theorem 1). The integrable cases for quadratic systems, $s = 2$, and cubic homogeneous systems, $s = 3$, have been studied by several authors Bautin [**1**], Chavarriga [**2**], Coppel [**5**], Lloyd [**6**], Lunkevich and Sibirskii [**7**], Schlomiuk [**9**] and Żoladek [**13**]. Some integrable cases of system (1.1) when $s = 4$ have been determinated by Chavarriga and

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Giné [**4**]. Poincaré [**8**] developed an important technique for the general solution of these problems. It consists on finding a formal power series of the form

$$H(x,y)=\sum_{n=2}^{\infty}H_n(x,y), \tag{1.2}$$

where $H_2(x,y)=\frac{(x^2+y^2)}{2}$, and $H_n(x,y)$ are homogeneous polynomials of degree n, so that

$$\dot{H}=\sum_{k=2}^{\infty}V_{2k}(x^2+y^2)^k,$$

where V_{2k} are real numbers called *Lyapunov constants*. The vanishing of all Lyapunov constants is a necessary condition for the integrability of the system (1.1); in this case the series $H(x,y)$ would be a first integral of the system if it converges. This is an open question today. On the other hand, in general it is not possible to express this first integral (if it exists) by means of elementary functions.

Lyapunov constants are polynomials in the variables given by the coefficients of the polynomials $X_s(x,y)$ and $Y_s(x,y)$. Thus the ideal generated by V_{2k} has a finite number of generators by Hilbert's Theorem. We denote by $M(s)$ the minimum number of generators of such an ideal. It has been proved by Shi Songling [**10**] that under certain hypotheses on the Lyapunov constants, the number of small amplitude limit cycles around the origin is at least $M(s)$. The vanishing of these generators is a sufficient condition for the vanishing of all Lyapunov constants and for the integrability of the system.

In Section 2 we give without proof some known results which are necessary for the proof of Theorem 1. In Section 3 we prove Theorem 1. This theorem characterizes the integrable cases by means of polar coordinates for $s=5$. Finally, we give an appendix on the computation of Lyapunov constants for $s=5$.

2. Some preliminary results

In the study of this problem we have used polar coordinates. In Lemma 1 we give the expression of system (1.1) in polar coordinates. In Proposition 1 we give the evaluation of series (1.2) in these coordinates.

Lemma 1. *In polar coordinates $x=r\cos(\varphi)$, $y=r\sin(\varphi)$ we can write system (1.1) as*

$$\begin{aligned}\dot{r}&=P_s(\varphi)r^s,\\ \dot{\varphi}&=1+Q_s(\varphi)r^{s-1},\end{aligned} \tag{2.1}$$

where $P_s(\varphi)$ and $Q_s(\varphi)$, are trigonometric polynomials of the form

$$P_s(\varphi) = R_{s+1}\cos((s+1)\varphi + \varphi_{s+1}) + R_{s-1}\cos((s-1)\varphi + \varphi_{s-1}) + \cdots + \begin{cases} R_1\cos(\varphi+\varphi_1) & \text{if } s \text{ is even;} \\ R_0 & \text{if } s \text{ is odd;} \end{cases}$$

$$Q_s(\varphi) = -R_{s+1}\sin((s+1)\varphi + \varphi_{s+1}) + r_{s-1}\sin((s-1)\varphi + \overline{\varphi}_{s-1}) + \cdots + \begin{cases} r_1\sin(\varphi+\overline{\varphi}_1) & \text{if } s \text{ is even;} \\ r_0 & \text{if } s \text{ is odd;} \end{cases}$$

being R_j, r_j, φ_j and $\overline{\varphi}_j$ arbitrary coefficients.

Proposition 1. *In polar coordinates series (1.2) for the system (1.1) is $H(r,\varphi) = \sum_{m=0}^{\infty} \overline{H}_m(\varphi) r^{m(s-1)+2}$ where $\overline{H}_0(\varphi) = \frac{1}{2}$ and $\overline{H}_m(\varphi)$, $m = 0,1,\dots,$ are homogeneous trigonometric polynomials of degree $m(s-1)+2$, satisfying the differential equations*

$$\frac{d\overline{H}_{m+1}}{d\varphi} + (m(s-1)+2)\overline{H}_m P_s(\varphi) + \frac{d\overline{H}_m}{d\varphi} Q_s(\varphi) = \begin{cases} 0 & \text{if } (m+1)(s-1)+2 \text{ is odd,} \\ V_{(m+1)(s-1)+2} & \text{if } (m+1)(s-1)+2 \text{ is even,} \end{cases} \tag{2.2}$$

where $V_{(m+1)(s-1)+2}$, $m = 0,1,\dots,$ are the Lyapunov constants.

Lemma 1 and Proposition 1 are proved in [**2**].

In particular for $s = 5$ system (2.1) takes the form

$$\begin{aligned} \dot{r} &= P_5(\varphi) r^5, \\ \dot{\varphi} &= 1 + Q_5(\varphi) r^4, \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} P_5(\varphi) &= R_6\cos(6\varphi+\varphi_6) + R_4\cos(4\varphi+\varphi_4) + R_2\cos(2\varphi+\varphi_2) + R_0, \\ Q_5(\varphi) &= -R_6\sin(6\varphi+\varphi_6) r_4\sin(4\varphi+\overline{\varphi}_4) + r_2\sin(2\varphi+\overline{\varphi}_2) + r_0. \end{aligned} \tag{2.4}$$

In this case the evaluation of $\dot{H}(r,\varphi)$ from system (2.3) yields

$$\frac{d\overline{H}_{m+1}}{d\varphi} + (4m+2)\overline{H}_m P_4(\varphi) + \frac{d\overline{H}_m}{d\varphi} Q_4(\varphi) = \overline{V}_{m+1} = \begin{cases} 0 & \text{if } 4(m+1)+2 \text{ is odd;} \\ V_{4(m+1)+2} & \text{if } 4(m+1)+2 \text{ is even;} \end{cases}$$

for $m = 0,1,\dots,$ with $\overline{H}_0(\varphi) = \frac{1}{2}$ and $\overline{H}_m(\varphi) = H_{4m+2}(\varphi)$.

In Proposition 2 we will give the general form of the symmetric integrable systems. In Proposition 3 we will give a class of integrable systems which have an integrant factor given by a quadratic polynomial in the variable r^{s-1} whose coefficients are functions in φ.

Proposition 2. *In the following two cases system (2.1) is integrable (in the sense that all its Lyapunov constants vanish):*

(i) $(s+1)P_s(\varphi) + \frac{dQ_s(\varphi)}{d\varphi} = 0$.

(ii) $P_s(\varphi)$ *and* $Q_s(\varphi)$ *are of the form*

$$\begin{aligned} P_s(\varphi) &= R_{s+1}\sin(s+1)\omega + R_{s-1}\sin(s-1)\omega \\ &\quad + \cdots + \begin{cases} R_1 \sin\omega & \textit{if } s \textit{ is even;} \\ R_2 \sin 2\omega & \textit{if } s \textit{ is odd;} \end{cases} \\ Q_s(\varphi) &= R_{s+1}\cos(s+1)\omega + r_{s+1}\cos(s-1)\omega \\ &\quad + \cdots + \begin{cases} r_1 \cos\omega & \textit{if } s \textit{ is even;} \\ r_2 \cos 2\omega + r_0 & \textit{if } s \textit{ is odd;} \end{cases} \end{aligned}$$

where $\omega = \varphi + \varphi_0$ *and* φ_0 *and the coefficients* R_j *and* r_j *are arbitrary.*

Systems satisfying (i) have null divergence and those satisfying (ii) have a certain resonance between the angular parameters φ_j, $\overline{\varphi}_j$.

Proposition 3. *For* $s \in \mathbb{N}$ *with* $s \geq 2$ *and arbitrary* k_1, k_2, $\varphi_0 \in \mathbb{R}$ *system (2.1) with*

$$\begin{aligned} P_s(\varphi) &= 2(-k_1\cos^{s-2}(\varphi+\varphi_0)\sin^3(\varphi+\varphi_0) \\ &\qquad + k_2\sin^{s-2}(\varphi+\varphi_0)\cos^3(\varphi+\varphi_0)), \\ Q_s(\varphi) &= (k_1\cos^{s-1}(\varphi+\varphi_0) - k_2\sin^{s-1}(\varphi+\varphi_0))\cos 2(\varphi+\varphi_0), \end{aligned} \tag{2.5}$$

is integrable.

In Cartesian coordinates $x = r\cos(\varphi+\varphi_0)$ and $y = r\sin(\varphi+\varphi_0)$ we can write system (2.5) in the form

$$\begin{aligned} \dot{x} &= -y - k_1 x^{s-1}y + k_2 y^{s-2}(2x^2 - y^2), \\ \dot{y} &= x + k_1 x^{s-2}(x^2 - 2y^2) + k_2 x y^{s-1}, \end{aligned} \tag{2.6}$$

with $s \geq 2$. We note that the origin is a center for system (2.6).

Proposition 2 is proved in [**2**] and Proposition 3 is proved in [**3**].

In order to establish the cases of Theorem 1, we have used a certain simplification expressed in the form of a conjecture. This conjecture simplifies the great number of factors that appear in the different Lyapunov constants. This assumption, which we think to be always satisfied, is stated as Conjecture 1.

Conjecture 1. *A necessary condition for all the Lyapunov constants of system (2.3) are zero is that the angular parameters* $(\varphi_6, \varphi_4, \overline{\varphi}_4, \varphi_2, \overline{\varphi}_2)$ *and the radial ones* $(R_6, R_4, r_4, R_2, r_2, R_0, r_0)$ *must be independent.*

With reference to the number of small amplitude limit cycles around the origin that appear in the system (2.3) we note that in the third integrable case (Case 9) the number of relations of parameters is nine. So if Conjecture 1 is true, we think that the number of small amplitude limit cycles is at least nine.

The Lyapunov constants were obtained by using the computer algebra system Mathematica.

3. The main result

Theorem 1. *System (2.3) is integrable in the following cases:*

(i) $R_0 = 0$, $\overline{\varphi}_2 = \varphi_2$, $\overline{\varphi}_4 = \varphi_4$, $6R_4 + 4r_4 = 0$ *and* $6R_2 + 2r_2 = 0$.

(ii) $R_0 = 0$, $\overline{\varphi}_2 = \varphi_2$, $\overline{\varphi}_4 = \varphi_4$, $\varphi_6 = 3\varphi_2$ *and* $\varphi_4 = 2\varphi_2 + \frac{\pi}{2}$.

(iii) $R_0 = 0$, $\overline{\varphi}_2 = \varphi_2$, $\overline{\varphi}_4 = \varphi_4$, $\varphi_6 = \varphi_4 + \varphi_2 + \frac{\pi}{2}$, $r_4 = R_4 = R_6$, $r_2 = R_2 = r_0$, *and* $|R_2| = |R_4|$.

(iv) $R_0 = 0$, $\overline{\varphi}_2 = \varphi_2$, $\overline{\varphi}_4 = \varphi_4$, $R_6 = 0$ *and*

$$\begin{cases} \text{(iv.1) } R_4 r_2 - 2r_4 R_2 = 0, \\ \text{(iv.2) } r_0 = 0,\ r_2 = R_2 = 2|R_4| \text{ and } r_4 = -2R_4. \\ \text{(iv.3) } r_0 = 0,\ r_2 = R_2,\ R_4 = 0 \text{ and } |R_2| = |r_4|. \end{cases}$$

This theorem is independent of Conjecture 1, but if Conjecture 1 is true then system (2.3) will *only* be integrable in the cases given by Theorem 1.

Proof of Theorem 1: The first not zero Lyapunov constant is $V_2 = -R_0$ therefore $R_0 = 0$. The next not zero Lyapunov constant is

$$V_{10} = -\frac{1}{2}(R_2 r_2 \sin(\varphi_2 - \overline{\varphi}_2) + R_4 r_4 \sin(\varphi_4 - \overline{\varphi}_4)).$$

In particular the previous constant vanishes when $\overline{\varphi}_2 = \varphi_2$ and $\overline{\varphi}_4 = \varphi_4$. On the other hand, if Conjecture 1 is true, we arrive at the same condition as above. With this assumption the next not zero Lyapunov constant is V_{14} given by

$$\begin{aligned} 12V_{14} = 3((R_2 - r_2)(2R_2 r_4 - r_2 R_4)) \cos(2\varphi_2 - \varphi_4) + ((6r_2 + 2R_2)r_4 \\ + (5r_2 - 9R_2)R_4)R_6 \cos(\varphi_2 + \varphi_4 - \varphi_6). \end{aligned}$$

The simultaneous vanishing of three factors of V_{14} respect to the radials parameters with arbitrary angular parameters leads to the following cases:

1. $6R_2 + 2r_2 = 0$ and $6R_4 + 4r_4 = 0$;
2. $R_2 = r_2 = 0$;
3. $R_4 = r_4 = 0$;
4. $R_6 = 0$, $R_4 r_2 - 2r_4 R_2 = 0$, $R_2^2 + r_2^2 \neq 0$, and $R_4^2 + r_4^2 \neq 0$;
5. $R_6 = 0$ and $r_2 - R_2 = 0$;

If we impose that the values of the angular parameters are not arbitrary, the possible dependence relations between them are the following:

6. $\varphi_4 - 2\varphi_2 = \frac{\pi}{2}$ and $\varphi_6 - 3\varphi_2 = 0$;
7. $2\varphi_2 - \varphi_4 = \frac{\pi}{2}$;
8. $\varphi_2 - 2\varphi_4 + \varphi_6 = 0$;
9. $\varphi_2 + \varphi_4 - \varphi_6 = \frac{\pi}{2}$;

Case 1: In this case it is easy to see that $6P_5 + Q_5' = 0$, where $' = \frac{d}{d\varphi}$, and the divergence of the vector field defined by system (2.3) is zero. So the system is integrable and all Lyapunov constants are zero (see Proposition 2).

Case 2: If $r_2 = R_2 = 0$ then the first not zero Lyapunov constant after V_{14} is V_{22}, that is

$$48V_{22} = ((3R_4 + 2r_4)(R_4 - 4r_4)(2R_4 - 3r_4))R_6^2 \cos(3\varphi_4 - 2\varphi_6).$$

The vanishing of the first factor corresponds to Case 1. The situation when $R_4 = 0$ and $r_4 = 0$ corresponds to a degenerate case of zero divergence. The vanishing of the rest of the factors of V_{22} allows to express r_4 in function of R_4. If we introduce these relations in the next non-zero constant V_{26}, see Appendix 1, we can see that this constant only vanishes when $r_0 = 0$. By substituting the expressions found in the next non-zero Lyapunov constant V_{30}, see Appendix 1, we can see that this constant never vanishes in the case $r_4 = \frac{2R_4}{3}$ and $r_0 = 0$ and in the case $r_4 = \frac{R_4}{4}$ and $r_0 = 0$ is

$$6144V_{30} = 7R_4^3 R_6^2 (2R_6 - R_4)(2R_6 + R_4)\cos(3\varphi_4 - 2\varphi_6).$$

In all cases V_{50}, see Appendix 2, never vanishes. Therefore there is no possible integrable cases. The vanishing of R_6 will be seen later. The vanishing of $\cos(3\varphi_4 - 2\varphi_6)$ corresponds to Case 6.

Case 3: If $r_4 = R_4 = 0$ then the first non-zero Lyapunov constant after V_{14} is V_{18}, that is

$$24V_{18} = ((r_2 + 3R_2)(3r_2 - 2R_2)(2R_2 - r_2))R_6 \sin(3\varphi_2 - \varphi_6).$$

The vanishing of the first factor corresponds to Case 1. The situation when $R_2 = 0$ and $r_2 = 0$ corresponds to a degenerate case of zero divergence. The vanishing of the rest of the factors of V_{18} allows to express r_2 in function of R_2. If we introduce these relations in the next non-zero constant V_{22}, see Appendix 1, we can see that this constant only vanishes when $r_0 = 0$. By substituting the expressions found in the next non-zero Lyapunov constant V_{26}, see Appendix 1, we can see that this constant never vanishes in the case $r_2 = 2R_2$ and $r_0 = 0$ and in the case $r_2 = \frac{2R_2}{3}$ and $r_0 = 0$ is

$$3888V_{26} = 11R_2^3 R_6(80R_2^2 - 201R_6^2)\sin(3\varphi_2 - \varphi_6).$$

The vanishing of V_{26} allows to express R_2 in function of R_6. Finally by substituting them in the next non-zero Lyapunov constant V_{34}, see Appendix 3, we can see that this constant does not vanish. Therefore there is no possible integrable cases. The vanishing of R_6 will be seen later. The vanishing of $\sin(3\varphi_2 - \varphi_6)$ corresponds to Case 6.

We now consider the case $R_6 = 0$, then

$$12V_{14} = 3((R_2 - r_2)(2R_2 r_4 - r_2 R_4))\cos(2\varphi_2 - \varphi_4).$$

If we impose that $V_{14} = 0$ we have three possibilities. The vanishing of $\cos(2\varphi_2 - \varphi_4)$ corresponds to Case 7. The others correspond to Cases 4 and 5 which we will be studied.

Case 4: In this case $R_6 = 0$ and $R_4 r_2 - 2r_4 R_2 = 0$. If $r_2 \neq 0$ and $r_4 \neq 0$, then $P_5(\varphi) = \lambda Q_5'(\varphi)$ where $\lambda = R_2/2r_2 = R_4/4r_4$. System (2.3) takes the form

$$\dot{r} = \lambda r^5 Q_5'(\varphi),$$
$$\dot{\varphi} = 1 + r^4 Q_5(\varphi).$$

The previous system is integrable and its first integral is

$$H(r,\varphi) = (1 + (1 + 4\lambda)Q_5(\varphi)r^4)r^{-\left(4+\frac{1}{\lambda}\right)}.$$

If $r_2 = 0$ or $r_4 = 0$ implies $\dot{\varphi} = 1$ and system (2.3) is trivially integrable.

Case 5: In this case $R_6 = 0$ and $r_2 - R_2 = 0$. We can suppose $r_2 = R_2 \neq 0$, because $r_2 = R_2 = 0$ corresponds to a particular case of Case 6. Then if we introduce these relations in the next non-zero constant V_{18}, see Appendix 1, we can see that this constant only vanishes when $r_0 = 0$. By substituting the expressions found in the next non-zero Lyapunov constant V_{22}, see Appendix 1, we can see that this constant is

$$48V_{22} = R_2^2(R_4 - 2r_4)(4R_2^2 - 4r_4^2 + r_4R_4 + 2R_4^2)\cos(2\varphi_2 - \varphi_4).$$

The vanishing of the first factor $R_4 - 2r_4$ corresponds to a particular case of $R_4r_2 - 2r_4R_2 = 0$ which has been studied in Case 4. The vanishing of other factor of V_{22} allows to express R_2 in function of R_4 and r_4. Then

$$\begin{aligned} 15360V_{30} &= (13r_4 - 10R_4)R_4(R_4 - 2r_4)(r_4 + 2R_4)(4r_4 + 3R_4) \\ &\quad (-4r_4^2 + r_4R_4 + 2R_4^2)\cos(2\varphi_2 - \varphi_4), \\ 36864V_{34} &= 7r_4R_4(2r_4 - R_4)(r_4 + 2R_4) \\ &\quad (-4r_4^2 + r_4R_4 + 2R_4^2)^2\sin(4\varphi_2 - 2v_4). \end{aligned}$$

The simultaneous vanishing of the two previous Lyapunov constants give rise to three possible cases either $r_4 + 2R_4 = 0$ $R_4 = 0$ or $R_4 - 2r_4 = 0$. The last one corresponds to a particular case of $R_4r_2 - 2r_4R_2 = 0$ which has been studied in Case 4. In the first case, that is $r_2 = R_2$, $r_4 = -2R_4$ and $R_2 = \pm 2R_4$, system (2.3) reduces to

$$\begin{aligned} \dot{r} &= r^5(R_4\cos(4\varphi + \varphi_4) \pm 2R_4\cos(2\varphi + \varphi_2)), \\ \dot{\varphi} &= 1 + r^4(-2R_4\sin(4\varphi + \varphi_4) \pm 2R_4\sin(2\varphi + \varphi_2)). \end{aligned}$$

If we make the change $R = r^4$ the system takes the form

$$\begin{aligned} \dot{R} &= 4R^2R_4(\cos(4\varphi + \varphi_4) \pm 2\cos(2\varphi + \varphi_2)), \\ \dot{\varphi} &= 1 + 2RR_4(-\sin(4\varphi + \varphi_4) \pm \sin(2\varphi + \varphi_2)), \end{aligned}$$

which corresponds to an integrable case of a homogeneous cubic system with linear part of center type, see for instance [**2**]. In the second case, that is $r_2 = R_2$, $R_4 = 0$ and $R_2 = \pm r_4$, give rise to a new possible integrable case. System (2.3) reduces to

$$\begin{aligned} \dot{r} &= \pm r^5 r_4\cos(2\varphi + \varphi_2), \\ \dot{\varphi} &= 1 + r^4(r_4\sin(4\varphi + \varphi_4) \pm r_4\sin(2\varphi + \varphi_2)). \end{aligned} \tag{3.1}$$

We will establish that this case is a true integrable case later.

Case 6: We call this case *angles resonance*. Then system (2.3) is integrable (see Proposition 2).

Case 7: Let $2\varphi_2 - \varphi_4 = \frac{\pi}{2}$. We also assume that $3\varphi_2 - \varphi_6 \neq 0$ (the opposite case has been studied in Case 6). Then a term that appears in V_{22} and which must be zero independently on the rest is

$$\frac{1}{48}((R_4 - 4r_4)(2R_4 - 3r_4)(3R_4 + 2r_4))R_6^2 \sin(6\varphi_2 - 2\varphi_6).$$

From the vanishing of the factors of the previous expression and their substitution in the constants V_{14}, V_{18}, V_{26} and the remaining terms of V_{22}, see Appendix 1, we obtain cases already studied.

Case 8: Let $\varphi_2 - 2\varphi_4 + \varphi_6 = 0$. We also assume $2\varphi_2 - \varphi_4 - \frac{\pi}{2} \neq 0$ (the opposite case has been studied in Case 6). Then a term that appears in V_{18} and which must be zero independently on the rest is

$$\frac{1}{24}((3r_2 - 2R_2)(2R_2 - r_2)(r_2 + 3R_2))R_6 \sin(4\varphi_2 - 2\varphi_4).$$

From the vanishing of the factors of the previous expression and their substitution into the constants V_{14}, V_{22}, V_{26} and the remaining terms of V_{18}, see Appendix 1, we obtain cases already studied.

Case 9: Let $\varphi_2 + \varphi_4 - \varphi_6 = \frac{\pi}{2}$. Then $4V_{14} = ((R_2 - r_2)(2R_2 r_4 - r_2 R_4))\cos(2\varphi_2 - \varphi_4)$. From the vanishing of the previous constant (we can assume that $2\varphi_2 - \varphi_4 - \frac{\pi}{2} \neq 0$, the opposite case has been studied in Case 6) we obtain two possible cases either $R_2 - r_2 = 0$ or $2R_2 r_4 - r_2 R_4 = 0$. From the last one we obtain only cases already studied. In the first case, by substituting this value in V_{18}, V_{22}, V_{26} and V_{30} and imposing that these Lyapunov constants to be zero, we obtain a system of four equations since V_{30} has two terms with independent trigonometric part, see Appendix 1. By solving this system we obtain some cases already studied, and a new solution given by $R_2 = r_2 = r_0$, $r_4 = R_4$, $R_6 = R_4$, and $R_4 = \pm R_2$. These relations give rise to a new possible integrable case. System (2.3) takes the form

$$\begin{aligned} \dot{r} &= r^5 R_6 \left(\cos\left(6\varphi + \varphi_4 + \varphi_2 + \frac{\pi}{2}\right)\right. \\ &\qquad \left. + \cos(4\varphi + \varphi_4) \pm \cos(2\varphi + \varphi_2)\right), \\ \dot{\varphi} &= 1 + r^4 R_6 \left(-\sin\left(6\varphi + \varphi_4 + \varphi_2 + \frac{\pi}{2}\right)\right. \\ &\qquad \left. + \sin(4\varphi + \varphi_4) \pm \sin(2\varphi + \varphi_2) \pm 1\right). \end{aligned} \tag{3.2}$$

In order to establish that this case is a true integrable case, we need remember some definitions.

If we do the change $R = r^{s-1}$, then system (2.1) becomes

$$\begin{aligned} \dot{R} &= (s-1)P_s(\varphi)R^2, \\ \dot{\varphi} &= 1 + Q_s(\varphi)R. \end{aligned} \tag{3.3}$$

In the study and determination of the first integrals for quadratic systems and homogeneous cubic systems (see [**2**]) and for homogeneous systems with $s > 3$ (see [**3**] and [**4**]), we used a technique consisting in the research of polynomial particular solutions of system (3.3) of the form

$$V(R,\varphi) = 1 + V_1(\varphi)R + V_2(\varphi)R^2 + \cdots + V_p(\varphi)R^p = 0, \tag{3.4}$$

where $V_k(\varphi)$, $k = 1, 2, \ldots, p$, are homogeneous trigonometrical polynomials of degree $k(s-1)$ in the variables $\cos\varphi$ and $\sin\varphi$.

Proposition 1. *Function (3.4) is a particular solution of system (3.3) if the homogeneous trigonometric polynomials $V_k(\varphi)$, $k = 1, 2, \ldots, p$, verify the following differential system*

$$\begin{aligned} V'_{k+1} + V'_k Q_s + k(s-1)V_k P_s &= V_k V'_1, \quad k = 1, 2, \ldots, p-1, \\ V'_p Q_s + p(s-1)V_p P_s &= V_p V'_1, \end{aligned} \tag{3.5}$$

where $' = \frac{d}{d\varphi}$.

A function $W(x,y)$ will be called a *null divergence factor* for system (1.1) if $W(x,y) = 0$ is a particular solution for this system and the divergence of the vector field

$$C = \left(\frac{-y + X_s(x,y)}{W(x,y)}, \frac{x + Y_s(x,y)}{W(x,y)} \right)$$

defined at $\mathbb{R}^2 \backslash \{(x,y) : W(x,y) = 0\}$ is zero.

We notice that if the divergence of a vector field is zero then system (1.1) defined for this vector field is integrable. In particular, if system (1.1) has a null divergence factor then this system is integrable and the origin is a center.

Proposition 2. *If*

$$W(R,\varphi) = (V(R,\varphi))^{\alpha}(U(R,\varphi))^{\beta} \tag{3.6}$$

is a null divergence factor for system (3.3) where $V(R,\varphi)$ and $U(R,\varphi)$ are particular solutions of system (3.3) of the form (3.4) and α and β real numbers then the homogeneous trigonometric polynomials $V_k(\varphi)$, $k = 1,2,\dots,p$ and $U_j(\varphi)$, $j = 1,2,\dots,q$, verify the following differential system

$$\begin{aligned} (s+1)P_s + Q_s' - \alpha V'1 - \beta U_1' &= 0,\\ V_{k-1}' + V_k' Q_s + k(s-1)V_k P_s &= V_k V_1', \quad k = 1,2,\dots,p-1,\\ V_p' Q_s + p(s-1)V_p P_s &= V_p V_1',\\ U_{j+1}' + U_j' Q_s + j(s-1)U_j P_s &= U_j U_1', \quad j = 1,2,\dots,q-1,\\ V_q' Q_s + q(s-1)V_q P_s &= V_q V_1', \end{aligned} \tag{3.7}$$

where $\prime = \frac{d}{d\varphi}$.

Proof: The function $W(R,\varphi)$ is a null divergence factor for system (3.3) if

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{P_s(\varphi)r^{s+1}}{W(R,\varphi)}\right) + \frac{\partial}{\partial\varphi}\left(\frac{1+Q_s(\varphi)r^{s-1}}{W(R,\varphi)}\right) = 0. \tag{3.8}$$

Now consider that the function $W(R,\varphi)$ is of the form given in (3.6) with $R = r^{s-1}$. Then if we develop the expression (3.8) with respect to the powers of R, we have differential system (3.7). ■

System (3.7) coincides with system (3.5) for each particular solution except in the first equation where the value of V_1 and U_1 are determined in function of $P_s(\varphi)$, $Q_s'(\varphi)$, β and α.

Using these methods we have found two particular solutions for system (3.2) of the form

$$\begin{aligned} U(R,\varphi) &= 1 + U_1 R,\\ V(R,\varphi) &= 1 + V_1 R + V_2 R^2, \end{aligned}$$

where

$$\begin{aligned} U_1 &= \pm 4R_6 \sin^2\left(2\varphi + \frac{\varphi_4}{2} \pm \frac{\pi}{4}\right),\\ V_1 &= 2R_6(\pm 1 \pm 2\sin(2\varphi+\varphi_2) + \sin(4\varphi+\varphi_4)),\\ V_2 &= 4R_6^2 \sin^2\left(2\varphi + \frac{\varphi_4}{2} \pm \frac{\pi}{4}\right)\left(\cos\left(\varphi + \frac{\varphi_2}{2}\right) + \sin\left(\varphi + \frac{\varphi_2}{2}\right)\right)^4. \end{aligned}$$

From these particular solutions is easy to see that $W(R,\varphi) = U(R,\varphi)^{\frac{1}{4}}V(R,\varphi)$ is a null divergence factor for system (3.2) which completes the proof and the system (3.2) is a new integrable case.

Using also these methods we have found a particular solution for system (3.1) of the form

$$V(R,\varphi) = 1 + 4r_4\cos^2\left(\frac{\varphi_2}{2} - \frac{\varphi_4}{2} - \varphi\right)\sin(2\varphi+\varphi_2)R + 4r_4^2\cos^4\left(\frac{\varphi_2}{2} - \frac{\varphi_4}{2} - \varphi\right)R^2.$$

From this particular solutions is easy to see that $W(R,\varphi) = V(R,\varphi)R$ is a null divergence factor for system (3.1) which completes the proof and system (3.1) is a new integrable case too. ■

Appendix 1

Lyapunov constants V_{14}, V_{18}, V_{22}, V_{26} and V_{30} if $\varphi_2 = \bar{\varphi}_2$ and $\varphi_4 = \bar{\varphi}_4$:

$$\begin{gathered}12V_{14} = 3(r_2 - R_2)(-2R_2r_4 + r_2R_4)\cos(2\varphi_2 - \varphi_4)\\ +(6r_2r_4 + 2R_2r_4 + 5r_2R_4 - 9R_2R_4)R_6\cos(\varphi_2+\varphi_4-\varphi_6)\end{gathered}$$

$$\begin{gathered}24V_{18} = 6r_0(-2r_2 + 3R_2)(-2R_2r_4 + r_2R_4)\cos(2\varphi_2 - \varphi_4)\\ +6r_0(-4r_2r_4 - 3r_2R_4 + 9R_2R_4)R_6\cos(\varphi_2+\varphi_4-\varphi_6)\\ +(3r_2 - 2R_2)(-r_2 + 2R_2)(r_2 + 3R_2)R_6\sin(3\varphi_2 - \varphi_6)\\ +(-9r_2r_4^2 + 25R_2r_4^2 - 14r_2r_4R_4 + 4R_2r_4R_4 + 13r_2R_4^2 - 9R_2R_4^2)\\ R_6\sin(\varphi_2 - 2\varphi_4 + \varphi_6)\end{gathered}$$

$$\begin{gathered}1440V_{22} = 3(-720r_0^2r_2R_2r_4 - 120r_2^3R_2r_4 + 1440r_0^2R_2^2r_4 - 160r_2^2R_2^2r_4\\ +4400r_2R_2^3r_4 - 4200R_2^4r_4 - 240r_2R_2r_4^3 + 320R_2^2r_4^3 + 360r_0^2r_2^2R_4\\ +60r_2^4R_4 - 720r_0^2r_2R_2R_4 + 80r_2^3R_2R_4 - 2200r_2^2R_2^2R_4 + 2100r_2R_2^3R_4\\ +120r_2^2r_4^2R_4 - 110r_2R_2r_4^2R_4 - 70R_2^2r_4^2R_4 - 25r_2^2r_4R_4^2 + 985r_2R_2r_4R_4^2\\ -990R_2^2r_4R_4^2 - 475r_2^2R_4^3 + 495r_2R_2R_4^3 - 18r_2^2r_4R_6^2 - 756r_2R_2r_4R_6^2\\ +1014R_2^2r_4R_6^2 + 333r_2^2R_4R_6^2 - 534r_2R_2R_4R_6^2 + 81R_2^2R_4R_6^2)\cos(2\varphi_2 - \varphi_4)\\ +30(3r_4 - 2R_4)(-4r_4 + R_4)(2r_4 + 3R_4)R_6^2\cos(3\varphi_4 - 2\varphi_6)\\ +R_6(2160r_0^2r_2r_4 + 720r_2^3r_4 - 720r_0^2R_2r_4 + 30r_2^2R_2r_4\\ -10920r_2R_2^2r_4 - 4230R_2^3r_4 + 720r_2r_4^3 + 336R_2r_4^3\end{gathered}$$

$$+1440r_0^2r_2R_4 + 435r_2^3R_4 - 6480r_0^2R_2R_4 - 1920r_2^2R_2R_4$$
$$-8415r_2R_2^2R_4 + 17820R_2^3R_4 + 282r_2r_4^2R_4 - 1194R_2r_4^2R_4$$
$$-3093r_2r_4R_4^2 - 747R_2r_4R_4^2 - 2259r_2R_4^3 + 4455R_2R_4^3$$
$$-444r_2r_4R_6^2 - 292R_2r_4R_6^2 - 406r_2R_4R_6^2 + 342R_2R_4R_6^2)\cos(\varphi_2 + \varphi_4 - \varphi_6)$$
$$+60r_0(r_2 + 3R_2)(9r_2^2 - 32r_2R_2 + 22R_2^2)R_6\sin(3\varphi_2 - \varphi_6)$$
$$30r_0(54r_2r_4^2 - 182R_2r_4^2 + 76r_2r_4R_4 - 8R_2r_4R_4 - 107r_2R_4^2 + 99R_2R_4^2)$$
$$R_6\sin(\varphi_2 - 2\varphi_4 + \varphi_6)$$

$$2880V_{26} = 2r_0(2880r_0^2r_2R_2r_4 + 1440r_2^3R_2r_4 - 7200r_0^2R_2^2r_4 + 2040r_2^2R_2^2r_4$$
$$-76920r_2R_2^3r_4 + 91320R_2^4r_4 + 2880r_2R_2r_4^3 - 4800R_2^2r_4^3$$
$$-1440r_0^2r_2^2R_4 - 720r_2^4R_4 + 3600r_0^2r_2R_2R_4 - 1020r_2^3R_2R_4$$
$$+38460r_2^2R_2^2R_4 - 45660r_2R_2^3R_4 - 1440r_2^2r_4^2R_4 + 1470r_2R_2r_4^2R_4$$
$$+1470R_2^2r_4^2R_4 + 465r_2^2r_4R_4^2 - 17175r_2R_2r_4R_4^2 + 21330R_2^2r_4R_4^2$$
$$+8220r_2^2R_4^3 - 10665r_2R_2R_4^3 + 270r_2^2r_4R_6^2 + 8972r_2R_2r_4R_6^2 - 15554R_2^2r_4R_6^2$$
$$-3865r_2^2R_4R_6^2 + 7858r_2R_2R_4R_6^2 - 1701R_2^2R_4R_6^2)\cos(2\varphi_2 - \varphi_4)$$
$$+40r_0(2r_4 + 3R_4)(81r_4^2 - 95r_4R_4 + 20R_4^2)R_6^2\cos(3\varphi_4 - 2\varphi_6)$$
$$+6r_0R_6(-960r_0^2r_2r_4 - 960r_2^3r_4 + 640r_0^2R_2r_4 + 310r_2^2R_2r_4$$
$$+21220r_2R_2^2r_4 + 4710R_2^3r_4 - 960r_2r_4^3 - 320R_2r_4^3$$
$$-560r_0^2r_2R_4 - 485r_2^3R_4 + 3600r_0^2R_2R_4 + 3190r_2^2R_2R_4$$
$$+14835r_2R_2^2R_4 - 42660R_2^3R_4 - 170r_2r_4^2R_4 + 2042R_2r_4^2R_4$$
$$+5879r_2r_4R_4^2 + 453R_2r_4R_4^2 + 4071r_2R_4^3 - 10665R_2R_4^3 + 1264r_2r_4R_6^2$$
$$+432R_2r_4R_6^2 + 1056r_2R_4R_6^2 - 1872R_2R_4R_6^2)\cos(\varphi_2 + \varphi_4 - \varphi_6)$$
$$+45(-r_2 + R_2)(-2R_2r_4 + r_2R_4)(4r_2^2r_4 + 3r_2R_2r_4$$
$$+113R_2^2r_4 + 13r_2^2R_4 - 300r_2R_2R_4 + 278R_2^2R_4)\sin(4\varphi_2 - 2\varphi_4)$$
$$+5(186r_2^2r_4^2 + 4032r_2R_2r_4^2 + 1494R_2^2r_4^2 + 1509r_2^2r_4R_4 + 3008r_2R_2r_4R_4$$
$$-6141R_2^2r_4R_4 + 1184r_2^2R_4^2 - 2226r_2R_2R_4^2 + 378R_2^2R_4^2)$$
$$R_6^2\sin(2\varphi_2 + 2\varphi_4 - 2\varphi_6)$$
$$+R_6(-2160r_0^2r_2^3 - 270r_2^5 + 3120r_0^2r_2^2R_2 - 300r_2^4R_2 + 20400r_0^2r_2R_2^2$$
$$+10710r_2^3R_2^2 - 25200r_0^2R_2^3 + 3180r_2^2R_2^3 - 60840r_2R_2^4 + 36720R_2^5$$
$$-1080r_2^3r_4^2 + 264r_2^2R_2r_4^2 - 28304r_2R_2^2r_4^2 + 15392R_2^3r_4^2 - 957r_2^3r_4R_4$$
$$+30683r_2^2R_2r_4R_4 - 28637r_2R_2^3r_4R_4 + 7623R_2^3r_4R_4 + 4942r_2^3R_4^2$$
$$+18488r_2^2R_2R_4^2 - 72684r_2R_2^2R_4^2 + 45630R_2^3R_4^2 - 450r_2^3R_6^2 - 40r_2^2R_2R_6^2$$
$$+2810r_2R_2^2R_6^2 - 3360R_2^3R_6^2)\sin(3\varphi_2 - \varphi_6)$$

$$
\begin{aligned}
&+R_6(-6480r_0^2r_2r_4^2-1620r_2^3r_4^2+25680r_0^2R_2r_4^2+2220r_2^2R_2r_4^2\\
&+39920r_2R_2^2r_4^2-100040R_2^3r_4^2-1260r_2r_4^4+4332R_2r_4^4\\
&-8160r_0^2r_2r_4R_4-2235r_2^3r_4R_4-2400r_0^2R_2r_4R_4+20395r_2^2R_2r_4R_4\\
&+22145r_2R_2^2r_4R_4-3105R_2^3r_4R_4-1626r_2r_4^3R_4-638R_2r_4^3R_4\\
&+16500r_0^2r_2R_4^2-1615r_2^3R_4^2-18900r_0^2R_2R_4^2+23545r_2^2R_2R_4^2\\
&-97740r_2R_2^2R_4^2+63990R_2^3R_4^2+8959r_2r_4^2R_4^2-20035R_2r_4^2R_4^2\\
&+9140r_2r_4R_4^3-1974R_2r_4R_4^3-9363r_2R_4^4+6615R_2R_4^4\\
&-1584r_2r_4^2R_6^2+5248R_2r_4^2R_6^2-3242r_2r_4R_4R_6^2+2074R_2r_4R_4R_6^2\\
&+976r_2R_4^2R_6^2-1872R_2R_4^2R_6^2)\sin(\varphi_2-2\varphi_4+\varphi_6)
\end{aligned}
$$

$$
\begin{aligned}
5806080V_{30}=3(&-4838400r_0^4r_2R_2r_4-4838400r_0^2r_2^3R_2r_4-302400r_2^5R_2r_4\\
&+14515200r_0^4R_2^2r_4-7257600r_0^2r_2^2R_2^2r_4-1582560r_2^4R_2^2r_4\\
&+354654720r_0^2r_2R_2^3r_4+39080160r_2^3R_2^3r_4-503677440r_0^2R_2^4r_4\\
&-1068480r_2^2R_2^4r_4-535036320r_2R_2^5r_4+509634720R_2^6r_4\\
&-9676800r_0^2r_2R_2r_4^3-1451520r_2^3R_2r_4^3+19353600r_0^2R_2^2r_4^3-1112832r_2^2R_2^2r_4^3\\
&+71968512r_2R_2^3r_4^3-80645376R_2^4r_4^3-967680r_2R_2r_4^5+1483776R_2^2r_4^5\\
&+2419200r_0^4r_2^2R_4+2419200r_0^2r_2^4R_4+151200r_2^6R_4-7257600r_0^4r_2R_2R_4\\
&+3628800r_0^2r_2^3R_2R_4+791280r_2^5R_2R_4-177327360r_0^2r_2^2R_2^2R_4\\
&-19540080r_2^4R_2^2R_4+251838720r_0^2r_2R_2^3R_4+534240r_2^3R_2^3R_4\\
&+267518160r_2^2R_2^4R_4-254817360r_2R_2^5R_4+4838400r_0^2r_2^2r_4^2R_4\\
&+725760r_2^4r_4^2R_4-5443200r_0^2r_2R_2r_4^2R_4-1628928r_2^3R_2r_4^2R_4\\
&-7620480r_0^2R_2^2r_4^2R_4-24284736r_2^2R_2^2r_4^2R_4-6975360r_2R_2^3r_4^2R_4\\
&+40678848R_2^4r_4^2R_4+483840r_2^2r_4^4R_4-122976r_2R_2r_4^4R_4-909216R_2^2r_4^4R_4\\
&-2116800r_0^2r_2^2r_4R_4^2+1092672r_2^4r_4R_4^2+79047360r_0^2r_2R_2r_4R_4^2\\
&+20456352r_2^3R_2r_4R_4^2-116847360r_0^2R_2^2r_4R_4^2-43319808r_2^2R_2^2r_4R_4^2\\
&-143887968r_2R_2^3r_4R_4^2+172428480R_2^4r_4R_4^2-309456r_2^2r_4^3R_4^2\\
&+12458376r_2R_2r_4^3R_4^2-14959224R_2^2r_4^3R_4^2-37618560r_0^2r_2^2R_4^3\\
&-13153056r_2^4R_4^3+58423680r_0^2r_2R_2R_4^3+33484416r_2^3R_2R_4^3\\
&+61774272r_2^2R_2^2R_4^3-86214240r_2R_2^3R_4^3-6001884r_2^2r_4^2R_4^3\\
&+5971644r_2R_2r_4^2R_4^3+2296224R_2^2r_4^2R_4^3+753984r_2^2r_4R_4^4\\
&-29546874r_2R_2r_4R_4^4+29765610R_2^2r_4R_4^4+14199381r_2^2R_4^5\\
&-14882805r_2R_2R_4^5-1088640r_0^2r_2^2r_4R_6^2-400176r_2^4r_4R_6^2\\
&-28831488r_0^2r_2R_2r_4R_6^2+12538680r_2^3R_2r_4R_6^2+62810496r_0^2R_2^2r_4R_6^2
\end{aligned}
$$

$$
\begin{aligned}
&-42077336r_2^2R_2^2r_4R_6^2 + 227367112r_2R_2^3r_4R_6^2 - 198677304R_2^4r_4R_6^2\\
&+1728r_2^2r_4^3R_6^2 + 32473536r_2R_2r_4^3R_6^2 - 43903616R_2^2r_4^3R_6^2\\
&+12129600r_0^2r_2^2R_4R_6^2 - 3575964r_2^4R_4R_6^2 - 30425472r_0^2r_2R_2R_4R_6^2\\
&+28853860r_2^3R_2R_4R_6^2 + 8817984r_0^2R_2^2R_4R_6^2 - 107740724r_2^2R_2^2R_4R_6^2\\
&+123174828r_2R_2^3R_4R_6^2 - 37751616R_2^4R_4R_6^2 - 19005240r_2^2r_4^2R_4R_6^2\\
&+55710608r_2R_2r_4^2R_4R_6^2 + 14042536R_2^2r_4^2R_4R_6^2 - 9749924r_2^2r_4R_4^2R_6^2\\
&+41382728r_2R_2r_4R_4^2R_6^2 - 108271908R_2^2r_4R_4^2R_6^2 - 23670494r_2^2R_4^3R_6^2\\
&+15788988r_2R_2R_4^3R_6^2 + 32537538R_2^2R_4^3R_6^2 - 146592r_2^2r_4R_6^4\\
&-2099712r_2R_2r_4R_6^4 + 4159392R_2^2r_4R_6^4 + 738672r_2^2R_4R_6^4\\
&-1967808r_2R_2R_4R_6^4 + 41157328R_2^2R_4R_6^4)\cos(2\varphi_2 - \varphi_4)\\
&+1008(-555r_2^4r_4 + 56608r_2^3R_2r_4 - 38349r_2^2R_2^2r_4 - 43890r_2R_2^3r_4\\
&+81402R_2^4r_4 + 4021r_2^4R_4 + 31012r_2^3R_2R_4 - 143285r_2^2R_2^2R_4\\
&+207714r_2R_2^3R_4 - 133830R_2^4R_4)R_6^2\cos(4\varphi_2 + \varphi_4 - 2\varphi_6)\\
&+6R_6^2(-6048000r_0^2r_4^3 - 952560r_2^2r_4^3 - 1878336r_2R_2r_4^3 + 39078192R_2^2r_4^3\\
&-774144r_4^5 - 403200r_0^2r_4^2R_4 - 12340812r_2^2r_4^2R_4 + 22732296r_2R_2r_4^2R_4\\
&+3831876R_2^2r_4^2R_4 - 102144r_4^4R_4 + 10886400r_0^2r_4R_4^2 - 10437666r_2^2r_4R_4^2\\
&+35813316r_2R_2r_4R_4^2 - 69863778R_2^2r_4R_4^2 + 5167008r_4^3R_4^2 - 3175200r_0^2R_4^3\\
&+2863653r_2^2R_4^3 - 18366426r_2R_2R_4^3 + 28959525R_2^2R_4^3 + 1682688r_4^2R_4^3\\
&-4843944r_4R_4^4 + 1025136R_4^5 - 203136r_4^3R_6^2 - 209024r_4^2R_4R_6^2\\
&+134816r_4R_4^2R_6^2 - 13056R_4^3R_6^2)\cos(3\varphi_4 - 2\varphi_6)\\
&+1008(-270r_2^5r_4 + 1557r_2^4R_2r_4 - 51674r_2^3R_2^2r_4 + 100173r_2^2R_2^3r_4\\
&-85786r_2R_2^4r_4 + 44880R_2^5r_4 - 1350r_2^5R_4 + 41713r_2^4R_2R_4 - 67686r_2^3R_2^2R_4\\
&-69103r_2^2R_2^3R_4 + 208086r_2R_2^4R_4 - 102060R_2^5R_4)R_6\cos(5\varphi_2 - \varphi_4 - \varphi_6)\\
&+R_6(14515200r_0^4r_2r_4 + 29030400r_0^2r_2^3r_4 + 2630880r_2^5r_4 - 14515200r_0^4R_2r_4\\
&-19958400r_0^2r_2^2R_2r_4 + 3994704r_2^4R_2r_4 - 879621120r_0^2r_2R_2^2r_4\\
&-196773696r_2^3R_2^2r_4 - 49714560r_0^2R_2^3r_4 + 90305712r_2^2R_2^3r_4\\
&+1121089536r_2R_2^4r_4 + 613164384R_2^5r_4 + 29030400r_0^2r_2r_4^3 + 8255520r_2^3r_4^3\\
&+5806080r_0^2R_2r_4^3 - 1542240r_2^2R_2r_4^3 - 200009376r_2R_2^2r_4^3 - 110673696R_2^3r_4^3\\
&+2903040r_2r_4^5 + 1852416R_2r_4^5 + 7257600r_0^4r_2R_4 + 11793600r_0^2r_2^3R_4\\
&+837648r_2^5R_4 - 65318400r_0^4R_2R_4 - 113944320r_0^2r_2^2R_2R_4 - 407232r_2^4R_2R_4\\
&-559742400r_0^2r_2R_2^2R_4 - 142654176r_2^3R_2^2R_4 + 2103252480r_0^2R_2^3R_4\\
&+563249232r_2^2R_2^3R_4 + 686257488r_2R_2^4R_4 - 2035892880R_2^5R_4\\
&-1088640r_0^2r_2r_4^2R_4 + 9820440r_2^3r_4^2R_4 - 75261312r_0^2R_2r_4^2R_4
\end{aligned}
$$

$$
\begin{aligned}
&-20488608r_2^2R_2r_4^2R_4 + 6927480r_2R_2^2r_4^2R_4 + 273134736R_2^3r_4^2R_4\\
&-654048r_2r_4^4R_4 - 5719968R_2r_4^4R_4 - 239246784r_0^2r_2r_4R_4^2\\
&-115636752r_2^3r_4R_4^2 + 21373632r_0^2R_2r_4R_4^2 + 122921820r_2^2R_2r_4R_4^2\\
&+748026216r_2R_2^2r_4R_4^2 - 85991220R_2^3r_4R_4^2 - 37911096r_2r_4^3R_4^2\\
&-12968424R_2r_4^3R_4^2 - 156836736r_0^2r_2R_4^3 - 69390678r_2^3R_4^3\\
&+525813120r_0^2R_2R_4^3 + 321576948r_2^2R_2R_4^3 - 13137390r_2R_2^2R_4^3\\
&-663390000R_2^3R_4^3 - 18524268r_2r_4^2R_4^3 + 59460444R_2r_4^2R_4^3\\
&+92465838r_2r_4R_4^4 + 21981402R_2r_4R_4^4 + 66988809r_2R_4^5 - 133945245R_2R_4^5\\
&-62625024r_0^2r_2r_4R_6^2 - 22453416r_2^3r_4R_6^2 - 7241472r_0^2R_2r_4R_6^2\\
&+67243224r_2^2R_2r_4R_6^2 - 33006264r_2R_2^2r_4R_6^2 - 5372280R_2^3r_4R_6^2\\
&-41942016r_2r_4^3R_6^2 - 15793152R_2r_4^3R_6^2 - 48779136r_0^2r_2R_4R_6^2\\
&-19430844r_2^3R_4R_6^2 + 124612992r_0^2R_2R_4R_6^2 + 93240516r_2^2R_2R_4R_6^2\\
&-153716436r_2R_2^2R_4R_6^2 + 132669900R_2^3R_4R_6^2 - 25863360r_2r_4^2R_4R_6^2\\
&+61119168R_2r_4^2R_4R_6^2 - 5121408r_2r_4R_4^2R_6^2 - 15689088R_2r_4R_4^2R_6^2\\
&-9476496r_2R_4^3R_6^2 + 30348432R_2R_4^3R_6^2 + 3611712r_2r_4R_6^4\\
&+2305216R_2r_4R_6^4 + 3285088r_2R_4R_6^4 - 2939616R_2R_4R_6^4)\cos(\varphi_2 + \varphi_4 - \varphi_6)\\
&+126(-7920r_2^3r_4^3 + 17568r_2^2R_2r_4^3 - 382384r_2R_2^2r_4^3 + 660736R_2^3r_4^3\\
&-17244r_2^3r_4^2R_4 + 1053168r_2^2R_2r_4^2R_4 - 3814476r_2R_2^2r_4^2R_4\\
&+2910712R_2^3r_4^2R_4 - 242128r_2^3r_4R_4^2 + 1520250r_2^2R_2r_4R_4^2\\
&-831804r_2R_2^2r_4R_4^2 - 568878R_2^3r_4R_4^2 - 4753r_2^3R_4^3 - 961706r_2^2R_2R_4^3\\
&+1652739r_2R_2^2R_4^3 - 609480R_2^3R_4^3)R_6\cos(3\varphi_2 - 3\varphi_4 + \varphi_6)\\
&+30240r_0(2R_2r_4 - r_2R_4)(-60r_2^3r_4 + 27r_2^2R_2r_4 - 2279r_2R_2^2r_4 + 2764R_2^3r_4\\
&-225r_2^3R_4 + 6443r_2^2R_2R_4 - 14199r_2R_2^2R_4 + 8473R_2^3R_4)\sin(4\varphi_2 - 2\varphi_4)\\
&+672r_0(-13050r_2^2r_4^2 - 411228r_2R_2r_4^2 - 99034R_2^2r_4^2 - 150156r_2^2r_4R_4\\
&-224420r_2R_2r_4R_4 + 743904R_2^2r_4R_4 - 116899r_2^2R_4^2 + 316008r_2R_2R_4^2\\
&-189945R_2^2R_4^2)R_6^2\sin(2\varphi_2 + 2\varphi_4 - 2\varphi_6)\\
&+672r_0R_6(10800r_0^2r_2^3 + 4050r_2^5 - 25200r_0^2r_2^2R_2 + 2430r_2^4R_2\\
&-111600r_0^2r_2R_2^2 - 216900r_2^3R_2^2 + 183600r_0^2R_2^3 + 38970r_2^2R_2^3\\
&+1429290r_2R_2^4 - 1131840R_2^5 + 16200r_2^3r_4^2 - 14112r_2^2R_2r_4^2\\
&+599490r_2R_2^2r_4^2 - 426306R_2^3r_4^2 + 15561r_2^3r_4R_4 - 627543r_2^2R_2r_4R_4\\
&+703578r_2R_2^2r_4R_4 - 188784R_2^3r_4R_4 - 104166r_2^3R_4^2 - 292170r_2^2R_2R_4^2\\
&+1689597r_2R_2^2R_4^2 - 1389825R_2^3R_4^2 + 4452r_2^3R_6^2 - 1142r_2^2R_2R_6^2\\
&-26036r_2R_2^2R_6^2 + 52374R_2^3R_6^2)\sin(3\varphi_2 - \varphi_6)
\end{aligned}
$$

$$+48r_0R_6(453600r_0^2r_2r_4^2+340200r_2^3r_4^2-2066400r_0^2R_2r_4^2-541800r_2^2R_2r_4^2$$
$$-11463228r_2R_2^2r_4^2+32507916R_2^3r_4^2+264600r_2r_4^4-1038744R_2r_4^4$$
$$+504000r_0^2r_2r_4R_4+423990r_2^3r_4R_4+504000r_0^2R_2r_4R_4$$
$$-5847282r_2^2R_2r_4R_4-2959656r_2R_2^2r_4R_4-3627540R_2^3r_4R_4$$
$$+292572r_2r_4^3R_4+342804R_2r_4^3R_4-1423800r_0^2r_2R_4^2+518238r_2^3R_4^2$$
$$+1927800r_0^2R_2R_4^2-7031766r_2^2R_2R_4^2+34151670r_2R_2^2R_4^2$$
$$-27380430R_2^3R_4^2-2501142r_2r_4^2R_4^2+6423102R_2r_4^2R_4^2-2429406r_2r_4R_4^3$$
$$+163338R_2r_4R_4^3+3281271r_2R_4^4-2841615R_2R_4^4+250704r_2r_4^2R_6^2$$
$$-977840R_2r_4^2R_6^2+607660r_2r_4R_4R_6^2-538316R_2r_4R_4R_6^2$$
$$-6164r_2R_4^2R_6^2+331956R_2R_4^2R_6^2)\sin(\varphi_2-2\varphi_4+\varphi_6)$$

Appendix 2

Lyapunov constants V_{34}, V_{38}, V_{42}, V_{46} and V_{50} if $\varphi_2=\overline{\varphi}_2$, $\varphi_4=\overline{\varphi}_4$ and $R_2=r_2=0$:

$$20160V_{34}=r_0(2r_4+3R_4)R_6^2(138600r_0^2r_4^2+52983r_4^4-235200r_0^2r_4R_4$$
$$-87248r_4^3R_4+66150r_0^2R_4^2-315000r_4^2R_4^2+390845r_4R_4^3$$
$$-81165R_4^4-8732r_4^2R_6^2+22992r_4R_4R_6^2-5120R_4^2R_6^2)\cos(3\varphi_4-2\varphi_6)$$

$$69672960V_{38}=(2r_4+3R_4)R_6^2(-914457600r_0^4r_4^2-696511872r_0^2r_4^4$$
$$-35334144r_4^6+1795046400r_0^4r_4R_4+1342510848r_0^2r_4^3R_4+67226976r_4^5R_4$$
$$-575769600r_0^4R_4^2+5169679200r_0^2r_4^2R_4^2+483433704r_4^4R_4^2$$
$$-7536052944r_0^2r_4R_4^3-625115880r_4^3R_4^3+1755710208r_0^2R_4^4$$
$$-1520820684r_4^2R_4^4+1603289781r_4R_4^5-297613278R_4^6$$
$$+461758464r_0^2r_4^2R_6^2+410379264r_4^4R_6^2-894424576r_0^2r_4R_4R_6^2$$
$$-1053694176r_4^3R_4R_6^2+210401152r_0^2R_4^2R_6^2+780043248r_4^2R_4^2R_6^2$$
$$-138347904r_4R_4^3R_6^2-421632R_4^4R_6^2-69988224r_4^2R_6^4$$
$$+60463808r_4R_4R_6^4-11648384R_4^2R_6^4)\cos(3\varphi_4-2\varphi_6)$$

$$1045094400V_{42}=2r_0(2r_4+3R_4)R_6^2(11887948800r_0^4r_4^2+15042827520r_0^2r_4^4$$
$$+2282994720r_4^6-26519270400r_0^4r_4R_4-33242711040r_0^2r_4^3R_4$$
$$-5014698336r_4^5R_4+9601804800r_0^4R_4^2-136170493200r_0^2r_4^2R_4^2$$

$$
\begin{gathered}
-38191940640r_4^4R_4^2 + 228712173480r_0^2r_4R_4^3 + 5729532100r_4^3R_4^3 \\
-59607394560r_0^2R_4^4 + 149395182813r_4^2R_4^4 - 182983090257r_4R_4^5 \\
+37337736348R_4^6 - 18529689600r_0^2r_4^2R_6^2 - 34470889632r_4^4R_6^2 \\
+37218090240r_0^2r_4R_4R_6^2 + 100626768240r_4^3R_4R_6^2 - 9736597440r_0^2R_4^2R_6^2 \\
-68442521256r_4^2R_4^2R_6^2 - 4211615760r_4R_4^3R_6^2 + 4136282208R_4^4R_6^2 \\
+8272065504r_4^2R_6^4-9313170928r_4R_4R_6^4+1912589824R_4^2R_6^4)\cos(3\varphi_4-2\varphi_6) \\
75(-4r_4 + R_4)(-3r_4 + 2R_4)(2r_4 + 3R_4)(1327296r_4^3 \\
+19843796r_4^2R_4 - 20383496r_4R_4^2 - 1728291R_4^3)R_6^4\sin(6\varphi_4 - 4\varphi_6)
\end{gathered}
$$

$$
\begin{gathered}
91968307200V_{46} = (2r_4 + 3R_4)R_6^2(-3379835289600r_0^6r_4^2 \\
-6397545369600r_0^4r_4^4 - 1937185113600r_0^2r_4^6 - 54305095680r_4^8 \\
+8449588224000r_0^6r_4R_4+15952669286400r_0^4r_4^3R_4+4826838677760r_0^2r_4^5R_4 \\
+135316583424r_4^7R_4 - 3420071424000r_0^6R_4^2 + 69313567276800r_0^4r_4^2R_4^2 \\
+38876560835904r_0^2r_4^4R_4^2+1498071421440r_4^6R_4^2-132079841308800r_0^4r_4R_4^3 \\
-66496774817088r_0^2r_4^3R_4^3 - 2348406553680r_4^5R_4^3 + 38312523849600r_0^4R_4^4 \\
-185006717902512r_0^2r_4^2R_4^4 - 13755736794888r_4^4R_4^4 \\
+258934879994472r_0^2r_4R_4^5+15681789602424r_4^3R_4^5-58177277073036r_0^2R_4^6 \\
+36609399850122r_4^2R_4^6 - 37938282905457r_4R_4^7 + 6998986202670R_4^8 \\
+36704130310656r_0^2R_4^4R_6^2 + 2541876069888r_4^6R_6^2 \\
-25916348620800r_0^4r_4R_4R_6^2 - 120263829386496r_0^2r_4^3R_4R_6^2 \\
-9046063347264r_4^5R_4R_6^2 + 7561283558400r_0^4R_4^2R_6^2 \\
+78635634936192r_0^2r_4^2R_4^2R_6^2 - 2374325413920r_4^4R_4^2R_6^2 \\
+25829007452160r_0^2r_4R_4^3R_6^2 + 37899940759488r_4^3R_4^3R_6^2 \\
-11128821558912r_0^2R_4^4R_6^2 - 26904377492424r_4^2R_4^4R_6^2 \\
+914580229356r_4R_4^5R_6^2 + 902107181376R_4^6R_6^2 - 12125300174592r_0^2r_4^2R_6^4 \\
-181709148672r_4^4R_6^4+16466967049984r_0^2r_4R_4R_6^4+3153124508160r_4^3R_4R_6^4 \\
-3689043404992r_0^2R_4^2R_6^4+956869371072r_4^2R_4^2R_6^4-3228272024736R_4R_4^3R_6^4 \\
+731160778176R_4^4R_6^4 + 415604386560r_4^2R_6^6 - 408352334720r_4R_4R_6^6 \\
+83307868160R_4^2R_6^6)\cos(3\varphi_4 - 2\varphi_6) \\
+2640r_0(2r_4 + 3R_4)(-455829696r_4^5 - 6767061984r_4^4R_4 \\
+16432034974r_4^3R_4^2 - 10466970455r_4^2R_4^3 + 1510245169r_4R_4^4 \\
+59824842R_4^5)R_6^4\sin(6\varphi_4 - 4\varphi_6)
\end{gathered}
$$

$$
\begin{aligned}
77253378048000V_{50} = {} & 16r_0(2r_4+3R_4)R_6^2(271593907200000r_0^6r_4^2\\
&+718003759334400r_0^4r_4^4+361591721568000r_0^2r_4^6+30352145947200r_4^8\\
&-752415713280000r_0^6r_4R_4-1994827071283200r_0^4r_4^3R_4\\
&-1008038270131200r_0^2r_4^5R_4-84902354515440r_4^7R_4\\
&+337480577280000r_0^6R_4^2-9167708919600000r_0^4r_4^2R_4^2\\
&-8569765028230560r_0^2r_4^4R_4^2-989688146568660r_4^6R_4^2\\
&+19572262150795200r_0^4r_4R_4^3+16482771673719120r_0^2r_4^3R_4^3\\
&+1753143714727020r_4^5R_4^3-6281488640750400r_0^4R_4^4\\
&+48735553839283080r_0^2r_4^2R_4^4+10876390621592265r_4^4R_4^4\\
&-76902061918900680r_0^2r_4R_4^5-14022391631378130r_4^3R_4^5\\
&+18997655805464490r_0^2R_4^6-35041944502919970r_4^2R_4^6\\
&+41245874064718785r_4R_4^7-8271309346572195R_4^8\\
&-1853048345702400r_0^4r_4^2R_6^2-8373709508693760r_0^2r_4^4R_6^2\\
&-1715393101496832r_4^6R_6^2+4352296354560000r_0^4r_4R_4R_6^2\\
&+30465818747844480r_0^2r_4^3R_4R_6^2+6786514275431328r_4^5R_4R_6^2\\
&-1409641647321600r_0^4R_4^2R_6^2-19772652431246400r_0^2r_4^2R_4^2R_6^2\\
&+2347272979697520r_4^4R_4^2R_6^2-11934312155740800r_0^2r_4R_4^3R_6^2\\
&-35274101084492904r_4^3R_4^3R_6^2+4784378140140480r_0^2R_4^4R_6^2\\
&+24948570463242372r_4^2R_4^4R_6^2+3224180449643040r_4R_4^5R_6^2\\
&-1961192324253024R_4^6R_6^2+3672315542206080r_0^2r_4^2R_6^4\\
&+801554059696320r_4^4R_6^4-5767137201893760r_0^2r_4R_4R_6^4\\
&-4841493356766864r_4^3R_4R_6^4+1420367473435680r_0^2R_4^2R_6^4\\
&+525140624175552r_4^2R_4^2R_6^4+4103356619127108r_4R_4^3R_6^4\\
&-1044257974466616R_4^4R_6^4-620889749612160r_4^2R_6^6\\
&+742218830691520r_4R_4R_6^6-154486688359360R_4^2R_6^6)\cos(3\varphi_4-2\varphi_6)\\
&+45(2r_4+3R_4)R_6^4(140299047060480r_0^2r_4^5+9659542007808r_4^7\\
&+2194955785989120r_0^2r_4^4R_4+142867869040512r_4^6R_4\\
&-6161335204837120r_0^2r_4^3R_4^2-511339958384640r_4^5R_4^2\\
&+4605825276236800r_0^2r_4^2R_4^3-530954773022520r_4^4R_4^3\\
&-898541589865920r_0^2r_4R_4^4+2105533556210700r_4^3R_4^4\\
&+13405333196640r_0^2R_4^5-1347003154420770r_4^2R_4^5\\
&+189985289577237r_4R_4^6+6318552230298R_4^7-57606305504256r_4^5R_6^2\\
&+77343332004864r_4^4R_4R_6^2+65699961724032r_4^3R_4^2R_6^2
\end{aligned}
$$

$$-91794931651968 r_4^2 R_4^3 R_6^2 + 18415533943584 r_4 R_4^4 R_6^2$$
$$-91829177856 R_4^5 R_6^2) \sin(6\varphi_4 - 4\varphi_6)$$

Appendix 3

Lyapunov constant V_{34} if $\varphi_2 = \overline{\varphi}_2$, $\varphi_4 = \overline{\varphi}_4$ and $R_4 = r_4 = 0$:

$$967680 V_{34} = 252(-r_2 - 3R_2)(r_2 - 2R_2)(3r_2 - 2R_2)$$
$$(-75r_2^3 + 3268 r_2^2 R_2 - 7058 r_2 R_2^2 + 8835 R_2^3) R_6^2 \sin(6\varphi_2 - 2\varphi_6)$$
$$+(r_2+3R_2)R_6(-1814400 r_0^4 r_2^2 - 1360800 r_0^2 r_2^4 - 68040 r_2^6 + 11289600 r_0^4 r_2 R_2$$
$$+3961440 r_0^2 r_2^3 R_2 + 6552 r_2^5 R_2 - 14112000 r_0^4 R_2^2 + 82474560 r_0^2 r_2^2 R_2^2$$
$$+7798644 r_2^4 R_2^2 - 310433760 r_0^2 r_2 R_2^3 - 16590168 r_2^3 R_2^3 + 233089920 r_0^2 R_2^4$$
$$-115120404 r_2^2 R_2^4 + 315596736 r_2 R_2^5 - 158623920 R_2^6 - 530208 r_0^2 r_2^2 R_6^2$$
$$+930753 r_2^4 R_6^2 + 1312192 r_0^2 r_2 R_2 R_6^2 - 8931882 r_2^3 R_2 R_6^2 - 4068512 r_0^2 R_2^2 R_6^2$$
$$+32406981 r_2^2 R_2^2 R_6^2 - 48002400 r_2 R_2^3 R_6^2 + 27025164 R_2^4 R_6^2 - 191688 r_2^2 R_6^4$$
$$+579488 r_2 R_2 R_6^4 - 260344 R_2^2 R_6^4) \sin(3\varphi_2 - \varphi_6)$$

Lyapunov constant V_{34} if $\varphi_2 = \overline{\varphi}_2$, $\varphi_4 = \overline{\varphi}_4$ and $R_6 = 0$:

$$92160 V_{34} = r_0(-2R_2 r_4 + r_2 R_4)(-138240 r_0^4 r_2 - 230400 r_0^2 r_2^3 - 43200 r_2^5$$
$$+483840 r_0^4 R_2 - 364800 r_0^2 r_2^2 R_2 - 250800 r_2^4 R_2 + 22191360 r_0^2 r_2 R_2^2$$
$$+7353600 r_2^3 R_2^2 - 36691200 r_0^2 R_2^3 - 636480 r_2^2 R_2^3 - 133370400 r_2 R_2^4$$
$$+148886640 R_2^5 - 460800 r_0^2 r_2 r_4^2 - 207360 r_2^3 r_4^2 + 1075200 r_0^2 R_2 r_4^2$$
$$-160512 r_2^2 R_2 r_4^2 + 13521024 r_2 R_2^2 r_4^2 - 17604736 R_2^3 r_4^2 - 138240 r_2 r_4^4$$
$$+247296 R_2 r_4^4 + 254400 r_0^2 r_2 r_4 R_4 - 345024 r_2^3 r_4 R_4 - 517440 r_0^2 R_2 r_4 R_4$$
$$+2179104 r_2^2 R_2 r_4 R_4 - 10570304 r_2 R_2^2 r_4 R_4 + 10004448 R_2^3 r_4 R_4$$
$$+112512 r_2 r_4^3 R_4 - 186224 R_2 r_4^3 R_4 + 4680960 r_0^2 r_2 R_4^2 + 4927872 r_2^3 R_4^2$$
$$-8467200 r_0^2 R_2 R_4^2 - 14659744 r_2^2 R_2 R_4^2 - 29497824 r_2 R_2^2 R_4^2$$
$$+49502880 R_2^3 R_4^2 + 2239792 r_2 r_4^2 R_4^2 - 3254100 R_2 r_4^2 R_4^2 - 376980 r_2 r_4 R_4^3$$
$$+615464 R_2 r_4 R_4^3 - 7028122 r_2 R_4^4 + 8631495 R_2 R_4^4) \cos(2\varphi_2 - \varphi_4)$$
$$+(-2R_2 r_4 + r_2 R_4)(-86400 r_0^2 r_2^3 r_4 - 8640 r_2^5 r_4 + 56160 r_0^2 r_2^2 R_2 r_4$$
$$-29952 r_2^4 R_2 r_4 - 4335360 r_0^2 r_2 R_2^2 r_4 + 269824 r_2^3 R_2^2 r_4 + 6060000 r_0^2 R_2^3 r_4$$
$$-762048 r_2^2 R_2^3 r_4 + 12286016 r_2 R_2^4 r_4 - 11845920 R_2^5 r_4 - 17280 r_2^3 r_4^3$$
$$+8520 r_2^2 R_2 r_4^3 - 666400 r_2 R_2^2 r_4^3 + 765880 R_2^3 r_4^3 - 367200 r_0^2 r_2^3 R_4$$
$$-34320 r_2^5 R_4 + 12192000 r_0^2 r_2^2 R_2 R_4 + 1106416 r_2^4 R_2 R_4$$

$$-31220640r_0^2r_2R_2^2R_4 - 583872r_2^3R_2^2R_4 + 22245600r_0^2R_2^3R_4$$
$$-31882336r_2^2R_2^3R_4 + 66701472r_2R_2^4R_4 - 35519040R_2^5R_4 - 76800r_2^3r_4^2R_4$$
$$+1908170r_2^2R_2r_4^2R_4 - 3778720r_2R_2^2r_4^2R_4 + 2136070R_2^3r_4^2R_4$$
$$+415040r_2^3r_4R_4^2 - 1877435r_2^2R_2r_4R_4^2 + 4832470r_2R_2^2r_4R_4^2$$
$$-3468915R_2^3r_4R_4^2 - 20185r_2^3R_4^3 - 5499785r_2^2R_2R_4^3$$
$$+13014180r_2R_2^2R_4^3 - 7600050R_2^3R_4^3)\sin(4\varphi_2 - 2\varphi_4)$$

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