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PLANAR VECTOR FIELD VERSIONS OF CARATHÉODORY'S AND LOEWNER'S CONJECTURES

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Abstract

Let $r = 3, 4, \dots, \infty, \omega$. The C^r-Carathéodory's Conjecture states that every C^r convex embedding of a 2-sphere into $\mathbb{R}^3$ must have at least two umbilics. The C^r-Loewner's conjecture (stronger than the one of Carathéodory) states that there are no umbilics of index bigger than one. We show that these two conjectures are equivalent to others about planar vector fields. For instance, if $r \neq \omega$, C^r-Carathéodory's Conjecture is equivalent to the following one:

Let $\rho > 0$ and $\beta : U \subset \mathbb{R}^2 \to \mathbb{R}$, be of class C^r, where U is a neighborhood of the compact disc $D(0,\rho) \subset \mathbb{R}^2$ of radius ρ centered at 0. If β restricted to a neighborhood of the circle $\partial D(0,\rho)$ has the form $\beta(x,y) = (ax^2 + by^2)/(x^2 + y^2)$, where $a < b < 0$, then the vector field (defined in U) that takes (x,y) to $(\beta_{xx}(x,y) - \beta_{yy}(x,y), 2\beta_{xy}(x,y))$ has at least two singularities in $D(0,\rho)$.

1. Introduction

The classical Carathéodory's Conjecture states that every smooth convex embedding of a 2-sphere in $\mathbb{R}^3$, i.e. an ovaloid, must have at least two umbilics. A well known approach to the problem is based on a "semi-local" argument. For any surface in $\mathbb{R}^3$, the eigenspaces of the second fundamental form define two orthogonal line fields (principal directions) whose singularities are exactly the umbilics. To each isolated umbilic we can attach the index of either one of the two fields, which is half of an integer, and the sum of these indexes is the Euler characteristic of the surface, if this is compact and all umbilics are isolated. So, if an ovaloid

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has only one umbilic, it must have index two. We just observe that, up to an inversion in $\mathbb{R}^3$, we can always suppose that the curvature at a given umbilic is positive and therefore the convexity hypothesis is not relevant for this argument. Examples of umbilics of index j are known for all $j \leq 1$ and a local conjecture, known as the Loewner conjecture, states that there are no umbilics of index bigger than one. Several authors, among whom are H. Hamburger [**Ham**], G. Bol [**Bol**], T. Klotz [**Klo**], C. J. Titus [**Tit**] and Scherbel [**Sch**] have asserted this conjecture for analytic surfaces to be true; implying therefore Carathéodory's Conjecture for analytic surfaces. Nevertheless, Klotz [**Klo**] pointed out a gap in Bol's proof; also, Scherbel [**Sch**] claims that there are gaps in the works of Klotz and Titus (see also [**Lan**], [**Yau**]). Related to the subject, we wish to mention the works [**GMS**] of Gutierrez, Mercuri and Sánchez-Bringas, [**GS1**], [**GS2**] of Gutierrez and Sánchez-Bringas, and [**SX1**], [**SX2**] of Smyth and Xavier.

In this paper we show that these two conjectures are equivalent to others about planar vector fields. We hope that our results help to obtain simpler solutions in the analytic case and to find ways to attack the C^r case.

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2. Preliminaries

Orient the sphere $S^2 \subset \mathbb{R}^3$ so that the positive unitary normal vector at $p \in S^2$ is p itself. Let $\mathcal{S} \subset \mathbb{R}^3$ be a *C^r-ovaloid.* By this we mean that $\mathcal{S}$ is an oriented C^r-embedded surface such that its Gauss map $N : \mathcal{S} \to S^2$ is an orientation preserving diffeomorphism. This definition implies that $\mathcal{S}$ is convex, compact and that its Gaussian curvature is positive everywhere. We define the *support function of* $\mathcal{S}$ as the map $\sigma : S^2 \to \mathbb{R}$ given by

$$\sigma(p) = p \cdot N^{-1}(p),$$

where the dot stands for the usual inner product.

Given $\delta \in \{-, +\}$, let $\Pi^\delta : \mathbb{R}^2 \to S^2 \setminus \{(0, 0, \delta(1))\}$ be the diffeomorphism given by

$$\Pi^\delta(x, y) = \left(\frac{2x}{1 + x^2 + y^2}, \frac{2y}{1 + x^2 + y^2}, \frac{\delta(x^2 + y^2 - 1)}{1 + x^2 + y^2} \right).$$

That is, Π^δ is the inverse map of the corresponding stereographic projection. The map

$$\Phi^\delta(x, y) = (X^\delta(x, y), Y^\delta(x, y), Z^\delta(x, y)) = N^{-1} \circ \Pi^\delta(x, y),$$

defined in $\mathbb{R}^2$, provides a global C^{r-1} parametrization of $\mathcal{S}\backslash\{N^{-1}(0,0,\delta(1))\}$ called *Bonnet chart associated to* $(\mathcal{S},\Pi^\delta)$. Given the support function σ of $\mathcal{S}$, associated to $(\mathcal{S},\Pi^\delta)$ we define the *Bonnet function*

$$\beta^\delta(x,y)=(1+x^2+y^2)\sigma(\Pi^\delta(x,y)).$$

Let $\Lambda^\delta(x,y)=(1+x^2+y^2)\Pi^\delta(x,y)$; that is,

$$\Lambda^\delta(x,y)=(2x,2y,\delta(x^2+y^2-1)).$$

As $\Lambda^\delta\cdot\Phi^\delta_x=\Lambda^\delta\cdot\Phi^\delta_y=0$, where the subindex means the partial derivative with respect to this variable, we have that $\Lambda^\delta_x\cdot\Phi^\delta=\beta^\delta_x$ and $\Lambda^\delta_y\cdot\Phi^\delta=\beta^\delta_y$. This together with $\Lambda^\delta\cdot\Phi^\delta=\beta^\delta$ can be written in matrix notation as $M^\delta\cdot\Phi^\delta=\mathcal{B}^\delta$, where

$$(1)\quad M^\delta=\begin{pmatrix}2x & 2y & \delta(x^2+y^2-1)\\ 2 & 0 & \delta(2x)\\ 0 & 2 & \delta(2y)\end{pmatrix},\ \Phi^\delta=\begin{pmatrix}X^\delta\\ Y^\delta\\ Z^\delta\end{pmatrix},\ \mathcal{B}^\delta=\begin{pmatrix}\beta^\delta\\ \beta^\delta_x\\ \beta^\delta_y\end{pmatrix}.$$

As N is of class C^{r-1}, Φ^δ is also of class C^{r-1}. Therefore, $M^\delta\cdot\Phi^\delta=\mathcal{B}^\delta$ implies that β^δ is of class C^r. Since, for all $(x,y)\in\mathbb{R}^2$, the determinant of M^δ is $-(\delta)4(1+x^2+y^2)\neq 0$, we may write $\Phi^\delta=(M^\delta)^{-1}\cdot\mathcal{B}^\delta$. From this, using a symbolic computer system we can obtain the first and second fundamental forms of Φ^δ and therefore the proof of proposition below (see [**GMS**], [**Dar**], [**Bon**]).

Proposition 2.1. *Let $\mathcal{S}\subset\mathbb{R}^3$ be a C^r ovaloid, $r\geq 3$. Then the support function σ of $\mathcal{S}$ is of class C^r and the differential equation of the principal lines of curvature of $\mathcal{S}$ in its Bonnet chart, associated to $(\mathcal{S},\Pi^\delta)$, is given by $\omega^\delta=0$ where*

$$(2)\qquad \omega^\delta=\beta^\delta_{xy}dx^2+(\beta^\delta_{yy}-\beta^\delta_{xx})dxdy-\beta^\delta_{xy}dy^2$$

and

$$\beta^\delta(x,y)=(1+x^2+y^2)\sigma(\Pi^\delta(x,y)).$$

The proof of the following proposition and theorem can be found in [**GMS**].

Proposition 2.2. *Let $\beta:\mathbb{R}^2\to\mathbb{R}$ be a C^r function, with $r\geq 3$. Suppose that the 2-jet $j^2\beta_{(0,0)}$ of β at $(0,0)$ has the form*

$$(3)\qquad j^2\beta_{(0,0)}(x,y)=a_{00}+a_{10}x+a_{01}y+a_{20}x^2+2a_{11}xy+a_{02}y^2$$

and that

$$(4)\qquad a_{00}^2+a_{00}(a_{02}+a_{20})-a_{11}^2+a_{02}a_{20}\neq 0.$$

Then there exists an open neighborhood $U\subset\mathbb{R}^2$ of $(0,0)$ such that $\beta|_U$ is the Bonnet function of an oriented C^r surface embedded in $\mathbb{R}^3$.

The following will be needed later

Lemma 2.3. *Let $\delta \in \{-,+\}$ and let*

$$\beta^\delta(x,y) = (1+x^2+y^2)\sigma(\Pi^\delta(x,y))$$

be the functions determined by σ as above. If $\mathcal{I} : \Re^2 \setminus \{0\} \to \Re^2 \setminus \{0\}$ is the inversion

$$\mathcal{I}(u,v) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right),$$

then, for all $(x,y) \in \Re^2 \setminus \{0\}$,

$$(x^2+y^2)\beta^- \circ \mathcal{I}(x,y) = \beta^+(x,y). \tag{5}$$

Proof: The result follows inmediately from the identity

$$\sigma(\Pi^+(x,y) = \sigma(\Pi^- \circ \mathcal{I}(x,y)). \quad \blacksquare$$

Let $\mathcal{F}$ be a one dimensional C^1-foliation defined on a neighborhood U of $0 \in \mathbb{R}^2$. Suppose that $\mathcal{F}$ has exactly one singularity which is 0. The index of 0 (i.e. of $\mathcal{F}$ at 0) is one-half of the degree of the map that takes each point q of a small circle centered at 0 to the element, of the projective circle $\mathbb{R}P$, which is tangent at q to the leaf of $\mathcal{F}$. Here, we identify $\mathbb{R}^2$ with the tangent space of U at q; also, the projective circle $\mathbb{R}P$ is the well known quotient space obtained from $\mathbb{R}^2 \setminus \{0\}$. If $\mathcal{F}$ is orientable, this definition coincides with the usual Hopf-Poincaré index.

If p is an isolated umbilic point of a C^r oriented surface $\mathcal{S} \subset \mathbb{R}^3$, with $r \geq 3$, the index of p is defined by using local coordinates and either one of the two foliations induced by the principal lines of curvature of $\mathcal{S}$. The umbilics are precisely the singularities of these foliations.

We say that a C^r vector field ξ on $\mathbb{R}^2$ fulfills a *Lojasiewicz-inequality at* $(0,0)$ if there exist $k \in \mathbb{N}^*$ and $\delta > 0$ such that $||\xi(x,y)|| \geq \delta||(x,y)||^k$ on some neighborhood of $(0,0)$. Under these circumstances, we will also say that ξ satisfies a *Lojasiewicz-inequality of order k (with associated constant δ) at* $(0,0)$. Suppose that a C^r oriented surface $\mathcal{S} \subset \mathbb{R}^3$, with $r \geq 3$, has an isolated umbilic point $p \in \mathcal{S}$. We will say that p *is an umbilic of Lojasiewicz-type* (of order k with $1 \leq k \leq r-2$) if there is a local C^r diffeomorphism φ of a neigborhood of $p \in \mathbb{R}^3$ onto an open set of $\mathbb{R}^3$ such that the image surface $\varphi(\mathcal{S}) = \tilde{\mathcal{S}}$ satisfies the following properties:

a) $\tilde{p} = \varphi(p)$ is an isolated umbilic of $\tilde{\mathcal{S}}$ with the same index of p and $\tilde{\mathcal{S}}$ has positive curvature in $\tilde{p}$ and unit normal vector $(0,0,-1)$.

b) The Bonnet function $\tilde{\beta}$ of $\tilde{\mathcal{S}}$ is such that the vector field $\tilde{\xi}(x,y) = (\tilde{\beta}_{xx} - \tilde{\beta}_{yy}, 2\tilde{\beta}_{xy})$ satisfies a Lojasiewicz-inequality of order k at $(0,0)$.

Remark 2.4.

(a) The composition of an appropriate rigid translation and the inversion $\mathcal{I}(p) = \frac{p}{||p||^2}$ preserves the principal lines of curvature, hence umbilics and their indexes as well. Thus an inversion may be used to transform a flat umbilic into an umbilic of positive curvature. Therefore, up to a conformal diffeomorphism, the first condition is always satisfied.

(b) With the notation right above, the index of ξ at $(0,0)$ is twice the index of the umbilic point p [**SX1**].

(c) If a vector field on $\mathbb{R}^2$ satisfies a Lojasiewicz-inequality at the singular point $(0,0)$, then $(0,0)$ is an isolated singularity of the vector field.

(d) Suppose that $Y : (U,(0,0)) \to (\mathbb{R}^2,(0,0))$ is an analytic vector field defined in an open set $U \subset \mathbb{R}^2$. Then $(0,0)$ is an isolated singular point of Y if, and only if, Y satisfies a Lojasiewicz-inequality at $(0,0)$ [**Loj**]. Therefore, using (a) above, an analytic surface immersed in $\mathbb{R}^3$ always satisfies a Lojasiewiecz-inequality at an isolated umbilic point.

The proofs of the following lemma and theorem right below are given in ([**GMS**]).

Lemma 2.5. *If $\xi : (U,(0,0)) \to (\mathbb{R}^2,(0,0))$ is a C^r vector field, $r \geq 1$, defined in a neighborhood U of $(0,0)$ and satisfying a Lojasiewicz-inequality of order k, $1 \leq k \leq r$ at $(0,0)$, then*

(a) *the k-jet $j^k\xi_0$ of ξ at $(0,0)$ satisfies a Lojasiewicz-inequality of order k at $(0,0)$*

(b) *both ξ and its k-jet $j^k\xi_0$ at $(0,0)$ have the same index at their common isolated singularity $(0,0)$.*

Theorem 2.6. *Assuming the truth of the Loewner's Conjecture for isolated umbilics on analytic surfaces, if a C^r surface $\mathcal{S} \subset \mathbb{R}^3$, with $r \geq 3$, satisfies a Lojasiewicz-inequality at an umbilic point p, then the index of p is at most 1. Therefore if a C^r immersion of a sphere has one umbilic of Lojasiewicz type, it must have at least one more umbilic.*

A proof of the following result can be found in [**GS1**]. See also [**D-G**], [**Fir**], [**LLR**], [**Nir**].

Theorem 2.7. *Let $\mathcal{S} \subset \mathbb{R}^3$ be a C^r ovaloid, $r \geq 3$. Then the inverse $N^{-1} : S^2 \to \mathcal{S}$, of the Gauss map N, can be written as follows:*

$$N^{-1}(u,v,w) = \sigma(u,v,w) \cdot (u,v,w) + \mathcal{A}(u,v,w) \cdot \nabla\sigma(u,v,w)$$

where $\sigma : S^2 \to \mathbb{R}$ denotes the support function of $\mathcal{S}$, $\nabla\sigma$ its gradient vector field and

$$\mathcal{A}(u,v,w) = \begin{pmatrix} v^2+w^2 & -uv & -uw \\ -uv & u^2+w^2 & -vw \\ -uw & -vw & u^2+v^2 \end{pmatrix}. \tag{6}$$

Conversely, given a C^r function $\sigma : S^2 \to \mathbb{R}$, $r \geq 3$, there exists a constant $c > 0$ such that $\sigma + c$ is the support function of an ovaloid of class C^r.

3. Equivalent conjectures

Let $r = 3, 4, \ldots, \infty, \omega$. The conjectures that we are interested in are the following ones:

C^r-Loewner's Conjecture (i.e. C^r-LC).

The index of an umbilic, of a surface C^r embedded in $\mathbb{R}^3$, is at most one.

C^r-Loewner's Conjecture* (i.e. C^r-LC*).

Let $\beta : U \subset \mathbb{R}^2 \to \mathbb{R}$, be a map of class C^r defined in a neighborhood U of $(0,0) \in \mathbb{R}^2$. If $(0,0)$ is an isolated singularity of the vector field

$$X : (x,y) \to (\beta_{xx} - \beta_{yy}, 2\beta_{xy}),$$

then the index of X at $(0,0)$ is less or equal than 2.

C^r-Loewner's Conjecture with Lojasiewicz condition (i.e. C^r-LC with LC).

The index of an umbilic of Lojasiewicz-type, of a surface C^r embedded in $\mathbb{R}^3$, is at most one.

C^r-Loewner's Conjecture* with Lojasiewicz condition (i.e. C^r-LC* with LC).

Let β and X be as in C^r-LC*. If X satisfies a Lojasiewicz Condition at the singularity 0, then the index of X at 0 is less or equal than 2.

C^r-Carathéodory's Conjecture (i.e. C^r-CC).

Every C^r convex embedding of a 2-sphere in $\mathbb{R}^3$ must have at least two umbilics.

C^r-Carathéodory's Conjecture* (i.e. C^r-CC*).

Let $\rho > 0$ and $\beta : U \subset \mathbb{R}^2 \to \mathbb{R}$, be of class C^r, where U is a neighborhood of the compact disc $D(0,\rho) \subset \mathbb{R}^2$ of radius ρ centered at 0. If β restricted to a neighborhood of the circle $\partial D(0,\rho)$ has the form

$$\beta(x,y) = \frac{ax^2 + by^2}{x^2 + y^2}$$

where $a < b < 0$, then the vector field (defined in U)

$$X : (x,y) \to (\beta_{xx} - \beta_{yy}, 2\beta_{xy})$$

has at least two singularities in $D(0,\rho)$.

Theorem 3.1. *Let $r = 3, 4, \ldots, \infty, \omega$.*

(a) *C^r-LC is equivalent to C^r-LC**

(b) *C^r-LC with LC is equivalent to C^r-LC* with LC*

(c) *The following conjectures are equivalent:*

(c1) C^ω-LC,*

*(c2) Polynomial-LC**

(c3) C^r-LC with LC.*

(d) *If $r \neq \omega$, C^r-CC is equivalent to C^r-CC**

Proof: The proofs of (a) and (b) are the same; they follow from Propositions 2.1, 2.2 and Remark 2.4(b).

The proof of (c) follows from Lemma 2.5 and Remark 2.4(d). See [**GMS**, Theorem 3.3].

Let us proceed to prove (d). We shall use the notations introduced in Section 2.

First, we shall see that if C^r-CC* is true then C^r-CC is true.

Without lost of generality, we may assume that the ovaloid is tangent to the xy-plane at $\bar{0} = (0,0,0)$, that $\bar{0}$ is not an umbilic point and that $N(\bar{0}) = (0,0,1)$. By using the formula $M \cdot \Phi^- = \mathcal{B}^-$ and the assumption that $\Phi(0,0) = \bar{0}$, we conclude that $\beta^-(0,0) = \beta^-_x(0,0) = \beta^-_y(0,0) = 0$. Therefore, by rotating the ovaloid around the z-axis if necessary, we may assume that

$$\beta^-(x,y) = ax^2 + by^2 + \text{higher order terms},$$

and that $|a| \leq |b|$. It is easy to see that the assumptions imply that β has a local maximun at $\bar{0}$. By the convexity of the ovaloid and as $\bar{0}$ is not umbilic, we obtain that $a < b < 0$.

By a small C^2-perturbation of the ovaloid, around $\bar{0}$, we may assume that, around $\bar{0}$,

$$\beta^-(x, y) = ax^2 + by^2.$$

As, by Lemma 2.3,

$$\beta^+(u, v) = (u^2 + v^2)\beta^- \left(\frac{u}{u^2 + v^2}, \frac{v}{u^2 + v^2} \right)$$

we get that, there exists $\rho > 0$ such that, for all (u, v) in an neighborhood of $\{(u, v) : u^2 + v^2 = \rho^2\}$,

$$\beta^+(u, v) = \frac{au^2 + bv^2}{u^2 + v^2}.$$

By the assumption this implies that the vector field

$$(u, v) \to (\beta^+_{uu} - \beta^+_{vv}, 2\beta^+_{uv})$$

has at least two singularities in $\{(u, v) : u^2 + v^2 \leq \rho^2\}$. Each of these singularities is taken by the parametrization Φ^+ to an umbilic of the ovaloid. This proves that C^r-CC is true.

Now, we shall see that if C^r-CC is true then C^r-CC* is true. Suppose that we have a function $\beta = \beta^+ = \beta^+(u, v)$, and real numbers ρ, a, b, with $a < b < 0$, as in the assumptions of C^r-CC*. In particular, we are assuming that β^+ restricted to a neighborhood of the circle $\partial D(0, \rho)$ has the form

$$\beta^+(u, v) = \frac{au^2 + bv^2}{u^2 + v^2}.$$

Without lost of generality, we may assume that β^+ is defined in the whole $\mathbb{R}^2$ and that, when restricted to a neighborhood of the set $\{(u, v) \in \mathbb{R}^2 : u^2 + v^2 \geq \rho\}$, is given by the expression right above. Define $\beta^- = \beta^-(x, y)$, in $\mathbb{R}^2 \setminus \{(0, 0)\}$ by the expression

$$\beta^-(x, y) = (x^2 + y^2)\beta^+ \left(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2} \right).$$

Let $\beta^-(0, 0) = 0$. It can be seen that β^- restricted to a neighborhood of

$$S = \left\{ (x, y) : x^2 + y^2 \leq \frac{1}{\rho^2} \right\}$$

has the form $\beta^-(x,y) = ax^2 + by^2$. By construction, β^+ and β^- can be made the Bonnet functions of the support function $\sigma : S^2 \to \mathbb{R}$ of class C^r. By Theorem 2.8, there exists $c \in \mathbb{R}$ such that we have an ovaloid associated to $\sigma + c$. By the assumptions, the ovaloid has at least two umbilics. These umbilics must be contained in $\Phi^+(D(0,\rho))$. Each of these umbilics is taken by $(\Phi^+)^{-1}$ to a singularity (contained in $D(0,\rho)$) of the vector field

$$(u,v) \to (\beta^+_{uu} - \beta^+_{vv}, 2\beta^+_{uv}).$$

This proves that if C^r-CC is true then C^r-CC* is also true. ■

Remark 3.2. Let $S \subset \mathbb{R}^3$ be a C^r ovaloid and $N^{-1} : S^2 \to S$ be inverse of the Gauss map N. There exists a quadratic differential form ω on S^2, as defined by V. Guíñez (see [**Gui**], [**Mic**]), such that its expression in the parametrization Π^δ, with $\delta \in \{-,+\}$, is the quadratic differential ω^δ of Proposition 2.1. The map N^{-1} takes the pair of foliations associated to ω to the pair of foliations tangent to the principal lines of S. The foliations associated to ω^δ, under the name of Hessian foliations, are considered in [**SX2**].

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