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Abstract

We study random mechanism design in an environment where the set of alternatives has a Cartesian product structure. We first show that all generalized random dictatorships are strategy-proof on a minimally rich domain if and only if the domain is a *top-separable domain*. We next generalize the notion of connectedness (Monjardet, 2009) to establish a particular class of top-separable domains: *connected$^+$ domains*, and show that in the class of minimally rich and connected$^+$ domains, the *multidimensional single-peakedness* restriction is necessary and sufficient for the design of a *flexible* random social choice function that is unanimous and strategy-proof. Such a flexible function is distinct from generalized random dictatorships in that it allows for a systematic notion of compromise. Our characterization remains valid (under an additional hypothesis) for a problem of voting with constraints where not all alternatives are feasible (Barberà et al., 1997).

Keywords: Generalized random dictatorships; Top-separable domains; Connected$^+$ domains; Multidimensional single-peaked domains; Constrained voting

JEL Classification: D71.

1 Introduction

Multidimensional models arise very naturally in economic environments as it is often the case that the object of choice consists of several attributes or components (commodities in consumer theory, positions in political economy, different levels of provision of distinct public goods, etc), with no dependence across choices in different components.[1] The set of alternatives thus has the structure of a Cartesian product set, i.e., $A \equiv \times_{s\in M} A^s$ where s is a component and A^s is a component set.[2] The underlying Cartesian product structure on the set of alternatives allows for a richer description of available alternatives and introduces furthermore the possibility of

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[1] See for instance, legislative, political and club-member elections (e.g., Border and Jordan, 1983; Barberà et al., 1991, 1993, 1999, 2005; Aswal et al., 2003) and public goods location and provision problems (e.g., Zhou, 1991; Peters et al., 1992; Chichilnisky and Heal, 1997; Le Breton and Sen, 1999; Le Breton and Weymark, 1999; Ehlers, 2002; Svensson and Torstensson, 2008; Reffgen and Svensson, 2012).

[2] We pick an element in each component set of the Cartesian product structure, and assemble these selected elements to form an alternative.

defining domains of restricted preferences which take cognizance of the multidimensional structure and allow positive results for aggregation and economic design. We explore the theoretical underpinnings of such multidimensional preference domains from the perspective of mechanism design. We first identify a particular condition, *top-separability*, which turns out to be fundamental in formulating multidimensional preferences that admit new possibilities for mechanism design.[3] Our principal finding is that within the class of top-separable preferences, *multidimensional single-peaked domains* (introduced by Barberà et al., 1993), a particular generalization of the notion of single-peakedness to a multidimensional setting, emerge as the unique preference domains that allow for the design of attractive random mechanisms. Thus the notion of single-peakedness, which is well-studied and prominent in aggregation theory, voting theory and political economy, turns out to be a particularly distinguished one in the context of multidimensional random mechanism design.

We focus on probabilistic mechanisms in multidimensional settings in the absence of monetary transfers where the set of alternatives is assumed to be finite.[4] We impose a strong version of the incentive compatibility requirement by requiring that truth-telling first order stochastically dominate every possible manipulation of preferences. We thus study Random Social Choice Functions (RSCFs) that satisfy the ordinal version of strategy-proofness formulated by Gibbard (1977).[5] We also impose throughout the assumption that RSCFs satisfy unanimity, which says that if an alternative is top ranked for every agent at a particular profile of preferences, then it receives probability one under the RSCFs at that profile.

An important class of RSCFs is the class of *random dictatorships*. These are defined by fixing a probability distribution over agents; the probability assigned to an alternative at a preference profile is then the sum of the weights of the agents who have this particular alternative as their top ranked alternative. Random dictatorships are strategy-proof and ex-post efficient (a strengthening of unanimity), and allow for a equitable distribution of power among agents which is precluded by a deterministic dictatorship. These are however not entirely satisfactory from the design point of view as they *lack flexibility*; indeed *any* alternative that is not top ranked for some agent at the profile in question, can never get strictly positive probability. In particular, such an alternative may be second ranked for all agents in a profile where agents disagree on peaks; we refer to such an alternative as a *compromise alternative* and suggest that it is desirable to design RSCFs that have the flexibility to give positive probability to such an alternative. [6]

Under a Cartesian product structure, random dictatorships can be naturally generalized to

[3]In a top-separable preference, when we compare two alternatives which disagree on one component but agree on all other components, the alternative that inherits the element of the top ranked alternative in that particular disagreed component is always preferred. Throughout the paper, the term *multidimensional preference* will refer to a preference which satisfies top-separability (and possibly some other restrictions).

[4]We focus on the classic voting model which we hope will be useful in formulating more general models where some of the dimensions include private goods or monetary transfers. Recent work (e.g., Morimoto and Serizawa, 2015; Kazumura et al., 2017) studies formulations with monetary transfers under non-quasi-linear preferences.

[5]This is equivalent to requiring that the expected utility of truth-telling be at least as large as the expected utility of manipulating, for *every* possible utility representation of the primitive ordinal preference.

[6]Gibbard (1977) proved that on the domain of unrestricted preferences, the only strategy-proof and unanimous RSCFs are random dictatorships. Recent literature has examined restricted preference domains where one may design more flexible RSCFs that are strategy-proof and unanimous. See for instance, almost random dictatorships (Chatterji et al., 2014), fixed probabilistic ballots rules (Ehlers et al., 2002) and probabilistic generalized median voter schemes (Peters et al., 2014).

accord with the multidimensional setting in the following way. Instead of fixing a probability distribution over agents, we fix a probability over *voter sequences*, where a voter sequence is an $|M|$-tuple of agents, and where each component is associated with an agent who can be viewed as the dictator of that component (and one agent can be associated to multiple components). At a preference profile, we can assemble, according to a voter sequence a unique alternative whose $k-th$ component is the $k-th$ component of the peak of the dictator for the $k-th$ component as specified by the voter sequence. The probability assigned to an alternative at a preference profile is then the sum of the weights of the voter sequences which can assemble this particular alternative. These random mechanisms are called *generalized random dictatorships* and were introduced by Chatterji et al. (2012). Generalized random dictatorships recognize the Cartesian product structure and allow for greater flexibility than do random dictatorships as at some preference profile, some non-peak alternatives can be assembled and receive strictly positive probability. In contrast to random dictatorships, certain preference restrictions must however be imposed to ensure strategy-proofness of a generalized random dictatorship. We show in Proposition 1 that top-separability is necessary and sufficient for the strategy-proofness of all generalized random dictatorships. However, due to the somewhat limited assembling capability of voter sequences, generalized random dictatorships sometimes ignore compromise alternatives.

This paper examines restricted domains of multidimensional preferences that allow us to construct strategy-proof RSCFs which are flexible in that they *systematically* admit compromise. The preference domains we study satisfy a particular "richness" property that is based on the idea of connectedness initially proposed by Grandmont (1978) and Monjardet (2009), and has been recently adopted to explore various issues which include the equivalence of local strategy-proofness and strategy-proofness (e.g., Sato, 2013; Cho, 2016; Mishra, 2016), the extent to which RSCFs can depend on agents' preferences (Chatterji and Zeng, 2017), and the characterization of preference restrictions that allow one to design attractive RSCFs (Chatterji et al., 2016). The notion of connectedness requires that one be able to reconcile the differences in two preferences via a sequence of preferences in the domain where each successive pair of preferences involves a "local switch" of two contiguously ranked alternatives. This richness condition restricts the probabilities received by alternatives that do not switch between two successive pairs of preferences, and plays a fundamental methodological role in deriving the results mentioned above.

However, this notion of connectedness does not apply to domains of multidimensional preferences, e.g., the top-separable domain, as it is often the case that multiple pairs of alternatives have to be switched simultaneously across two successive preferences. We introduce a new notion of a connectedness which permits the requisite simultaneous local switches and allows us to investigate systematically domains of multidimensional preferences that permit the design of nice strategy-proof RSCFs. The domains we consider are termed *connected$^+$ domains*; these are subsets of the top-separable domain, and include the well studied instances of separable preferences (Barberà et al., 1991; Le Breton and Sen, 1999), multidimensional single-peaked preferences (Barberà et al., 1993), and their intersection and unions. Connected$^+$ domains also possess the requisite generality and structure that would in principle allow one to investigate other issues being studied in the literature (like the equivalence of local strategy-proofness and strategy-proofness, etc, alluded to above) and can presumably be exploited beyond this paper.

In the class of connected$^+$ domains, multidimensional single-peaked domains are an important and well studied class. These are a particular generalization of the idea of single-peaked preferences to a multidimensional setting using the Cartesian product structure and the city block metric. Our first theorem characterizes multidimensional single-peaked domains as the unique domains that permit the design of strategy-proof and unanimous RSCFs that systematically depart from random dictatorships/generalized random dictatorships, in that they admit compromises, wherein compromise alternatives necessarily receive strictly positive probability whenever they appear (see Theorem 1). Our version of multidimensional single-peaked domains allows elements of each component set to be arranged on a tree which is a generalization of multidimensional single-peakedness initiated by Barberà et al. (1993).[7] In the special case where the connected$^+$ domain contains two complete reversals preferences, we refine the domain characterization to the more familiar formulation of Barberà et al. (1993). We next provide a characterization result for multidimensional single-peaked domains using deterministic social choice functions (see Theorem 2). We do so by replacing the compromise property by the familiar axiom of anonymity.[8]

We finally turn to the setup of voting under constraints originally proposed by Barberà et al. (1997). Here, not all alternatives in the underlying Cartesian product structure are feasible. We investigate what structure on the set of feasible alternatives and preferences (applicable now only to the restriction of the original preferences to the feasible alternatives) would allow us to define RSCFs which satisfy our requirements of strategy-proofness, compromise, etc on connected$^+$ domains. We deduce that the set of feasible alternatives must be factorizable as a Cartesian product of trees and the preferences must satisfy a particular version of multidimensional single-peakedness w.r.t. the peak of the set of feasible alternatives (see Theorem 3). Our results are therefore robust to restrictions on feasibility.

The rest of the paper is organized as follows. The remainder of the Introduction explains in greater detail the relation of this paper to the literature. Section 2 describes the model and introduces generalized random dictatorships. Section 3 presents the domain characterization results for multidimensional single-peaked preferences, while Section 4 concludes. The Appendix gathers proofs, examples and verifications not included in the main text.

1.1 Related Literature

Much of the literature on multidimensional models has focused on deterministic social choice functions (DSCFs). The early literature proved impossibility results for various generalizations of single-peakedness to cases where the set of alternatives are a convex subset of $\mathbb{R}^M$ (e.g., Border and Jordan, 1983; Bordes et al., 1990; Zhou, 1991; Peters et al., 1992). The case of separable preferences over a convex subset of alternatives was analyzed by Le Breton and Weymark (1999) while general results on formulations where the set of alternatives is a subset of a metric space were presented by Weymark (2008). Barberà et al. (1991) provide a possibility result of *voting by committees* when the number of elements in each component set is two, while in the general case with finitely many elements in each component studied by Le Breton and Sen (1999), strategy-proof DSCFs degenerate to *generalized dictatorships* which are the deterministic counterparts of generalized random dictatorships. Positive characterization results for generalized median

[7] In the formulation of multidimensional single-peakedness of Barberà et al. (1993), all elements of each component set are located on a line.

[8] Anonymity implies that the social choice is immune to the identities of agents.

voter schemes have been introduced by Barberà et al. (1993) who proposed the restriction of multidimensional single-peakedness, and by Barberà et al. (1997) who introduced the intersection property on generalized median voter schemes to accord with voting under constraints. A comprehensive survey of these results is provided by Sprumont (1995) and Barberà (2010). Besides the characterizations of strategy-proof DSCFs on multidimensional domains, several papers also verify the necessity of separable preferences (see Hatsumi et al., 2014), and various versions of multidimensional single-peaked preferences for the existence of particular generalized median voter schemes (e.g., Barberà et al., 1993, 1999) or strategy-proof DSCFs satisfying various well-behavedness criteria (e.g., neutrality and anonymity in Nehring and Puppe, 2007, and tops-only property and anonymity in Chatterji and Massó, 2016).[9]

The literature on random mechanism design on restricted domains arising from multidimensional models is not as large. An early paper by Dutta et al. (2002) studies lotteries defined on a convex subset of $\mathbb{R}^M$ where preferences are convex, continuous and single-peaked, and establishes a random dictatorship result. More recently, Chatterji et al. (2012) characterize generalized random dictatorships on the lexicographically separable domain, which is a particular subset of the separable domain (and is excluded by the class of connected$^+$ domains), while Chatterji and Zeng (2017) characterize random dictatorships using strategy-proofness and ex-post efficiency on the multidimensional single-peaked domain. Recently, Chatterji et al. (2016) characterize single-peaked preferences on a tree in the class of connected domains. The characterization of the multidimensional single-peaked domain in this paper differs from their result in two important ways. First, Chatterji et al. (2016) uses an extra tops-only axiom on the RSCFs and secondly, as mentioned earlier, their connectedness assumption excludes the multidimensional domains studied in this paper. In the present paper, the tops-only property emerges *endogenously* (Proposition 3) from our richness condition. Our richness condition is a strengthening of the "Interior and Exterior" properties of Chatterji and Zeng (2017) that was shown to precipitate the tops-only property. Their results do not apply for the connected$^+$ domains we study in this paper. We extend their tops-only result to our setting by postulating the existence of sufficiently many separable preferences that allow the sort of multiple switches of alternatives we alluded to earlier. This strengthening is critical for establishing that the alternatives of the Cartesian product structure be embedded in a product of trees and that preferences be multidimensional single-peaked as stated in Theorem 1. Similarly, our characterization result for multidimensional single-peaked domains using deterministic social choice functions extends the analysis of Chatterji et al. (2013) to multidimensional domains (which were excluded by their hypothesis of connected domains) and does so by endogenizing the tops-only property (which was imposed as an axiom in their paper). For the set up of voting under constraints, Barberà et al. (1997) characterized all unanimous and strategy-proof DSCFs on the multidimensional single-peaked domain for arbitrary feasible sets. We investigate and provide an answer to the converse question for RSCFs: What can be inferred about the structure of the set of feasible alternatives and the preferences from the existence of a well-behaved strategy-proof RSCF satisfying the properties of compromise, tops-onlyness, etc, on a connected$^+$ domain?

[9]Neutrality implies that the social choice is immune to relabelings of alternatives. The tops-only property implies that when each agent has the same preference peak across two preference profiles, the social choices remain identical at two profiles.

2 Preliminaries

Let A be a *finite* set of alternatives with $|A| \geq 3$. We assume that the alternative set can be represented as a **Cartesian product** of a finite number of sets, each of which contains finitely many elements. Formally, let $A = \times_{s \in M} A^s$ where $M = \{1, 2, \dots, m\}$, $m \geq 2$ is an integer; and $|A^s| \geq 2$ is an integer for each $s \in M$.[10] Each $s \in M$ is called a **component**; A^s is referred to as a **component set**, and an element in A^s is denoted as a^s. Accordingly, an alternative is represented by a m-tuple, i.e., $a \equiv (a^1, a^2, \dots, a^m) \equiv (a^s)_{s \in M}$. Given a nonempty strict subset $S \subset M$, let $A^S = \times_{s \in S} A^S$, $a^S \equiv (a^s)_{s \in S} \in A^S$; $A^{-S} \equiv \times_{s \notin S} A^s$ and $a^{-S} \equiv (a^s)_{s \notin S} \in A^{-S}$.[11] Therefore, we also write alternative $a \equiv (a^s, a^{-s}) \equiv (a^S, a^{-S})$. In particular, we say a pair of alternatives $a, b \in A$ is **similar** if they disagree on *exactly one* component, i.e., $a^s \neq b^s$ and $a^{-s} = b^{-s}$ for some $s \in M$. For notational convenience, given non-empty $S \subset M$, $X^S \subseteq A^S$ and $Y^{-S} \subseteq A^{-S}$, let $(X^S, Y^{-S}) = \{a \in A | a^S \in X^S \text{ and } a^{-S} \in Y^{-S}\}$.[12] Let $\Delta(A)$ denote the space of lotteries over A. An element of $\Delta(A)$ is thus a lottery or probability distribution over A. In particular, $e_a \in \Delta(A)$ is a degenerate lottery where alternative a is chosen with probability one.

Let $I = \{1, \dots, N\}$ be a *finite* set of voters with $N \geq 2$. Each voter i has a preference order P_i over A which is complete, antisymmetric and transitive, i.e., a *linear order*. For any $a, b \in A$, aP_ib is interpreted as "a is *strictly* preferred to b according to P_i".[13] Two preferences P_i, P_i' are **complete reversals** if $[aP_ib] \Leftrightarrow [bP_i'a]$ for all $a, b \in A$. We use $\boldsymbol{aP_i!b}$ to denote that a is *contiguously* ranked above b in P_i, i.e., aP_ib and there exists no $c \in A$ such that aP_ic and cP_ib. Given a preference P_i, let $r_k(P_i)$ denote the kth ranked alternative in P_i, $1 \leq k \leq |A|$. Let $\mathbb{P}$ denote the set containing *all* linear orders over A. The set of all admissible preferences is a set $\mathbb{D} \subseteq \mathbb{P}$, referred to as a **preference domain**. We call $\mathbb{P}$ *the complete domain*. When $\mathbb{D} \neq \mathbb{P}$, $\mathbb{D}$ is referred to as a **restricted domain**. For notational convenience, given $a \in A$, let $\mathbb{D}^a = \{P_i \in \mathbb{D} | r_1(P_i) = a\}$ denote a preference subdomain where each preference's peak is a. Correspondingly, a domain $\mathbb{D}$ is **minimally rich** if $\mathbb{D}^a \neq \emptyset$ for every $a \in A$.

Each voter presents a preference, and all reported preferences are collected to formulate a preference profile $P \equiv (P_1, P_2, \dots, P_N) \equiv (P_i, P_{-i}) \in \mathbb{D}^N$. A **Random Social Choice Function** (or RSCF) is a map $\varphi : \mathbb{D}^N \to \Delta(A)$, which associates each preference profile $P \in \mathbb{D}^N$ to a "socially desirable" lottery $\varphi(P)$. For any $a \in A$, $\varphi_a(P)$ is the probability with which the alternative a will be chosen in $\varphi(P)$. Thus, $\varphi_a(P) \geq 0$ for all $a \in A$ and $\sum_{a \in A} \varphi_a(P) = 1$. A **Deterministic Social Choice Function** (or DSCF) is a particular RSCF where a degenerate lottery is specified under each preference profile, i.e., $\varphi(P) = e_a$ for some $a \in A$ at profile P.[14] An RSCF satisfies unanimity if it assigns probability one to an alternative that is top ranked by all voters, i.e., an RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ is **unanimous** if $[r_1(P_i) = a \text{ for all } i \in I] \Rightarrow [\varphi_a(P) = 1]$ for all $a \in A$ and $P \in \mathbb{D}^N$. Next, an RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ is **strategy-proof** if for all $i \in I$, $P_i, P_i' \in \mathbb{D}$ and $P_{-i} \in \mathbb{D}^{N-1}$, the lottery $\varphi(P_i, P_{-i})$ first-order stochastically dominates $\varphi(P_i', P_{-i})$ according to P_i, i.e., $\sum_{k=1}^{t} \varphi_{r_k(P_i)}(P_i, P_{-i}) \geq \sum_{k=1}^{t} \varphi_{r_k(P_i)}(P_i', P_{-i})$, $t = 1, \dots, |A|$.

[10] To make sure all components indispensable, we assume $|A^s| \geq 2$ for every $s \in M$.

[11] In this paper, $\subseteq$ and $\subset$ denote the weak inclusion relation and the strict inclusion relation respectively.

[12] For instance, $(x^s, A^{-s}) \equiv \{a \in A | a^s = x^s\}$ and $(A^s, x^{-s}) \equiv \{a \in A | a^{-s} = x^{-s}\}$ are frequently used henceforth.

[13] In a table, we specify a preference "vertically". In a sentence, we specify a preference "horizontally". For instance, preference P_i: $a \rightharpoonup b \rightharpoonup c \rightharpoonup \cdots$ represents that a is at the top, b is the second best, c is the third ranked alternative while the rest of rankings in P_i are arbitrary.

[14] We sometimes simply write a DSCF as $f : \mathbb{D}^N \to A$.

A prominent class of unanimous and strategy-proof RSCFs is the class of random dictatorships (Gibbard, 1977). Each voter is assigned a non-negative weight such that the sum of the weights accross the voters add up to one. In a random dictatorship, at each preference profile, the probability received by an alternative is determined by the set of voters who prefer this alternative the most and equals the sum of these voters' weights. Formally, an RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ is a **random dictatorship** if there exists $\varepsilon_i \geq 0$ for each $i \in I$ with $\sum_{i \in I} \varepsilon_i = 1$ such that for all $P \in \mathbb{D}^N$ and $a \in A$, $\varphi_a(P) = \sum_{i \in I: r_1(P_i)=a} \varepsilon_i$.[15] Note that a random dictatorship is strategy-proof on any arbitrary preference domain.

2.1 Generalized random dictatorships

Under the Cartesian product setting, one may consider the following generalization of a random dictatorship. We associate each component $s \in M$ to a voter $i^s \in I$. Thus, a m-tuple of voters $\underline{i} = (i^s)_{s \in M} \in I^m$ forms a *voter sequence*. A voter sequence can be viewed as a combination of m dictators (one voter may appear multiple times); on each component $s \in M$, voter i^s is the dictator over the component set A^s. Given a profile $P \in \mathbb{D}^N$, for notational convenience, assume $r_1(P_i) = x_i \equiv (x_i^s)_{s \in M}$, $i \in I$. We say that an alternative $a \equiv (a^s)_{s \in M}$ is *assembled* by a voter sequence $\underline{i} \equiv (i^s)_{s \in M}$ at profile P, if $a^s = x^s_{i^s} \equiv r_1(P_{i^s})^s$ for all $s \in M$. Analogously to random dictatorships, we associate a non-negative weight to each voter sequence, denoted $\gamma(\underline{i}) \geq 0$, $\underline{i} \in I^m$, and let the sum of all voter sequences' weights equal to one, i.e., $\sum_{\underline{i} \in I^m} \gamma(\underline{i}) = 1$. Last, at a preference profile, the probability assigned to an alternative is determined by the set of voter sequences who can assemble this alternative. Such an RSCF is referred to as a generalized random dictatorship (Chatterji et al., 2012). Formally, an RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ is a **generalized random dictatorship**, if there exists $\gamma(\underline{i}) \geq 0$ for each $\underline{i} \in I^m$ with $\sum_{\underline{i} \in I^m} \gamma(\underline{i}) = 1$, such that for all $P \in \mathbb{D}^N$ and $a \in A$, $\varphi_a(P) = \sum\limits_{\underline{i} \equiv (i^s)_{s \in M} \in I^m:\, a = (r_1(P_{i^s})^s)_{s \in M}} \gamma(\underline{i})$.

Evidently, every generalized random dictatorship satisfies unanimity. If only *identity voter sequences* (i.e., one voter dictates all components) receive positive weights, a generalized random dictatorship degenerates to a random dictatorship, while on the other hand, if every voter sequence receives a strictly positive weight, we have a *strict generalized random dictatorship*, which prescribes a maximal support for the social lottery under each preference profile compared to other generalized random dictatorships. The characterization of random dictatorships Gibbard (1977) implies that a generalized random dictatorship where some voter sequence other than an identity voter sequence receives strictly positive weight fails to be strategy-proof.

Definition 1 *A preference P_i is **top-separable** if given $s \in M$ and $b^s \in A^s$, we have*

$$\left[r_1(P_i)^s \equiv a^s \neq b^s\right] \Rightarrow \left[(a^s, y^{-s}) P_i (b^s, y^{-s}) \text{ for all } y^{-s} \in A^{-s}\right].$$

Let $\mathbb{D}_{TS}$ denote **the top-separable domain** which contains all top-separable preferences. Henceforth, we use the term "*multidimensional domains*" to refer to subdomains of $\mathbb{D}_{TS}$.[16]

[15]In particular, if $\varepsilon_i = 1$ for some $i \in I$, the random dictatorship degenerates to a **dictatorship**.

[16]The Cartesian product structure would be redundant if it is not involved in establishing preference restrictions. The restriction of top-separability is indeed formulated w.r.t. the Cartesian product structure, and therefore distinguishes our model from the one-dimensional models in the literature (e.g., Gibbard, 1977). The tops-separable domain includes all restricted domains studied in this paper.

Note that the restriction of top-separability applies only to particular pairs of similar alternatives where one of the two disagreed elements is inherited from the preference peak. For a pair of similar alternatives where neither of the two disagreed elements coincides with the peak of the preference, their relative ranking in a top-separable preference is arbitrary. A significant strengthening of top-separability is *separability* which imposes restrictions on every pair of similar alternatives.

Definition 2 *A preference P_i is* ***separable*** *if given $s \in M$ and $a^s, b^s \in A^s$, we have*

$$\big[(a^s, x^{-s})P_i(b^s, x^{-s}) \text{ for some } x^{-s} \in A^{-s}\big] \Rightarrow \big[(a^s, y^{-s})P_i(b^s, y^{-s}) \text{ for all } y^{-s} \in A^{-s}\big].$$

The domain of separable preferences which includes all separable preferences is referred to as **the separable domain**, denoted $\mathbb{D}_S$. Evidently, $\mathbb{D}_S = \mathbb{D}_{TS}$ if $|A^s| = 2$ for all $s \in M$, and $\mathbb{D}_S \subset \mathbb{D}_{TS}$ if $|A^s| > 2$ for some $s \in M$. Given a separable preference P_i, we can induce a *marginal preference* on each component set, denoted $[P_i]^s$ over A^s, $s \in M$.[17]

The top-separable domain includes many multidimensional domains widely studied in the literature. We use Figure 1 below to summarize the relations across some important multidimensional domains: The top-separable domain includes the separable domain, the lexicographically separable domain (Chatterji et al., 2012)[18], and two multidimensional single-peaked domains (Barberà et al., 1993).[19]

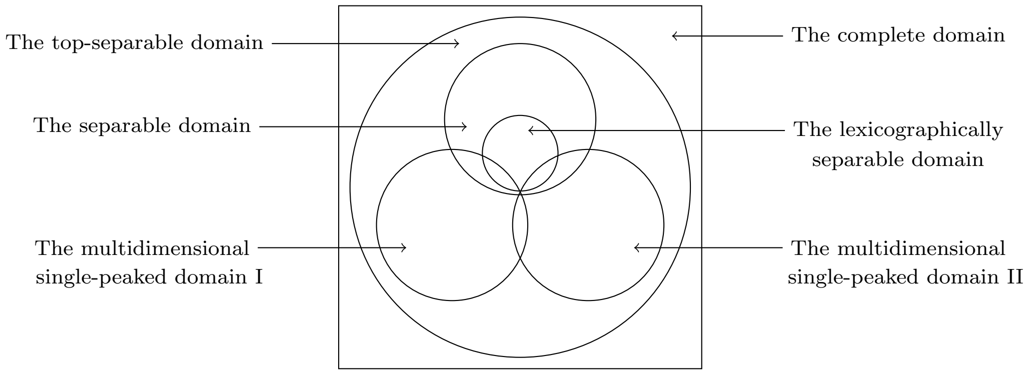


Figure 1: The relations among several domains

The proposition below implies that the top-separable domain is the maximal minimally rich domain for the strategy-proofness of *all* generalized random dictatorships.

Proposition 1 *Let $\mathbb{D}$ be a minimally rich domain. All generalized random dictatorships are strategy-proof on $\mathbb{D}$ if and only if $\mathbb{D} \subseteq \mathbb{D}_{TS}$.*

[17] For more detailed studies on separable preferences, please refer to Le Breton and Sen (1999), Barberà et al. (2005) and Reffgen and Svensson (2012).

[18] A preference $P_i \in \mathbb{D}_S$ is **lexicographically separable** if there exists a lexicographic order (a linear order) $\succ$ over M such that for all $x, y \in A$, we have $[x^s[P_i]^s y^s$ and $x^\tau = y^\tau$ for all $\tau \in M$ with $\tau \succ s] \Rightarrow [xP_iy]$. Let $\mathbb{D}_{LS}$ denote **the lexicographically separable domain** which contains merely but all lexicographically separable preferences. Evidently, $\mathbb{D}_{LS} \subset \mathbb{D}_S$.

[19] For instance, see two different multidimensional single-peaked domains of Examples 1 and 4 in Section 3 below. We will discuss multidimensional single-peaked preferences in Section 3 in detail.

Proof: Let $\mathbb{D} \subseteq \mathbb{D}_{TS}$, and $\varphi : \mathbb{D}^N \to \Delta(A)$ be a generalized random dictatorship, i.e., there exists $\gamma(\underline{i}) \geq 0$ for each $\underline{i} \in I^m$ with $\sum_{\underline{i}\in I^m} \gamma(\underline{i}) = 1$, such that for all $P \in \mathbb{D}^N$ and $a \in A$, $\varphi_a(P) = \sum_{\underline{i}\equiv(i^s)_{s\in M}\in I^m:\, a=(r_1(P_{i^s})^s)_{s\in M}} \gamma(\underline{i})$. Given a voter sequence $\underline{i} \equiv (i^s)_{s\in M} \in I^m$, let $f^{\underline{i}} : \mathbb{D}^N \to A$ be a DSCF called a *generalized dictatorship* such that for all $P \in \mathbb{D}^N$, $f^{\underline{i}}(P) = \big(r_1(P_{i^s})^s\big)_{s\in M}$. Then, φ can be re-expressed as a convex combination of generalized dictatorships, i.e., for all $P \in \mathbb{D}^N$, $\varphi(P) = \sum_{\underline{i}\in I^m} \gamma(\underline{i})\, f^{\underline{i}}(P)$. Therefore, to verify the strategy-proofness of φ, it suffices to show that every generalized dictatorship $f^{\underline{i}}$ is strategy-proof.

Fix a voter sequence $\underline{i} \equiv (i^s)_{s\in M} \in I^m$ and $i \in I$. Given $P_i, P_i' \in \mathbb{D}$ and $P_{-i} \in \mathbb{D}^{N-1}$, assume $f^{\underline{i}}(P_i, P_{-i}) = x \equiv (x^s)_{s\in M}$ and $f^{\underline{i}}(P_i', P_{-i}) = y \equiv (y^s)_{s\in M}$. We show either $x = y$ or xP_iy. Assume $r_1(P_i) = a \equiv (a^s)_{s\in M}$ and $r_1(P_i') = b \equiv (b^s)_{s\in M}$. Furthermore, in the voter sequence $\underline{i} \equiv (i^s)_{s\in M}$, we assume that there exists $S \subseteq M$ such that $i^s = i$ for all $s \in S$ and $i^\tau \neq i$ for all $\tau \notin S$. Consequently, $x^s = a^s$ and $y^s = b^s$ for all $s \in S$ and $x^{-S} = y^{-S}$. Evidently, if $S = \emptyset$, $x = y$. Similarly, if $S \neq \emptyset$ and $x^s = y^s$ for all $s \in S$, we also have $x = y$. Last, we assume that $S \neq \emptyset$, and there exists a non-empty $S^+ \subseteq S$ such that $a^s \neq b^s$ for all $s \in S^+$ and $a^\tau = b^\tau$ for all $\tau \in S\backslash S^+$. Consequently, $x^s = a^s$ and $y^s = b^s$ for all $s \in S^+$ and $x^{-S^+} = y^{-S^+} \equiv z^{-S^+}$. For notational simplicity, assume $S^+ = \{1, \ldots, s\}$. We identify alternatives $a_k = (a^1, \ldots, a^k, b^{k+1}, \ldots, b^s, z^{-S^+})$, $k = 0, 1, \ldots, s$. Thus, $a_0 = b$ and $a_s = a$. Since $\mathbb{D} \subseteq \mathbb{D}_{TS}$, top-separability implies $a_kP_ia_{k-1}$, $k = 1, \ldots, s$. Consequently, aP_ib by transitivity. This completes the verification of strategy-proofness of $f^{\underline{i}}$, as required.

Conversely, let all generalized random dictatorships be strategy-proof on domain $\mathbb{D}$. We show $\mathbb{D} \subseteq \mathbb{D}_{TS}$. Suppose that it is not true. Thus, there exist $\bar{P}_i \in \mathbb{D}$, $s \in M$, $b^s \in A^s$ and $z^{-s} \in A^{-s}$ such that $a^s = r_1(\bar{P}_i)^s$ and $(b^s, z^{-s})\bar{P}_i(a^s, z^{-s})$. We pick a strict generalized random dictatorship $\varphi : \mathbb{D}^N \to \Delta(A)$ and construct a particular preference profile $(\bar{P}_i, P_{-i})$ where $r_1(P_j) = (b^s, z^{-s})$ for all $j \neq i$. Given a voter sequence $\underline{i}$ with $i^s = i$ and $i^\tau \neq i$ for all $\tau \neq s$, we know that (a^s, z^{-s}) can be assembled by the voter sequence $\underline{i}$. Consequently, $\varphi_{(a^s,z^{-s})}(\bar{P}_i, P_{-i}) \geq \gamma(\underline{i}) > 0$. Given $P_i' \in \mathbb{D}^{(b^s,z^{-s})}$, it is evident that $\varphi_{(b^s,z^{-s})}(P_i', P_{-i}) = 1$. Since $(b^s, z^{-s})\bar{P}_i(a^s, z^{-s})$, voter i will manipulate at $(\bar{P}_i, P_{-i})$ via P_i'. Therefore, $\mathbb{D} \subseteq \mathbb{D}_{TS}$. $\square$

3 Main results

As mentioned earlier, random dictatorships never admit compromise as probabilities are assigned only to peak alternatives in every preference profile. Generalized random dictatorships improve upon random dictatorships in this respect by diversifying social lotteries. However, they do not *systematically* admit compromise since not every compromise alternative can be assembled by the peaks of some preference profile, and hence such an alternative is ignored by generalized random dictatorships.

For instance, two voters may disagree strongly on each other's most preferred alternatives but may nonetheless have a common second best alternative, e.g., $r_1(P_i) = (a^1, a^2) \neq (b^1, b^2) = r_1(P_j)$ and $r_2(P_i) = (a^1, b^2) = r_2(P_j)$ or $r_2(P_i) = (b^1, a^2) = r_2(P_j)$ where $a^1 \neq b^1$ and $a^2 \neq b^2$. This commonly second best alternative (a^1, b^2) or (b^1, a^2) can naturally be viewed as a compromise alternative at profile (P_i, P_j); it is however ignored by a random dictatorship. Next, assume $\{a^s, b^s, c^s\} \subseteq A^s$ for some $s \in M$, and consider a two-voter strict generalized random dictatorship φ. Given two groups of three alternatives: (1) $a \equiv (a^s, a^\tau, z^{-\{s,\tau\}})$, $b \equiv (b^s, b^\tau, z^{-\{s,\tau\}})$ and

$c \equiv (a^s, b^\tau, z^{-\{s,\tau\}})$; and (2) $a' \equiv (a^s, z^{-s})$, $b' \equiv (b^s, z^{-s})$ and $c' \equiv (c^s, z^{-s})$, we identify two profiles of separable preferences: (P_i, P_j) where $P_i \in \mathbb{D}_S^a$, $P_j \in \mathbb{D}_S^b$ and $r_2(P_i) = r_2(P_j) = c$, and (P_i', P_j') where $P_i' \in \mathbb{D}_S^{a'}$, $P_j' \in \mathbb{D}_S^{b'}$ and $r_2(P_i') = r_2(P_j') = c'$. At profile (P_i, P_j), since the compromise alternative c can be assembled by the voter sequence $(i^s, i^\tau, i^{-\{s,\tau\}}) \equiv (i, j, i, \dots, i)$, we have $\varphi_c(P_i, P_j) > 0$. However, the compromise alternative c' cannot be assembled by any voter sequence at (P_i', P_j'), and therefore $\varphi_{c'}(P_i', P_j') = 0$.

We are interested in identifying a class of unanimous and strategy-proof RSCFs which differ from random dictatorships in a "minimal" but significant degree by systematically admitting compromise. Recently, Chatterji et al. (2016) have introduced *the compromise property* on an RSCF which guarantees that a compromise alternative receives a strictly positive probability whenever it appears.

Definition 3 *An RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ satisfies* ***the compromise property*** *if there exists $\hat{I} \subseteq I$ with $|\hat{I}| = \frac{N}{2}$ if N is even and $|\hat{I}| = \frac{N+1}{2}$ if N is odd, such that for all $P_i, P_j \in \mathbb{D}$ with $r_1(P_i) \neq r_1(P_j)$ and $r_2(P_i) = r_2(P_j) \equiv a$, we have $\varphi_a\left(\frac{P_i}{\hat{I}}, \frac{P_j}{I \backslash \hat{I}}\right) > 0$.*[20]

We ask what multidimensional domains admit unanimous and strategy-proof RSCFs satisfying the compromise property. To address this question, we restricting attention to a broad class of multidimensional domains: *connected$^+$ domains*. We show that the existence of a unanimous and strategy-proof RSCF satisfying the compromise property on a connected$^+$ domain implies that the domain must be a multidimensional single-peaked domain, and conversely, we construct a particular RSCF, a mixed multidimensional projection rule, satisfying unanimity, strategy-proofness and the compromise property on an arbitrary multidimensional single-peaked domain. Finally we generalize our analysis to the case of voting under constraints.

3.1 Connected$^+$ domains

We start the investigation with two particular preferences P_i and P_i' where *every* pair of oppositely ranked alternatives is also contiguously ranked, i.e., for all $a, b \in A$, we have $[aP_ib$ and $bP_i'a] \Rightarrow [aP_i!b$ and $bP_i'!a]$. Then the relation between P_i and P_i' can be elaborated further: We can identify $t \geq 1$ pair(s) of distinct alternatives $\{a_k, a_k'\}$, $k = 1, \dots, t$, and t integers $1 \leq l(1) < \cdots < l(k) < l(k+1) < \cdots < l(t) \leq |A|$, such that the following three conditions hold:

(i) $\{a_k, a_k'\} \cap \{a_l, a_l'\} = \emptyset$ for all $k \neq l$.

(ii) $a_k = r_{l(k)}(P_i) = r_{l(k)+1}(P_i')$ and $a_k' = r_{l(k)+1}(P_i) = r_{l(k)}(P_i')$, $k = 1, \dots, t$.

(iii) $\left[a \notin \cup_{k=1}^t \{a_k, a_k'\}\right] \Rightarrow \left[a = r_q(P_i) = r_q(P_i') \text{ for some } 1 \leq q \leq |A|\right]$.

Observe here that *every* pair $\{a_k, a_k'\}$ is locally switched, and hence, the relative rankings of an alternative in $\{a_k, a_k'\}$ and *every* alternative *not* in $\{a_k, a_k'\}$ remain identical in both P_i and P_i'. Thus, each pair $\{a_k, a_k'\}$ is referred to as a **local switching pair**, and preferences P_i and P_i' are referred to as a pair of ***t*-adjacent** preferences and denoted $P_i \sim^t P_i'$.[21] To be consistent

[20] The notation $\left(\frac{P_i}{\hat{I}}, \frac{P_j}{I \backslash \hat{I}}\right)$ denotes a preference profile where all voters of $\hat{I}$ report preference P_i while all voters not in $\hat{I}$ report preference P_j.

[21] Henceforth, to avoid confusion, when we say $\{a, b\}$ is a local switching pair in some t-adjacent preferences P_i and P_i', we also presume that $aP_i!b$ and $bP_i'!a$.

with the literature (e.g., Sato, 2013), the notion 1-adjacency here is simply called **adjacency**, and the notation $\sim^1$ is simplified to $\sim$.

Given two distinct preferences P_i and P_i', a sequence of preferences $\{P_i^k\}_{k=1}^q$, $q \geq 2$, is referred to as a (general) *path* connecting P_i and P_i' if (i) $P_i = P_i^1$ and $P_i' = P_i^q$, and (ii) given $1 \leq k \leq q-1$, there exists an integer $t \geq 1$ such that $P_i^k \sim^t P_i^{k+1}$. This indicates that the differences between the two preferences P_i and P_i' can be reconciled via a sequence of one-pair or multiple-pair local switchings. In particular, if every consecutive pair of preferences in a path is adjacent, this path is referred to as a **simple path**.

In multidimensional domains however, differences in preferences cannot always be reconciled via simple paths, and one may have to resort to paths where successive preferences are t-adjacent, $t > 1$. Note that the notion of t-adjacency is in fact independent of the Cartesian product setting. Since we have introduced a Cartesian product structure on the alternative set, we now turn to a way of systematically describing relations among all local switching pairs by imposing separability on some particular t-adjacent preferences.

Consider two particular separable preferences P_i and P_i' which are t-adjacent in a particular way: There exist $s \in M$ and $a^s, b^s \in A^s$ such that the local switching pairs in P_i and P_i' are $\big\{(a^s, z^{-s}), (b^s, z^{-s})\big\}_{z^{-s} \in A^{-s}}$. Thus, P_i and P_i' are in fact $\big|A^{-s}\big|$-adjacent.

Henceforth, we say that a pair of preferences P_i and P_i' is **adjacent$^+$**, denoted $P_i \sim^+ P_i'$, if the following two conditions are satisfied: (i) P_i and P_i' are separable preferences, and (ii) P_i and P_i' are $|A^{-s}|$-adjacent for some $s \in M$. Note that in the adjacent$^+$ preferences P_i and P_i', their marginal preferences on component set A^s are adjacent while all other marginal preferences are identical, i.e., $[P_i]^s \sim [P_i']^s$ and $[P_i]^\tau = [P_i']^\tau$ for all $\tau \neq s$.

We make two observations regarding a pair of adjacent$^+$ preferences. First, the multiple local switchings in P_i and P_i' are driven by the restriction of separability. Second, each local switching pair here is a pair of similar alternatives. More importantly, due to separability, after observing one pair of similar alternatives locally switched in P_i and P_i', we obtain information on *all* local switching pairs. Therefore, similarly to adjacency, adjacency$^+$ maintains the feature that the transition from one separable preference to another involves a minimal number of switches.

In a strategy-proof RSCF, if one voter *unilaterally* changes her preference to an adjacent or adjacent$^+$ preference, the probability associated to an alternative in a local switching pair whose ranking is lifted up from one preference to the other, might increase, while the sum of two probabilities in each local switching pair, and the probability received by every alternative excluded from the set of local switching pairs remain fixed (see Lemma 1 below). This makes the variation of two corresponding social lotteries in a strategy-proof RSCF more tractable.

LEMMA 1 *Let $\varphi : \mathbb{D}^N \to \Delta(A)$ be a strategy-proof RSCF. Fix $i \in I$, $P_i, P_i' \in \mathbb{D}$ and $P_{-i} \in \mathbb{D}^{N-1}$. The following two statements hold.*

1. *If $P_i \sim P_i'$ and $\{a, b\}$ is the corresponding local switching pair, then we have*
   - (i) $\varphi_a(P_i, P_{-i}) \geq \varphi_a(P_i', P_{-i})$ *and* $\varphi_b(P_i, P_{-i}) \leq \varphi_b(P_i', P_{-i})$;
   - (ii) $\varphi_a(P_i, P_{-i}) + \varphi_b(P_i, P_{-i}) = \varphi_a(P_i', P_{-i}) + \varphi_b(P_i', P_{-i})$;
   - (iii) $\varphi_z(P_i, P_{-i}) = \varphi_z(P_i', P_{-i})$ *for all* $z \notin \{a, b\}$.

2. *If $P_i \sim^+ P'_i$ and $\big\{(a^s, z^{-s}), (b^s, z^{-s})\big\}_{z^{-s}\in A^{-s}}$ are the corresponding local switching pairs, then for all $c^s \notin \{a^s, b^s\}$ and $z^{-s} \in A^{-s}$, we have*

   (i) $\varphi_{(a^s,z^{-s})}(P_i, P_{-i}) \geq \varphi_{(a^s,z^{-s})}(P'_i, P_{-i})$ *and* $\varphi_{(b^s,z^{-s})}(P_i, P_{-i}) \leq \varphi_{(b^s,z^{-s})}(P'_i, P_{-i})$;

   (ii) $\varphi_{(a^s,z^{-s})}(P_i, P_{-i}) + \varphi_{(b^s,z^{-s})}(P_i, P_{-i}) = \varphi_{(a^s,z^{-s})}(P'_i, P_{-i}) + \varphi_{(b^s,z^{-s})}(P'_i, P_{-i})$;

   (iii) $\varphi_{(c^s,z^{-s})}(P_i, P_{-i}) = \varphi_{(c^s,z^{-s})}(P'_i, P_{-i})$.

The verification of Lemma 1 is routine and we hence omit it.

If one confines attention to simple paths, especially the adjacency of two preferences with distinct peaks, multidimensional domains stand excluded,[22] and the model degenerates to a one dimensional setting. To avoid this, we require the presence of adjacency$^+$ (correspondingly, the presence of separable preferences) in a path whenever a consecutive pair of preferences differs in peaks. To simplify the structure, we only allow the appearance of either adjacency or adjacency$^+$.

Formally, a path $\{P_i^k\}_{k=1}^q$ is referred to as a **simple$^+$ path** if for all $1 \leq k \leq q-1$, (i) either $P_i^k \sim P_i^{k+1}$ or $P_i^k \sim^+ P_i^{k+1}$, and (ii) $\big[r_1(P_i^k) \neq r_1(P_i^{k+1})\big] \Rightarrow [P_i^k \sim^+ P_i^{k+1}]$.[23]

Now, we use simple$^+$ paths to specify the class of domains studied in this paper. Our domain has two properties: the Interior$^+$ Property and the Exterior$^+$ property. First, we partition the domain into several subdomains of preferences according to the peaks of the preferences. The Interior$^+$ property is established on each subdomain, and requires two preferences in one subdomain be connected via a simple$^+$ path in this subdomain. The Exterior$^+$ property imposes conditions on two preferences in two distinct subdomains. When these two preferences share the same relative ranking of some pair of alternatives, we can construct a simple$^+$ path in the domain connecting them, while preserving the relative ranking of this particular pair of alternatives along this simple$^+$ path. In particular, when the two preferences have similar peaks, say (a^s, z^{-s}) and (b^s, z^{-s}), an additional condition is imposed so that the peak of each preference in the corresponding simple$^+$ path lies in the set (A^s, z^{-s}).

Definition 4 *Domain $\mathbb{D}$ satisfies the **Interior$^+$ property** if given distinct $P_i, P'_i \in \mathbb{D}$ with $r_1(P_i) = r_1(P'_i) \equiv a$, there exists a simple$^+$ path $\{P_i^k\}_{k=1}^q \subseteq \mathbb{D}^a$ connecting P_i and P'_i.*

Definition 5 *Domain $\mathbb{D}$ satisfies the **Exterior$^+$ property** if given $P_i, P'_i \in \mathbb{D}$ with $r_1(P_i) \neq r_1(P'_i)$, and $a, b \in A$ with aP_ib and aP'_ib, there exists a simple$^+$ path $\{P_i^k\}_{k=1}^q \subseteq \mathbb{D}$ connecting P_i and P'_i such that aP_i^kb, $k = 1, \ldots, q$. In addition, when $r_1(P_i)$ and $r_1(P'_i)$ are similar, the simple$^+$ path $\{P_i^k\}_{k=1}^q$ satisfies the **no-detour property**, i.e., $\big[r_1(P_i) = (a^s, z^{-s})$ and $r_1(P'_i) = (b^s, z^{-s})\big] \Rightarrow \big[r_1(P_i^k) \in (A^s, z^{-s})$ for all $1 \leq k \leq q\big]$.*[24]

A domain satisfying the Interior$^+$ property and the Exterior$^+$ property is referred to as a **connected$^+$ domain**. It turns out that every minimally rich and connected$^+$ domain is a subset of the top-separable domain, and therefore a multidimensional domain.

[22] If two preferences are adjacent and disagree on peaks, one of them must violate top-separability.

[23] The co-existence of adjacency and adjacency$^+$ is critical in the construction of a simple$^+$ path. We first use adjacency to adjust preferences to reach an appropriate separable preference where then we can make simultaneous multiple local switchings required by an adjacency$^+$ (see Example 5 in Appendix D.3).

[24] Appendix D.1 provides two examples of simple$^+$ paths which satisfy and violate the no-detour property respectively.

Proposition 2 *Let $\mathbb{D}$ be a minimally rich and connected$^+$ domain. Then, $\mathbb{D} \subseteq \mathbb{D}_{TS}$.*

Proof: Suppose that $\mathbb{D} \nsubseteq \mathbb{D}_{TS}$. Thus, there exists $P_i \in \mathbb{D}\backslash\mathbb{D}_{TS}$ such that $r_1(P_i)^s = a^s$ and $(b^s, z^{-s})P_i(a^s, z^{-s})$ for some $s \in M$, $b^s \in A^s\backslash\{a^s\}$ and $z^{-s} \in A^{-s}$. Pick another preference $P_i' \in \mathbb{D}^{(b^s,z^{-s})}$ by minimal richness. The Exterior$^+$ property implies that there exists a simple$^+$ path $\{P_i^k\}_{k=1}^q \subseteq \mathbb{D}$ connecting P_i and P_i' such that $(b^s, z^{-s})P_i^k(a^s, z^{-s})$ for all $1 \leq k \leq q$. Since $r_1(P_i^1) = a \neq (b^s, z^{-s}) = r_1(P_i')$, there must exist $1 \leq k < q$ such that $r_1(P_i^k) = a \neq r_1(P_i^{k+1})$. Consequently, $P_i^k \sim^+ P_i^{k+1}$, and hence, P_i^k is a separable preference which implies $(a^s, z^{-s})P_i^k(b^s, z^{-s})$ for all $z^{-s} \in A^{-s}$. Contradiction! □

Remark 1 The top-separable domain, the separable domain, multidimensional single-peaked domains and their intersection and unions are all included in the class of connected$^+$ domains. The detailed verifications are available in Appendices D.3 - D.7. The lexicographically separable domain however fails connectedness$^+$ due to the non-existence of preferences that deliver adjacency.[25] The class of connected$^+$ domains also excludes domains studied in the one-dimensional setting (e.g., Gibbard, 1977; Moulin, 1980; Saporiti, 2009; Sato, 2013; Chatterji et al., 2016). □

We next turn to an important property of unanimous and strategy-proof RSCFs on connected$^+$ domains which plays a critical role in the subsequent analysis: The social lottery at every preference profile depends only on voters' peaks. We say that such an RSCF satisfies the tops-only property. Formally, an RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ satisfies **the tops-only property** if for every pair of *tops-equivalent* profiles $P, P' \in \mathbb{D}^N$, i.e., $r_1(P_i) = r_1(P_i')$ for all $i \in I$, we have $\varphi(P) = \varphi(P')$.

Proposition 3 *Every unanimous and strategy-proof RSCF on a connected$^+$ domain satisfies the tops-only property.*

The proof of Proposition 3 is available in Appendix A.

Remark 2 We add the superscript "+" to highlight the role of simple$^+$ paths in our two properties, and thereby distinguish our two properties from the Interior and Exterior properties of Chatterji and Zeng (2017). The connected$^+$ domains here fail to satisfy their Interior and Exterior properties: The Interior property is a strengthening of the Interior$^+$ property since it is established by using simple paths which cannot be generally applied to multidimensional domains, like the separable domain (see Example 5 of Appendix D.3), while the Exterior property is significantly weaker than the Exterior$^+$ property as it is defined by using the notion of *isolation* which is weaker than both adjacency and adjacency$^+$. The verification of Proposition 3 is similar to the proof of the Theorem of Chatterji and Zeng (2017), but requires an additional step that specifically applies to adjacent$^+$ preferences (Lemma 10 of Appendix A). Finally, we note that Proposition 3 still holds even when the no-detour property fails. We believe that Proposition 3 is of some independent interest for the study RSCFs' in the voting model. □

We next use Proposition 3 is to generalize an existing characterization result of generalized random dictatorships on all connected$^+$ supersets of the lexicographically separable domain (recall footnote 18), like the separable domain and the top-separable domain.

[25]In the lexicographically separable domain, we know that (i) there exists no pair of adjacent preferences, (ii) even though adjaceny$^+$ exists, every pair of adjacent$^+$ preferences shares the same lexicographic order, and (iii) therefore, the difference in a pair of lexicographically separable preferences with distinct lexicographic orders can never be reconciled via a simple$^+$ path in the lexicographically separable domain.

Corollary 1 *Let $|A^s| \geq 3$ for each $s \in M$, and $\mathbb{D}$ be a connected$^+$ domain that includes the lexicographically separable domain. A unanimous RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ is strategy-proof if and only if it is a generalized random dictatorship.*

Proof: Since the lexicographically separable domain is minimally rich and included in $\mathbb{D}$, we know that $\mathbb{D}$ is also minimally rich and hence $\mathbb{D} \subseteq \mathbb{D}_{TS}$ by Proposition 2. The sufficiency part of Corollary 1 thus is implied by Proposition 1. We show the necessity part. First, recall that Theorem 3 of Chatterji et al. (2012) shows that every unanimous and strategy-proof RSCF on the lexicographically separable domain is a generalized random dictatorship. Next, by Proposition 3, RSCF φ satisfies the tops-only property. Last, since the lexicographically separable domain is included in $\mathbb{D}$, tops-onlyness implies that φ must be a generalized random dictatorship. $\square$

We conclude this section with an implication of Proposition 3 that will play in important role in our analysis. Since the tops-only property emerges endogenously, every unanimous and strategy-proof RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ simplifies to a random voting rule $\varphi : A^N \to \Delta(A)$. We hence simplify the notation of a preference profile $(P_1, \ldots, P_N)$ to $(x_1, \ldots, x_N)$, where $r_1(P_i) = x_i$, $i = 1, \ldots, N$. We also mix the notation of alternatives and preferences, e.g., (a, P_j) represents a two-voter preference profile where the peak of voter i's preference is a and voter j's preference is P_j. More importantly, henceforth, we can simply focus on the peaks in each pair of adjacent$^+$ preferences with distinct peaks since the peaks determine the social lotteries because of the tops-only property. Accordingly, we induce an adjacency$^+$ relation between alternatives from the adjacency$^+$ relation between preferences as follows. We say that a pair of *alternatives* $a, b \in A$ is **adjacent$^+$**, denoted $a \sim^+ b$, if there exist $P_i \in \mathbb{D}^a$ and $P_i' \in \mathbb{D}^b$ such that $P_i \sim^+ P_i'$. Given distinct $a, b \in A$, let $\{x_k\}_{k=1}^t$ denote an **adjacent$^+$ path** (of alternatives) connecting a and b if $x_1 = a$, $x_t = b$ and $x_k \sim^+ x_{k+1}$, $k = 1, \ldots, t-1$. Consequently, we can now specify a geometric relation on all alternatives that will be useful in the subsequent analysis.

3.2 Multidimensional single-peakedness

Corollary 1 shows that connected$^+$ domains that contain the domain of lexicographically separable preferences, like for instance, the separable domain, do not allow us to construct unanimous and strategy-proof RSCFs satisfying the compromise property. In this section we prove that if a minimally rich and connected$^+$ domain admits an RSCF which satisfies the aforementioned properties, then it must be a multidimensional single-peaked domain.

The version of multidimensional single-peakedness we derive is a generalization of the one studied by Barberà et al. (1993). We first introduce our notion of multidimensional single-peakedness. Besides the Cartesian product setting, we impose an additional condition on the alternative set: for each $s \in M$, all elements in A^s are located on a *tree*, denoted $G(A^s)$.[26] Let $\langle a^s, b^s \rangle$ denote the *unique* graph path between a^s and b^s in $G(A^s)$.[27] Combining all trees $G(A^s)$, $s \in M$, we generate a **product of trees** $\times_{s \in M} G(A^s)$ where the set of vertices is A, and two alternatives a and b form an edge if and only if a and b are similar, say $a^{-s} = b^{-s}$ for some $s \in M$, and moreover, a^s and b^s form an edge in $G(A^s)$. Given $a, b \in A$, let $\langle a, b \rangle = \{x \in A | x^s \in$

[26] A *graph* is a combination of vertices and edges. A *graph path* is a sequence of vertices with each consecutive pair forming an edge. A tree is a graph where each pair of vertices is connected via a unique graph path.

[27] If $a^s = b^s$, $\langle a^s, b^s \rangle = \{a^s\}$ is a singleton set.

$\langle a^s, b^s\rangle$ for each $s \in M\}$ denote the **minimal box** containing *all* alternatives located between a and b in each component.

Definition **6** *A preference P_i is **multidimensional single-peaked** on a product of trees $\times_{s\in M} G(A^s)$ if for all distinct $a, b \in A$, we have $\big[a \in \langle r_1(P_i), b\rangle\big] \Rightarrow [aP_ib]$.*

Therefore, a domain is multidimensional single-peaked if there exists a product of trees on which every preference in the domain is multidimensional single-peaked. Given a product of trees $\times_{s\in M} G(A^s)$, a multi-dimensional domain on $\times_{s\in M} G(A^s)$ containing all admissible multi-dimensional single-peaked preferences is referred to as **the multidimensional single-peaked domain**, denoted $\mathbb{D}_{MSP}$.[28] We provide the following example to illustrate.

Example **1** Let $A = A^1 \times A^2$, $A^1 = \{a^1, b^1, c^1, d^1\}$ and $A^2 = \{0, 1\}$. Let $G(A^1)$ be a tree in part (i) of Figure 2, and $G(A^2)$ be a line in part (ii) of Figure 2. Then, we have a product of trees $G(A^1) \times G(A^2)$ specified in part (iii) of Figure 2.

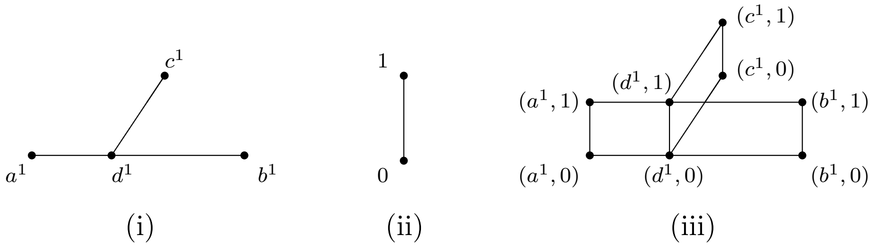


Figure 2: A tree, a line and a product of trees

Let domain $\mathbb{D}_{MSP}$ be the multidimensional single-peaked domain on $G(A^1) \times G(A^2)$. Consider a multidimensional single-peaked preference P_i with $r_1(P_i) = (a^1, 0)$. For instance, since $(d^1, 0) \in \langle (a^1, 0), (c^1, 1)\rangle = \{(a^1, 0), (a^1, 1), (d^1, 0), (d^1, 1), (c^1, 0), (c^1, 1)\}$, we have $(d^1, 0)P_i(c^1, 1)$; and since $(d^1, 0) \notin \langle (a^1, 0), (a^1, 1)\rangle = \{(a^1, 0), (a^1, 1)\}$, we may have $(a^1, 1)P_i(d^1, 0)$. For instance, we have $P_i :\ (a^1, 0) \rightharpoondown (a^1, 1) \rightharpoondown (d^1, 0) \rightharpoondown (d^1, 1) \rightharpoondown (b^1, 0) \rightharpoondown (c^1, 0) \rightharpoondown (b^1, 1) \rightharpoondown (c^1, 1)$. $\square$

Remark **3** In the multidimensional single-peaked domain, some preferences are separable and some preferences are not separable. Note that a separable preference $P_i \in \mathbb{D}_S$ is multi-dimensional single-peaked on a product of trees $\times_{s\in M} G(A^s)$ if and only if for every $s \in M$, the marginal preference $[P_i]^s$ is **single-peaked** on the tree $G(A^s)$, i.e., for all distinct $a^s, b^s \in A^s$, we have $\big[a^s \in \langle r_1([P_i]^s), b^s\rangle\big] \Rightarrow \big[a^s [P_i]^s b^s\big]$.[29] $\square$

Now, we formally state the main result.

[28]Throughout this paper, any strict subset of the multidimensional single-peaked domain is just referred to as "a multi-dimensional single-peaked domain". Two distinct product graphs of trees induces two distinct the multidimensional single-peaked domains.

[29]To show the necessity part, according to $a^s, b^s \in A^s$, pick arbitrary $x^{-s} \in A^{-s}$. Thus, $(a^s, x^{-s}) \in \langle r_1(P_i), (b^s, x^{-s})\rangle$ and multidimensional single-peakedness of P_i implies $(a^s, x^{-s})P_i(b^s, x^{-s})$. Furthermore, separability of P_i implies $a^s[P_i]^s b^s$. For the sufficiency part, we assume $r_1(P_i) = x$ and pick $a, b \in A$ with $a \in \langle x, b\rangle$. Thus, for every $s \in M$, either $a^s = b^s$ or $a^s \neq b^s$ and $a^s \in \langle x^s, b^s\rangle$. Then, single-peakedness of marginal preferences implies that for every component which a and b disagree on, alternative a always has the element marginally preferred to that of b, i.e., $[a^s \neq b^s] \Rightarrow \big[a^s[P_i]^s b^s\big]$. Therefore, it must be the case aP_ib, as required.

Theorem 1 *Let $\mathbb{D}$ be a minimally rich and connected$^+$ domain. If it admits a unanimous and strategy-proof RSCF satisfying the compromise property, it is multidimensional single-peaked. Conversely, a multidimensional single-peaked domain admits a unanimous and strategy-proof RSCF satisfying the compromise property.*

Proof: We start from the verification of the necessity part. Let $\phi : \mathbb{D}^N \to \Delta(A)$ be a unanimous and strategy-proof RSCF satisfying the compromise property. First, Proposition 3 implies that ϕ satisfies the tops-only property. Since ϕ satisfies the compromise property, we can separate voters into two groups $\hat{I}$ and $I\backslash\hat{I}$ with $|\hat{I}| = \frac{N}{2}$ if N is even, and $|\hat{I}| = \frac{N+1}{2}$ if N is odd. We induce a two-voter RSCF: For all $P_i, P_j \in \mathbb{D}$, $\varphi(P_i, P_j) = \phi\big(\frac{P_i}{\hat{I}}, \frac{P_j}{I\backslash\hat{I}}\big)$. It is easy to verify that φ is unanimous, tops-only and strategy-proof, and satisfies the compromise property.

In Lemma 2 below, we show that every pair of preferences with similar peaks cannot be complete reversals (recall the definition in Section 2). Therefore, every pair of similar alternatives (a^s, x^{-s}) and (b^s, x^{-s}) is connected via an adjacent$^+$ path in (A^s, x^{-s}).

Lemma 2 *Given $s \in M$, $a^s, b^s \in A^s$ and $x^{-s} \in A^{-s}$, there exists an adjacent$^+$ path $\{x_k\}_{k=1}^q \subseteq (A^s, x^{-s})$ connecting (a^s, x^{-s}) and (b^s, x^{-s}).*

Proof: Since $\mathbb{D}^{(a^s,x^{-s})} \neq \emptyset$ and $\mathbb{D}^{(b^s,x^{-s})} \neq \emptyset$ by minimal richness, there are two exclusive situations to consider: (i) There exist $P_i \in \mathbb{D}^{(a^s,x^{-s})}$ and $P_i' \in \mathbb{D}^{(b^s,x^{-s})}$ such that they agree on the relative ranking of some pair of alternatives, and (ii) both $\mathbb{D}^{(a^s,x^{-s})}$ and $\mathbb{D}^{(b^s,x^{-s})}$ are singleton sets, and $P_i \in \mathbb{D}^{(a^s,x^{-s})}$ and $P_i' \in \mathbb{D}^{(b^s,x^{-s})}$ are complete reversals.

In the first situation, the no-detour property in the Exterior$^+$ Property implies that there exists a simple$^+$ path $\{P_i^k\}_{k=1}^q \subseteq \mathbb{D}$ connecting P_i and P_i' such that $r_1(P_i^k) \in (A^s, x^{-s})$ for all $1 \leq k \leq q$. By sorting all preferences of $\{P_i^k\}_{k=1}^q$ according to the peaks of preferences, and removing those repetitions of top alternatives, we can elicit an adjacent$^+$ path $\{x_k\}_{k=1}^q \subseteq (A^s, x^{-s})$ such that $x_1 = (a^s, x^{-s})$, $x_q = (b^s, x^{-s})$ and $x_k \sim^+ x_{k+1}$, $k = 1, \ldots, q-1$.

We next show that the second situation is invalid. Suppose that it is not true. Thus, the worst alternative in P_i is (b^s, x^{-s}). Pick an arbitrary $\tau \in M\backslash\{s\}$ and $z^\tau \in A^\tau\backslash\{x^\tau\}$. We have an alternative $(a^s, z^\tau, x^{-\{s,\tau\}})$ and $\bar{P}_i \in \mathbb{D}^{(a^s,z^\tau,x^{-\{s,\tau\}})}$ by minimal richness. Since P_i and P_i' are complete reversals and $P_i' \neq \bar{P}_i$, preferences P_i and $\bar{P}_i$ must agree on the relative ranking of some pair of alternatives. Thus, the no-detour property in the Exterior$^+$ property implies that there exists a simple$^+$ path $\{P_i^k\}_{k=1}^q \subseteq \mathbb{D}$ connecting P_i and $\bar{P}_i$ such that $r_1(P_i^k) \in (a^s, A^\tau, x^{-\{s,\tau\}})$ for all $1 \leq k \leq q$. Since $\mathbb{D}^{(a^s,x^{-s})}$ is singleton and $r_1(P_i^1) \neq r_1(P_i^q)$, preferences P_i^1 and P_i^2 must disagree on peaks. Therefore, $P_i^1 \sim^+ P_i^2$ and hence, $P_i = P_i^1$ is a separable preference. Consequently, by separability, $r_1(P_i) = (a^s, x^\tau, x^{-\{s,\tau\}})$ implies $(b^s, x^\tau, x^{-\{s,\tau\}}) P_i (b^s, z^\tau, x^{-\{s,\tau\}})$. This implies that (b^s, x^{-s}) is not the worst alternative in P_i. Contradiction! $\square$

Lemma 3 *Given $s \in M$ and $x^{-s} \in A^{-s}$, let $\{x_k\}_{k=1}^q \subseteq (A^s, x^{-s})$ be an adjacent$^+$ path. There exist $0 \leq \alpha_1 < \cdots < \alpha_{q-1} \leq 1$ such that for all $1 \leq k < k' \leq q$, $\varphi(x_k, x_{k'}) = \alpha_k e_{x_k} + \sum_{l=k+1}^{k'-1}(\alpha_l - \alpha_{l-1}) e_{x_l} + (1 - \alpha_{k'-1}) e_{x_{k'}}$. Moreover, for every $P_i \in \mathbb{D}^{x_1}$, $x_k P_i x_{k+1}$, $k = 1, \ldots, q-1$.*

Proof: Given $1 \leq k \leq q-1$, since $x_k \sim^+ x_{k+1}$, we have $P_i \in \mathbb{D}^{x_k}$ and $P_j \in \mathbb{D}^{x_{k+1}}$ with $P_i \sim^+ P_j$. Thus, $r_1(P_i) = r_2(P_j) = x_k$ and $r_2(P_i) = r_1(P_j) = x_{k+1}$. Then, by tops-onlyness, item 2(ii) of Lemma 1 and unanimity, we have $\varphi_{x_k}(x_k, x_{k+1}) + \varphi_{x_{k+1}}(x_k, x_{k+1}) = \varphi_{x_k}(P_i, P_j) +$

$\varphi_{x_{k+1}}(P_i, P_j) = \varphi_{x_k}(P_i, P_i) + \varphi_{x_{k+1}}(P_i, P_i) = \varphi_{x_k}(P_i, P_i) = 1$. Let $\varphi_{x_k}(x_k, x_{k+1}) = \alpha_k$ and $\varphi_{x_{k+1}}(x_k, x_{k+1}) = 1 - \alpha_k$ where $0 \leq \alpha_k \leq 1$. Thus, $\varphi(x_k, x_{k+1}) = \alpha_k \, e_{x_k} + (1-\alpha_k)e_{x_{k+1}}$. Next, we adopt an induction argument.

Induction Hypothesis: Given $l \geq 2$, for all $1 \leq k < k' \leq q$ with $0 < k' - k < l$, we have $\varphi(x_k, x_{k'}) = \alpha_k e_{x_k} + \sum_{\nu=k+1}^{k'-1}(\alpha_\nu - \alpha_{\nu-1})e_{x_\nu} + (1-\alpha_{k'-1})e_{x_{k'}}$.

Let $k' - k = l$. We show $\varphi(x_k, x_{k'}) = \alpha_k e_{x_k} + \sum_{\nu=k+1}^{k'-1}(\alpha_\nu - \alpha_{\nu-1})e_{x_\nu} + (1-\alpha_{k'-1})e_{x_{k'}}$. Since $x_k \sim^+ x_{k+1}$, we have $P_i \in \mathbb{D}^{x_k}$ and $P_i' \in \mathbb{D}^{x_{k+1}}$ with $P_i \sim^+ P_i'$. Then, according to the induction hypothesis, the following equalities hold.

(i) $\varphi_{x_k}(P_i, x_{k'}) + \varphi_{x_{k+1}}(P_i, x_{k'}) = \varphi_{x_k}(P_i', x_{k'}) + \varphi_{x_{k+1}}(P_i', x_{k'}) = \alpha_{k+1}$ by item 2(ii) of Lemma 1;

(ii) $\varphi_{x_\nu}(P_i, x_{k'}) = \varphi_{x_\nu}(P_i', x_{k'}) = \alpha_\nu - \alpha_{\nu-1}$, $\nu = k+2, \ldots, k'-1$ by item 2(iii) of Lemma 1;

(iii) $\varphi_{x_{k'}}(P_i, x_{k'}) = \varphi_{x_{k'}}(P_i', x_{k'}) = 1 - \alpha_{k'-1}$ by item 2(iii) of Lemma 1.

Similarly, since $x_{k'} \sim^+ x_{k'-1}$, we have $P_j \in \mathbb{D}^{x_{k'}}$ and $P_j' \in \mathbb{D}^{x_{k'-1}}$ such that $P_j \sim^+ P_j'$. Then, item 2(iii) of Lemma 1 and induction hypothesis imply $\varphi_{x_k}(x_k, P_j) = \varphi_{x_k}(x_k, P_j') = \alpha_k$. Thus, $\varphi_{x_{k+1}}(x_k, x_{k'}) = \varphi_{x_k}(P_i, x_{k'}) + \varphi_{x_{k+1}}(P_i, x_{k'}) - \varphi_{x_k}(x_k, P_j) = \alpha_{k+1} - \alpha_k$. Therefore, $\varphi(x_k, x_{k'}) = \alpha_k e_{x_k} + \sum_{\nu=k+1}^{k'-1}(\alpha_\nu - \alpha_{\nu-1})e_{x_\nu} + (1-\alpha_{k'-1})e_{x_{k'}}$. This completes the verification of the induction hypothesis.

Next, we show $\alpha_k < \alpha_{k+1}$, $k = 1, \ldots, q-2$. Given $1 \leq k \leq t-2$, since $x_k \sim^+ x_{k+1}$ and $x_{k+1} \sim^+ x_{k+2}$, we have $P_i \in \mathbb{D}^{x_k}$ and $P_j \in \mathbb{D}^{x_{k+2}}$ such that $r_2(P_i) = r_2(P_j) = x_{k+1}$. Thus, $\alpha_{k+1} - \alpha_k = \varphi_{x_{k+1}}(P_i, P_j) > 0$ by the compromise property.

Last, given $P_i \in \mathbb{D}^{x_1}$, we show $x_k P_i x_{k+1}$, $k = 1, \ldots, q-1$. Given $1 \leq k \leq q-1$, suppose $x_{k+1} P_i x_k$. Evidently, $1 < k < q$. At the profile (P_i, x_{k+1}), we have $\varphi_{x_k}(P_i, x_{k+1}) = \alpha_k - \alpha_{k-1} > 0$. Assume $a_{k+1} = r_\eta(P_i)$. Consequently, $\sum_{t=1}^{\eta} \varphi_{r_t(P_i)}(P_i, x_{k+1}) \leq 1 - \varphi_{x_k}(P_i, x_{k+1}) < 1 = \varphi_{x_{k+1}}(x_{k+1}, x_{k+1}) = \sum_{t=1}^{\eta} \varphi_{r_t(P_i)}(x_{k+1}, x_{k+1})$. Thus, voter i will manipulate at (P_i, x_{k+1}) via a preference with peak x_{k+1}. Therefore, $x_k P_i x_{k+1}$, $k = 1, \ldots, q-1$. □

Given $s \in M$ and $x^{-s} \in A^{-s}$, we induce a graph $G_{\sim^+}\big((A^s, x^{-s})\big)$ where (A^s, x^{-s}) is the set of vertices, and two alternatives form an edge if they are adjacent$^+$. By Lemma 2, we know that in $G_{\sim^+}\big((A^s, x^{-s})\big)$, there exists a graph path between any pair of vertices.

Lemma 4 *Given $s \in M$ and $x^{-s} \in A^{-s}$, $G_{\sim^+}\big((A^s, x^{-s})\big)$ is a tree.*

Proof: Suppose not, i.e., there exists a cycle $\{x_k\}_{k=1}^t \subseteq (A^s, x^{-s})$, $t \geq 3$, such that $x_k \sim^+ x_{k+1}$, $k = 1, \ldots, t$, where $x_{t+1} = x_1$. According to the sequence $\{x_k\}_{k=1}^t$, Lemma 3 implies $\varphi_{x_1}(x_1, x_t) + \varphi_{x_t}(x_1, x_t) < 1$. However, $x_1 \sim^+ x_t$ implies $\varphi_{x_1}(x_1, x_t) + \varphi_{x_t}(x_1, x_t) = 1$. Contradiction! Therefore, $G_{\sim^+}\big((A^s, x^{-s})\big)$ is a tree. □

We are going to show that two trees $G_{\sim^+}\big((A^s, x^{-s})\big)$ and $G_{\sim^+}\big((A^s, y^{-s})\big)$ are "identical" in the sense that for all $a^s, b^s \in A^s$, (a^s, x^{-s}) and (b^s, x^{-s}) form an edge in $G_{\sim^+}\big((A^s, x^{-s})\big)$ if and only if (a^s, y^{-s}) and (b^s, y^{-s}) form an edge in $G_{\sim^+}\big((A^s, y^{-s})\big)$. With this result, we can generate a tree $G(A^s)$ on the component set A^s.

For the next lemma, we fix the following four alternatives: $a = (x^s, x^\tau, z^{-\{s,\tau\}})$, $b = (y^s, y^\tau, z^{-\{s,\tau\}})$, $c = (x^s, y^\tau, z^{-\{s,\tau\}})$ and $d = (y^s, x^\tau, z^{-\{s,\tau\}})$ where $x^s \neq y^s$ and $x^\tau \neq y^\tau$.

Lemma **5** *If $a \sim^+ c$ and $a \sim^+ d$, then $b \sim^+ c$ and $b \sim^+ d$.*

Proof: Since $b, c \in (A^s, y^\tau, z^{-\{s,\tau\}})$ and $b, d \in (y^s, A^s, z^{-\{s,\tau\}})$, Lemma 4 implies that there exists a unique adjacent$^+$ path $\{x_k\}_{k=1}^p \subseteq (y^s, A^\tau, z^{-\{s,\tau\}})$ connecting b and d, and a unique adjacent$^+$ path $\{y_k\}_{k=1}^q \subseteq (A^s, y^\tau, z^{-\{s,\tau\}})$ connecting b and c. We use the following diagram to illustrate the geometric relations among a, b, c and d.

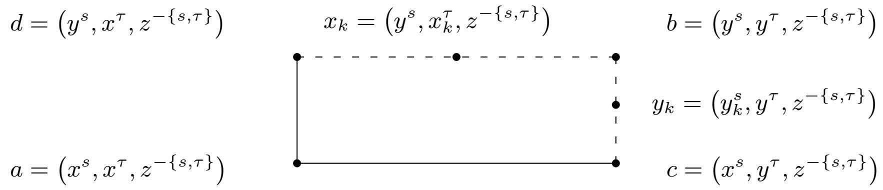


Figure 3: The geometric relations among a, b, c and d [30]

To verify this lemma, we show $q = 2$ and $p = 2$ (equivalently, $b \sim^+ c$ and $b \sim^+ d$). Suppose not, i.e., either $q > 2$ or $p > 2$. Assume $q > 2$. The verification related to $p > 2$ is symmetric and we hence omit it. Thus, $y_2 \equiv (y_2^s, y^\tau, z^{-\{s,\tau\}})$, $y_2 \notin \{b, c\}$ and $y_2^s \notin \{x^s, y^s\}$.

Since $a \sim^+ c$, we have $P_i \in \mathbb{D}^a$ and $P_i' \in \mathbb{D}^c$ with $P_i \sim^+ P_i'$. According to $\{y_k\}_{k=1}^q$, $\varphi_{y_2}(P_i', b) = \varphi_{y_2}(y_q, y_1) > 0$ by Lemma 3. Let $z_2 \equiv (y_2^s, x^\tau, z^{-\{s,\tau\}})$. Thus, $\{z_2, y_2\}$ is a local switching pair of P_i and P_i', and hence item 2(ii) of Lemma 1 implies $\varphi_{z_2}(P_i, b) + \varphi_{y_2}(P_i, b) = \varphi_{z_2}(P_i', b) + \varphi_{y_2}(P_i', b) > 0$. On the other hand, since $a \sim^+ d$, we have $\bar{P}_i \in \mathbb{D}^a$ and $\bar{P}_i' \in \mathbb{D}^d$ with $\bar{P}_i \sim^+ \bar{P}_i'$. Since $y_2, z_2 \notin \{x_k\}_{k=1}^p \subseteq (y^s, A^\tau, z^{-\{s,\tau\}})$, Lemma 3 implies $\varphi_{y_2}(\bar{P}_i', b) = \varphi_{y_2}(x_p, x_1) = 0$ and $\varphi_{z_2}(\bar{P}_i', b) = \varphi_{z_2}(x_p, x_1) = 0$. Furthermore, since $y_2, z_2 \notin (x^s, A^{-s}) \cup (y^s, A^{-s})$, item 2(iii) of Lemma 1 implies $\varphi_{y_2}(\bar{P}_i, b) = \varphi_{y_2}(\bar{P}_i', b) = 0$ and $\varphi_{z_2}(\bar{P}_i, b) = \varphi_{z_2}(\bar{P}_i', b) = 0$. Thus, $\varphi_{z_2}(\bar{P}_i, b) + \varphi_{y_2}(\bar{P}_i, b) = 0$. Consequently, $\varphi(P_i, b) \neq \varphi(\bar{P}_i, b)$ which contradicts the tops-only property. Therefore, $q = 2$. By a similar argument, $p = 2$. $\square$

Lemma **6** *Given $s \in M$ and $a^s, b^s \in A^s$, if $(a^s, x^{-s}) \sim^+ (b^s, x^{-s})$ for some $x^{-s} \in A^{-s}$, then $(a^s, y^{-s}) \sim^+ (b^s, y^{-s})$ for all $y^{-s} \in A^{-s}$.*

Proof: Given $y^{-s} \in A^{-s} \backslash \{x^{-s}\}$ and $\tau \in M \backslash \{s\}$ with $x^\tau \neq y^\tau$, we show $(a^s, y^\tau, x^{-\{s,\tau\}}) \sim^+ (b^s, y^\tau, x^{-\{s,\tau\}})$. By switching $x^{-\{s,\tau\}}$ to $y^{-\{s,\tau\}}$ component by component and applying the symmetric argument, we can complete the verification of the lemma.

Since $G_{\sim^+}\big((a^s, A^\tau, x^{-\{s,\tau\}})\big)$ is a tree, there exists a unique adjacent$^+$ path $\{a_k\}_{k=1}^q \subseteq (a^s, A^\tau, x^{-\{s,\tau\}})$ such that $a_1 = (a^s, x^\tau, x^{-\{s,\tau\}})$, $a_q = (a^s, y^\tau, x^{-\{s,\tau\}})$ and $a_k \sim^+ a_{k+1}$, $k = 1, \ldots, q-1$. Accordingly, we construct another sequence $\{b_k\}_{k=1}^q \subseteq (b^s, A^\tau, x^{-\{s,\tau\}})$ such that $b_k = (b^s, a_k^\tau, x^{-\{s,\tau\}})$, $k = 1, \ldots, q$. Thus, $b_1 = (b^s, x^\tau, x^{-\{s,\tau\}})$ and $b_q = (b^s, y^\tau, x^{-\{s,\tau\}})$. Note that the sequence $\{b_k\}_{k=1}^q$ is *not* necessarily an adjacent$^+$ path in $(b^s, A^\tau, x^{-\{s,\tau\}})$ so far.

Since $a_1 = (a^s, x^\tau, x^{-\{s,\tau\}}) = (a^s, x^{-s}) \sim^+ (b^s, x^{-s}) = (b^s, x^\tau, x^{-\{s,\tau\}}) = b_1$ by hypothesis, and $a_1 \sim^+ a_2$, Lemma 5 implies $b_2 \sim^+ b_1$ and $b_2 \sim^+ a_2$. Following the adjacent$^+$ path $\{a_k\}_{k=1}^t$ and repeatedly applying Lemma 5, we have $b_k \sim^+ b_{k-1}$ and $b_k \sim^+ a_k$, $k = 2, \ldots, q$. Eventually, $(a^s, y^\tau, x^{-\{s,\tau\}}) = a_q \sim^+ b_q = (b^s, y^\tau, x^{-\{s,\tau\}})$. $\square$

By Lemmas 4 and 6, we can induce a tree $G(A^s)$ over A^s for each $s \in M$ such that for all $a^s, b^s \in A^s$, (a^s, b^s) is an edge in $G(A^s)$ if and only if $(a^s, x^{-s}) \sim^+ (b^s, x^{-s})$ for *all* $x^{-s} \in A^{-s}$.

[30] The dash line represents an adjacency$^+$ path connecting two alternatives.

Thus, we have a product of trees $\times_{s\in M} G(A^s)$. According to $\times_{s\in M} G(A^s)$, we know that a pair of alternatives a and b is adjacent$^+$ if and only if they are similar, i.e., $a^s \neq b^s$ and $a^{-s} = b^{-s}$ for some $s \in M$, and moreover, a^s and b^s form an edge in $G(A^s)$.

Lemma 7 *Given $P_i \in \mathbb{D}$, if P_i is separable, it is multidimensional single-peaked on $\times_{s\in M} G(A^s)$.*

Proof: Assume $r_1(P_i) = a \equiv (a^s)_{s\in M}$. To verify this lemma, it suffices to show that for every $s \in M$, the marginal preference $[P_i]^s$ is single-peaked on the tree $G(A^s)$ (recall Remark 3).

Suppose that $P_i \in \mathbb{D}$ is not multidimensional single-peaked on $\times_{s\in M} G(A^s)$. Thus, there exist $s \in M$ and $x^s, y^s \in A^s$ such that $x^s \in \langle a^s, y^s\rangle$ but $y^s[P_i]^s x^s$. Let $\langle a^s, y^s\rangle = \{x_k^s\}_{k=1}^q$ where $x_1^s = a^s$ and $x_q^s = y^s$. Thus, $x^s = x_l^s$ for some $1 < l < q$. Pick arbitrary $x^{-s} \in A^{-s}$, and let $x_k = (x_k^s, x^{-s})$, $k = 1, \ldots, q$. Thus, $\{x_k\}_{k=1}^q$ is a graph path in $G_{\sim^+}\big((A^s, x^{-s})\big)$. By separability, $y^s[P_i]^s x^s$ implies $x_q P_i x_l$. Accordingly, assume $x_q = r_\nu(P_i)$ and $x_l = r_{\nu'}(P_i)$ where $\nu < \nu'$. According to $\{x_k\}_{k=1}^q$, Lemma 3 implies $\varphi_{x_l}(P_i, x_q) > 0$. Consequently, $\sum_{t=1}^{\nu} \varphi_{r_t(P_i)}(P_i, x_q) < 1 = \sum_{t=1}^{\nu} \varphi_{r_t(P_i)}(x_q, x_q)$, and hence voter i manipulates at (P_i, x_q) via a preference with peak x_q. Therefore, P_i is multidimensional single-peaked on $\times_{s\in M} G(A^s)$. □

Lemma 8 *Domain $\mathbb{D}$ is multidimensional single-peaked on $\times_{s\in M} G(A^s)$.*

Proof: Given $P_i \in \mathbb{D}$, suppose that it is not multidimensional single-peaked on $\times_{s\in M} G(A^s)$. Assume $r_1(P_i) = a \equiv (a^s)_{s\in M}$. Thus, there exist distinct $x, y \in A$ such that $x \in \langle a, y\rangle$ but $yP_i x$. Evidently, $a \neq y$. Since $\mathbb{D}$ is minimally rich, we have $P_i' \in \mathbb{D}^y$. Thus, P_i and P_i' differ on peaks but agree on the relative ranking of y and x. Then, the Exterior$^+$ implies that there exists a simple$^+$ path $\{P_i^k\}_{k=1}^q \subseteq \mathbb{D}$ connecting P_i and P_i' such that $yP_i^k x$ for all $k = 1, \ldots, q$. Note that since $r_1(P_i^1) = a \neq y = r_1(P_i^q)$, there must exist $1 \leq k < q$ such that $r_1(P_i^k) = a \neq r_1(P_i^{k+1})$. Consequently, it is true that $P_i^k \sim^+ P_i^{k+1}$ and P_i^k is a separable preference. Then, Lemma 7 implies that P_i^k is multidimensional single-peaked on $\times_{s\in M} G(A^s)$, and hence $x \in \langle a, y\rangle$ implies $xP_i^k y$. Contradiction! This completes the verification of the necessity part of Theorem 1. □

Now, we turn to the sufficiency part of Theorem 1. Given a product of trees $\times_{s\in M} G(A^s)$, let $\mathbb{D}$ be a multidimensional single-peaked domain, and $\mathbb{D}_{MSP}$ be the multidimensional single-peaked domain. Evidently, $\mathbb{D} \subseteq \mathbb{D}_{MSP}$. For notational convenience, let $\bar{\mathbb{D}}_{MSP} = \mathbb{D}_S \cap \mathbb{D}_{MSP}$ denote the intersection of the separable domain and the multidimensional single-peaked domain, and $\bar{\mathbb{D}}_{MSP}^s = \{[P_i]^s : P_i \in \bar{\mathbb{D}}_{MSP}\}$ denote the induced marginal domain on A^s for each $s \in M$. Evidently, given $s \in M$, $\bar{\mathbb{D}}_{MSP}^s$ is the single-peaked (marginal) domain on the tree $G(A^s)$. We will construct an RSCF on $\mathbb{D}_{MSP}$ by three steps.

Step 1. We introduce a class of DSCFs on each marginal domain. Fix $s \in M$. Given a N-tuple of elements $(x_1^s, \ldots, x_N^s) \in [A^s]^N$, let $G(x_1^s, \ldots, x_N^s)$ denote the *minimal subgraph* of $G(A^s)$ containing $x_1^s, \ldots, x_N^s$ as vertices.[31] Given $a^s \in A^s$, we have the *projection* of a^s on $G(x_1^s, \ldots, x_N^s)$, denoted $\pi^s\big(a^s, G(x_1^s, \ldots, x_N^s)\big)$, which is unique.[32] Thus, we have a particular marginal function $\pi^s : [A^s]^N \to A^s$.

[31] For details of minimal subgraph, please refer to Chatterji et al. (2013).

[32] Fix a tree G, a subtree $G' \subseteq G$ and a vertex a. If a belongs to the vertex set of G', the projection of a on G' is a itself. If a does not belong to the vertex set of G', there exists an *unique* vertex a' in G' which lies in every path connecting a and every vertex of G'. Thus, a' is referred to as the projection of a on G'. Let $\pi(a, G')$ denote the projection of a on G'.

STEP 2. We assemble all marginal functions $(\pi^s)_{s\in M}$ to construct a DSCF on $\mathbb{D}_{MSP}$. Fixing $a \equiv (a^s)_{s\in M} \in A$, given $P \equiv (P_1, \dots, P_N) \in \mathbb{D}^N_{MSP}$, assuming $r_1(P_i) = x_i \equiv (x_i^s)_{s\in M}$, $i \in I$, for notational convenience, let $f^a(P_1, \dots, P_N) = \big(\pi^s\big(a^s, G(x_1^s, \dots, x_N^s)\big)\big)_{s\in M}$. DSCF f^a is called a *multidimensional projection rule*, and the alternative a is referred to as the *projector* of f^a.

STEP 3. Last, we construct an RSCF on $\mathbb{D}_{MSP}$ by a mixture of all multidimensional projection rules. We associate each alternative/projector to a strictly positive real weight, i.e., $\lambda_a > 0$ for all $a \in A$, and let $\sum_{a\in A} \lambda_a = 1$. Then, for all $P \in \mathbb{D}^N$, let $\varphi(P) = \sum_{a\in A} \lambda_a f^a(P)$. RSCF φ is called a *mixed multidimensional projection rule*. Evidently, RSCF φ is well-defined and satisfies unanimity.

CLAIM 1: RSCF φ is strategy-proof.

According to the construction, we know that if all multidimensional projection rules are strategy-proof, then RSCF φ is strategy-proof on $\mathbb{D}_{MSP}$.

To show strategy-proofness of all multidimensional projection rules, we first recall an important characterization result established by Barberà et al. (1993) and Le Breton and Sen (1999): Every unanimous DSCF $f : \mathbb{D}^N_{MSP} \to A$ is strategy-proof on $\mathbb{D}_{MSP}$ if and only if the following two conditions are satisfied.

(i) DSCF f is *decomposable*, i.e., for each $s \in M$, there exists a marginal function $f^s : [A^s]^N \to A^s$ such that for all $P \in \mathbb{D}^N_{MSP}$, assuming $r_1(P_i) = x_i \equiv (x_i^s)_{s\in M}$, $i \in I$, for notational convenience, we have $f(P) = \big(f^s(x_1^s, \dots, x_N^s)\big)_{s\in M}$.

(ii) For each $s \in M$, according to the marginal function f^s, inducing a marginal DSCF $\bar{f}^s : [\bar{\mathbb{D}}^s_{MSP}]^N \to A^s$ such that for every $([P_1]^s, \dots, [P_N]^s) \in [\bar{\mathbb{D}}^s_{MSP}]^N$, $\bar{f}^s([P_1]^s, \dots, [P_N]^s) = f^s\big(r_1([P_1]^s), \dots, r_1([P_N]^s)\big)$, we know that $\bar{f}^s$ is strategy-proof.

According to the construction in Step 2, we know that every multidimensional projections rule f^a, $a \in A$, is decomposable. Next, fixing a multidimensional projection rule f^a, according to each marginal function π^s constructed in Step 1, we induce a marginal DSCF $f^{a^s} : [\bar{\mathbb{D}}^s_{MSP}]^N \to A^s$ such that for all $([\bar{P}_1]^s, \dots, [\bar{P}_N]^s) \in [\bar{\mathbb{D}}^s_{MSP}]^N$, $f^{a^s}\left([\bar{P}_1]^s, \dots, [\bar{P}_N]^s\right) = \pi^s\big(r_1([P_1]^s), \dots, r_1([P_N]^s)\big)$. Furthermore, according to the proof of the sufficiency part of the Theorem of Chatterji et al. (2013), we know that each marginal DSCF f^{a^s}, $s \in M$, is strategy-proof on the marginal domain $\bar{\mathbb{D}}^s_{MSP}$. Therefore, all multidimensional projection rules are strategy-proof, as required. This completes the verification of the claim.

CLAIM 2: RSCF φ satisfies the compromise property.

Let $\hat{I} \subseteq I$ be a subset of voters with $|\hat{I}| = \frac{N}{2}$ if N is even, and $|\hat{I}| = \frac{N+1}{2}$ if N is odd. Given $P_i, P_j \in \mathbb{D}$, assume $r_1(P_i) = x \neq y = r_1(P_j)$ and $r_2(P_i) = r_2(P_j) = a$. We claim $a \in \langle x, y\rangle$. Since $r_2(P_i) = a$, it is true that x and a are similar, e.g., $x^s \neq a^s$ and $x^{-s} = a^{-s}$ for some $s \in M$, and moreover, x^s and a^s form an edge in $G(A^s)$. Similarly, y and a are similar too, e.g., $y^\tau \neq a^\tau$ and $y^{-\tau} = a^{-\tau}$ for some $\tau \in M$, and moreover, y^τ and a^τ form an edge in $G(A^\tau)$. If $s = \tau$, it must be the case that $x^s \neq y^s$ and $a^s \in \langle a^s, b^s\rangle \backslash \{x^s, y^s\}$. Thus, $x^{-s} = a^{-s} = y^{-s}$ and hence $a \in \langle x, y\rangle$. If $s \neq \tau$, it must be the case that $a^s = y^s$, $a^\tau = x^\tau$ and $x^{-\{s,\tau\}} = a^{-\{s,\tau\}} = y^{-\{s,\tau\}}$. Thus, $a \in \langle x, y\rangle$, and hence, $f^a\left(\frac{P_i}{\hat{I}}, \frac{P_j}{I\backslash\hat{I}}\right) = \big(\pi^s(a^s, \langle x^s, y^s\rangle)\big)_{s\in M} = a$. Consequently, $\varphi_a\left(\frac{P_i}{\hat{I}}, \frac{P_j}{I\backslash\hat{I}}\right) =$

$\sum_{z \in A} \lambda_z f_a^z\left(\frac{P_i}{\hat{I}}, \frac{P_j}{I \backslash \hat{I}}\right) \geq \lambda_a > 0$. This completes the verification of the claim, and hence proves the sufficiency part of Theorem 1. $\square$

Remark 4 The Theorem of Chatterji et al. (2016) shows that in the class of minimally rich and *connected* domains (i.e., the difference of two preferences can be reconciled via a simple path of the domain), the existence of a unanimous, tops-only and strategy-proof RSCF satisfying the compromise property implies that the domain must be single-peaked on a tree. Theorem 1 significantly generalizes their result in two ways. First, all the multidimensional domains studied here (and in particular the multidimensional single-peakedness we induce) are *excluded* by their domain richness condition. Second, we *endogenize* the tops-only property in the class of connected$^+$ domains. Even though the Interior and Exterior properties of Chatterji and Zeng (2017) include multidimensional single-peaked domains, the characterization in Theorem 1 cannot be achieved in their model. In their Interior and Exterior properties, the boundary for distinguishing the one-dimensional models and the multidimensional models is not clear. On the contrary, the key notion of this paper, adjacency$^+$, brings sufficiently many separable preferences into consideration which not only clearly separate domains in question from the one-dimensional setting (see Proposition 2), but also create the basis for embodying the restriction of multidimensional single-peakedness (see Lemma 7) and spreading the restriction to other preferences (see the proof of Lemma 8) in establishing the necessity part of Theorem 1. More importantly, in a connected$^+$ domain, we utilize the notion of adjacency$^+$ between preferences with distinct peaks to induce a general geometric relation among alternatives which is eventually refined (via Lemmas 5 and 6) to be a product of trees, a necessary step for establishing multidimensional single-peakedness in Theorem 1. $\square$

If the minimally rich and connected$^+$ domain happens to include two complete reversals preferences (recall the definition in Section 2), we refine the necessity part of Theorem 1 by showing that the domain is multidimensional single-peaked on a product of lines.[33]

Corollary 2 *Let $\mathbb{D}$ be a minimally rich and connected$^+$ domain. If it contains two complete reversals preferences and admits a unanimous and strategy-proof RSCF satisfying the compromise property, it is multidimensional single-peaked on a product of lines.*

Proof: By Theorem 1, we know that domain $\mathbb{D}$ is multidimensional single-peaked on a product of trees $\times_{s \in M} G(A^s)$. Let $\underline{P}_i$ and $\overline{P}_i$ be a pair of complete reversals preferences in $\mathbb{D}$. Assume $r_1(\underline{P}_i) = \underline{x}$ and $r_1(\overline{P}_i) = \overline{x}$. Evidently, $\underline{x} \neq \overline{x}$. We show that $\times_{s \in M} G(A^s)$ is a product of lines. Note that if $\times_{s \in M} G(A^s)$ is not a product of lines, there exists *no* pair of alternatives whose induced minimal box contains the whole alternative set (for instance, recall Figure 2 of Example 1). Therefore, to complete the verification, it suffices to show $\langle \underline{x}, \overline{x} \rangle = A$. Suppose not, i.e., there exists $a \notin \langle \underline{x}, \overline{x} \rangle$. Thus, we have the projection of a on $\langle \underline{x}, \overline{x} \rangle$, say a'. Since $a' \in \langle \underline{x}, a \rangle$ and $a' \in \langle \overline{x}, a \rangle$, multidimensional single-peakedness implies $a' \underline{P}_i a$ and $a' \overline{P}_i a$. Contradiction! $\square$

[33]Let $\times_{s \in M} G(A^s)$ be a product graph of lines, and $\mathbb{D}_{MSP}$ be the multidimensional single-peaked domain on $\times_{s \in M} G(A^s)$. Given $s \in M$, according to $G(A^s)$, we can arrange all elements in A^s on a linear order $>^s$. Thus, we have $\underline{x}^s, \overline{x}^s \in A^s$ such that $\overline{x}^s >^s x^s >^s \underline{x}^s$ for all $x^s \in A^s \backslash \{\underline{x}^s, \overline{x}^s\}$. Then, we can identify $P_i \in \mathbb{D}_{MSP}$ with $r_1(P_i) = (\underline{x}_s)_{s \in M}$ and $P_i' \in \mathbb{D}_{MSP}$ with $r_1(P_i) = (\overline{x}_s)_{s \in M}$ which are complete reversals.

As all generalized random dictatorships are strategy-proof on the top-separable domain by Proposition 1, the violation of the compromise property in generalized random dictatorships explained in the beginning of this section demonstrates the indispensability of the compromise property in Theorem 1. We provide two other examples below to illustrate the indispensable role of our richness condition in Theorem 1.

Example **2** Let $A = A^1 \times A^2$ and $A^1 = A^2 = \{0,1\}$. Let domain $\mathbb{D}$ be consist of the following four preferences.

P_1	P_2	P_3	P_4
$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
$(1,0)$	$(0,0)$	$(1,1)$	$(0,1)$
$(1,1)$	$(1,1)$	$(0,0)$	$(1,0)$
$(0,1)$	$(0,1)$	$(1,0)$	$(0,0)$

Table 1: Domain $\mathbb{D}$

Domain $\mathbb{D}$ is not a connected$^+$ domain, as we only have $P_1 \sim P_2$ and $P_3 \sim^+ P_4$. We cannot construct any product of trees and show that domain $\mathbb{D}$ is correspondingly multidimensional single-peaked.[34] Every random dictatorship is unanimous and strategy-proof on $\mathbb{D}$, and moreover, satisfies the compromise property *vacuously* since no profile shares a compromise alternative. This indicates that connectedness$^+$ provides an appropriate environment and ensures that the compromise property acts effectively. □

Example **3** Let $A \equiv A^1 \times A^2 \times A^3$ and $A^1 = A^2 = A^3 = \{0,1\}$. Let $I = \{i,j\}$. Let $G(A^1) \times G(A^2) \times G(A^3)$ be a product graph of three lines specified in the diagram below. Let $\mathbb{D}_{MSP}$ be the multidimensional single-peaked domain on on $G(A^1) \times G(A^2) \times G(A^3)$.

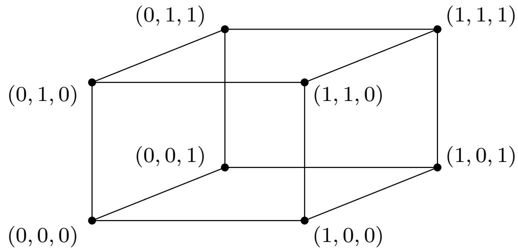


Figure 4: A product of lines $G(A^1) \times G(A^2) \times G(A^3)$

We specify a particular preference below which is *not* included in $\mathbb{D}_{MSP}$:

$$P_i^*: \quad (0,0,0) \rightharpoonup (0,0,1) \rightharpoonup (1,0,0) \rightharpoonup (1,0,1) \rightharpoonup (0,1,0) \rightharpoonup (0,1,1) \rightharpoonup (1,1,1) \rightharpoonup (1,1,0).$$ [35]

Note that P_i^* is not a top-separable preference either, e.g., $r_1(P_i^*) = (0,0,0)$ but $(1,1,1)P_i^*(1,1,0)$. Thus, domain $\mathbb{D} = \mathbb{D}_{MSP} \cup \{P_i^*\}$ is *not* a connected$^+$ domain by Proposition 2.

We first identify six multidimensional projection rules f^a, $a \notin \{(1,1,0),(1,1,1)\}$. Next, we highlight two voter sequences (j,i,i) and (i,j,j), and construct two generalized dictatorships

[34] In a multidimensional single-peaked domain, no pair of preferences with distinct peaks is adjacent. However, in domain $\mathbb{D}$ of this example, P_1 and P_2 are adjacent and disagree on peaks.

[35] Since $(1,1,0) \in \langle (0,0,0),(1,1,1) \rangle$ but $(1,1,1)P_i^*(1,1,0)$, we know $P_i^* \notin \mathbb{D}_{MSP}$.

$f^{(j,i,i)}$ and $f^{(i,j,j)}$ (recall the proof of Proposition 1). Last, we construct an RSCF by a mixture of these eight DSCFs: For all $P_i, P_j \in \mathbb{D}$,

$$\varphi(P_i, P_j) = \frac{1}{8} \sum_{a \notin \{(1,1,0),(1,1,1)\}} f^a(P_i, P_j) + \frac{1}{8} f^{(j,i,i)}(P_i, P_j) + \frac{1}{8} f^{(i,j,j)}(P_i, P_j).$$

We assert that RSCF φ is unanimous and strategy-proof, and satisfies the compromise property. All detailed verifications are available in Appendix D.2. $\square$

3.3 Deterministic voting

In this section we provide a characterization of multidimensional single-peaked domains using deterministic social choice functions. The axiom of anonymity is appropriate for distinguishing deterministic social choice functions from dictatorships, while in the random setting, requiring an RSCF to satisfy anonymity does not help in distinguishing the RSCF from the class of random dictatorships as the particular random dictatorship that gives equal weights to all voters satisfies anonymity. We replace the compromise property by anonymity and obtain a characterization result using deterministic social choice functions assuming an even number of voters.

Formally, an RSCF $\varphi : \mathbb{D}^N \to A$ is **anonymous** if for all $(P_1, \ldots, P_N) \in \mathbb{D}^N$ and every permutation $\sigma : N \to N$, we have $\varphi(P_1, \ldots, P_N) = \varphi\big(P_{\sigma(1)}, \ldots, P_{\sigma(N)}\big)$.

Theorem **2** *Let $\mathbb{D}$ be a minimally rich and connected$^+$ domain. If it admits an unanimous, anonymous and strategy-proof DSCF for an even number of voters, it is multidimensional single-peaked. Conversely, a multidimensional single-peaked domain admits an unanimous, anonymous and strategy-proof DSCF for an arbitrary number of voters.*

The proof of Theorem 2 is available in Appendix B.

Remark **5** Theorem 2 generalizes the Theorem of Chatterji et al. (2013) to the multidimensional setting and does so without imposing the axiom of tops-onlyness on the DSCF, since this property emerges endogenously in our set up as mentioned in Remark 4 above. This characterization of single-peakedness in the multidimensional setting can be interpreted as further evidence in favour of the Gul conjecture (see Section 6.5.2 of Barberà, 2010).[36] In comparing Theorems 1 and 2, we realize that randomization helps us avoid the restriction on the number of voters, and moreover, significantly simplifies the proof. In particular, the 4 situations (Figure 6) considered in Appendix B are *simultaneously* covered in the random setting. For instance, loosely speaking, we can view $\varphi_c(a, c) \times \varphi_a(a, d)$ as the probability of Situation 1 in Figure 6. $\square$

3.4 Random voting under constraints

Barberà et al. (1997) first studied the model where not all alternatives are feasible. Thus, the feasible alternative set becomes a strict subset of the Cartesian product structure. In such a

[36] Chatterji et al. (2013) investigate the same class of rich domains as Chatterji et al. (2016), and show that the existence of a unanimous, anonymous, tops-only and strategy-proof DSCF for an even number of voters implies that the domain must be *semi-single-peaked* on a tree, which is weaker than the restriction of single-peakedness. Here we characterize full single-peakedness.

setup, our result is not valid. In particular, the necessity part of Theorem 1 fails once invalid alternative appears,[37] while the sufficiency part of Theorem 1 may not hold as the multidimensional projection rule may select infeasible alternatives.

In this section, we adapt our model to accord with the infeasible alternatives problem in the following three ways: (1) Modify RSCFs to constrained RSCFs which only specify probabilities to feasible alternatives, (2) adjust the axioms of unanimity and the compromise property w.r.t. feasible alternatives, (3) strengthen the well-behavedness of the strategy-proof RSCF in question by exogenously imposing the tops-only property. Then, we show that without any change of the domain richness condition: minimal richness and connectedness$^+$, the existence of a unanimous, tops-only and strategy-proof constrained RSCF satisfying the compromise property implies that the domain must be multidimensional single-peaked w.r.t. feasible alternatives, i.e., (i) the set of feasible alternatives is factorizable (in other words, the feasible set itself is a Cartesian product) and moreover, located on a product of trees, (ii) every pair of preferences with an identical peak must have a same most preferred feasible alternative, and (iii) for each preference over A, the induced preference over the feasible alternatives is multidimensional single-peaked on the product of trees consisting of feasible alternatives. With these modifications, every multidimensional projection rule with a projector of feasible alternative (recall the proof of the sufficiency part of Theorem 1) is well-defined, and the mixture of these particular multidimensional projection rules satisfies the requirements of unanimity, tops-onlyness, strategy-proofness and the compromise property. This indicates that our characterization of the restriction of multidimensional single-peakedness is robust w.r.t. voting under constraints.

Let $\bar{A} \subset A \equiv \times_{s\in M} A^s$ denote the set of **feasible** alternatives. Given a preference P_i over A, let $P_{i|\bar{A}}$ denote the *induced preference* over $\bar{A}$ which preserves the relative rankings of feasible alternatives in preference P_i. Accordingly, let $\mathbb{D}_{|\bar{A}} = \cup_{P_i \in \mathbb{D}}\{P_{i|\bar{A}}\}$ denote the domain of induced preferences over $\bar{A}$. Note that if there exists $s \in M$ such that $a^s = b^s$ for all $a, b \in \bar{A}$, then the component set s becomes redundant and hence can be eliminated. Hence, we impose an assumption to make all components indispensable.

Assumption **1** *For each $s \in M$, there exist $a, b \in \bar{A}$ such that $a^s \neq b^s$.*

Under Assumption 1, we say that the feasible set $\bar{A}$ is **factorizable** if there exists $\bar{A}^s \subseteq A^s$ for each $s \in M$ such that $\bar{A} = \times_{s\in M}\bar{A}^s$.

A **constrained** RSCF is a map $\varphi : \mathbb{D}^N \to \Delta(\bar{A})$ which assigns positive probabilities only to feasible alternatives. We modify the axioms of unanimity and the compromise property to accord with feasibility. Formally, a constrained RSCF $\varphi : \mathbb{D}^N \to \Delta(\bar{A})$ is **unanimous** (w.r.t. feasibility) if for all $a \in \bar{A}$ and $P \in \mathbb{D}^N$, $[r_1(P_{i|\bar{A}}) = a$ for all $i \in I] \Rightarrow [\varphi_a(P) = 1]$. Next, a constrained RSCF $\varphi : \mathbb{D}^N \to \Delta(\bar{A})$ satisfies the **compromise property** (w.r.t. feasibility) if there exists $\hat{I} \subseteq I$ with $|\hat{I}| = \frac{N}{2}$ if N is even, and $|\hat{I}| = \frac{N+1}{2}$ if N is odd, such that for all $P_i, P_j \in \mathbb{D}$ with $r_1(P_i) \neq r_1(P_j)$ and $r_2(P_i) = r_2(P_j) \equiv a \in \bar{A}$, we have $\varphi_a\left(\frac{P_i}{\hat{I}}, \frac{P_j}{I\backslash\hat{I}}\right) > 0$. The definition of strategy-proofness is not affected by the feasibility issue.

For voting under constraints, the definition of multidimensional single-peaked domain is modified as follows.

[37] For instance, the verification of Lemma 5 relies on the feasibility of the four alternatives a, b, c and d.

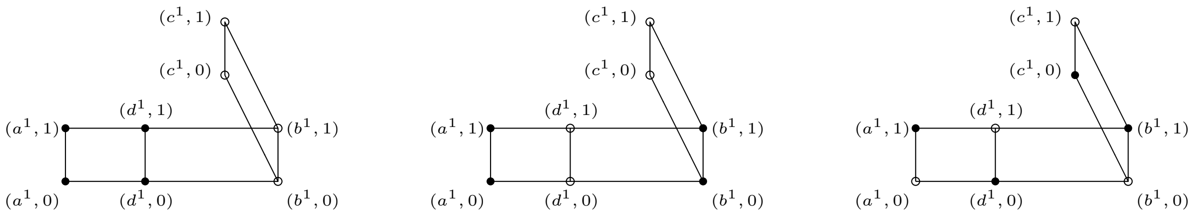
Definition 7 *A domain $\mathbb{D}$ is* ***multidimensional single-peaked w.r.t.*** $\bar{A}$ *if $\mathbb{D}_{|\bar{A}}$ is multidimensional single-peaked, i.e., the following two conditions are satisfied:*

(i) *The feasible set $\bar{A}$ is factorizable, i.e., $\bar{A} = \times_{s\in M}\bar{A}^s$.*

(ii) *There exists a product of trees $\times_{s\in M}G(\bar{A}^s)$ such that for every $P_i \in \mathbb{D}$, the induced preference $P_{i|\bar{A}}$ is multidimensional single-peaked, i.e., given distinct $x, y \in \bar{A}$, $\big[x \in \langle r_1(P_{i|\bar{A}}), y\rangle\big] \Rightarrow [xP_iy]$.*[38]

We provide an example to illustrate multidimensional single-peakedness w.r.t. feasibility.

Example 4 Let $A = A^1 \times A^2$, $A^1 = \{a^1, b^1, c^1, d^1\}$ and $A^2 = \{0, 1\}$. We specify a product of two lines $G(A^1)\times G(A^2)$, induce the multidimensional single-peaked domain $\hat{\mathbb{D}}_{MSP}$ and consider three cases of the feasible set (see Figure 5 below).



(1) (2) (3)

Figure 5: Three cases of feasible alternatives

In the first case, the feasible set $\bar{A} \equiv \big\{(a^1,0),(d^1,0),(a^1,1),(d^1,1)\big\} = \{a^1,d^1\} \times \{0,1\}$ is factorizable (see diagram (1)), and located on a product of lines $G\left(\{a^1,d^1\}\right) \times G\left(\{0,1\}\right)$. To verify the second condition of Definition 7, we consider for instance preferences $P_i \in \hat{\mathbb{D}}_{MSP}$ with $r_1(P_i) = (b^1,1)$. The induced preference $P_{i|\bar{A}}$ is either $(d^1,1)\rightharpoonup(a^1,1)\rightharpoonup(d^1,0)\rightharpoonup(a^1,0)$ or $(d^1,1)\rightharpoonup(d^1,0)\rightharpoonup(a^1,1)\rightharpoonup(a^1,0)$, as required.

In the second case, the feasible set $\bar{A} \equiv \big\{(a^1,0),(b^1,0),(a^1,1),(b^1,1)\big\} = \{a^1,b^1\} \times \{0,1\}$ is factorizable (see diagram (2)), and located on a product of lines $G\left(\{a^1,b^1\}\right) \times G\left(\{0,1\}\right)$. To verify the second condition of Definition 7, we consider for instance preferences $P_i \in \hat{\mathbb{D}}_{MSP}$ with $r_1(P_i) = (d^1,1)$. The induced preference $P_{i|\bar{A}}$ is either $(a^1,1)\rightharpoonup(a^1,0)\rightharpoonup(b^1,1)\rightharpoonup(b^1,0)$, or $(a^1,1)\rightharpoonup(b^1,1)\rightharpoonup(a^1,0)\rightharpoonup(b^1,0)$, or $(b^1,1)\rightharpoonup(b^1,0)\rightharpoonup(a^1,1)\rightharpoonup(a^1,0)$, or $(b^1,1)\rightharpoonup(a^1,1)\rightharpoonup(b^1,0)\rightharpoonup(a^1,0)$, as required.

In either case (1) or case (2), we can still construct 4 multidimensional projection rules $f^a : \big[\hat{\mathbb{D}}_{MSP|\bar{A}}\big]^N \to \bar{A}$, $a \in \bar{A}$. Then, the mixed multidimensional projection rule $\phi(P) \equiv \sum_{a\in\bar{A}} \lambda_a f^a(P)$ for all $P \in \big[\hat{\mathbb{D}}_{MSP|\bar{A}}\big]^N$, where $\lambda_a \geq 0$ for each $a \in \bar{A}$ and $\sum_{a\in\bar{A}} = 1$, is unanimous and strategy-proof on $\hat{\mathbb{D}}_{MSP|\bar{A}}$ and satisfies the compromise property. Furthermore, we extend ϕ to a constrained RSCF $\varphi : \big[\hat{\mathbb{D}}_{MSP}\big]^N \to \Delta(\bar{A})$ such that $\varphi(P_1,\ldots,P_N) = \phi\big(P_{1|\bar{A}},\ldots,P_{N|\bar{A}}\big)$ for all $P \in \big[\hat{\mathbb{D}}_{MSP}\big]^N$. Thus, the constrained RSCF φ remains strategy-proof and satisfies unanimity and the compromise property w.r.t. feasibility.

[38]In the product of trees $\times_{s\in M}G(\bar{A}^s)$, the vertex set is $\bar{A} = \times_{s\in M}\bar{A}^s$. Graph $G(\bar{A}^s)$, $s \in M$, is a tree on the vertex set $\bar{A}^s$.

In the last case, the feasible set $\bar{A} \equiv \{(a^1,1),(b^1,1),(c^1,0),(d^1,0)\}$ is not factorizable (see diagram (3)). Thus, no product of trees can be elicited for $\bar{A}$. Therefore, $\hat{\mathbb{D}}_{MSP}$ is not multidimensional single-peaked w.r.t. $\bar{A}$. Indeed, the induced domain $\mathbb{D}_{MSP|\bar{A}}$ only admits random dictatorships for unanimity and strategy-proofness.[39] Consequently, no unanimous and strategy-proof constrained RSCFs on $\hat{\mathbb{D}}_{MSP}$ other than constrained random dictatorships (i.e., the modification of random dictatorships w.r.t. the peaks of induced preferences over $\bar{A}$) can be constructed.

Last, observe that although domain $\hat{\mathbb{D}}_{MSP}$ is distinct to domain $\mathbb{D}_{MSP}$ of Example 1, they both multidimensional single-peaked w.r.t. $\bar{A}$ of the first case. Thus, the union $\hat{\mathbb{D}}_{MSP} \cup \mathbb{D}_{MSP}$ is multidimensional single-peaked w.r.t. $\bar{A}$ of the first case. □

Even though domain $\hat{\mathbb{D}}_{MSP}$ of Example 4 is multidimensional single-peaked w.r.t. the feasible set of the first two cases, there is an important difference between the two cases: In case (1), domain $\hat{\mathbb{D}}_{MSP}$ satisfies the **unique feasible peaks condition**, i.e., $[r_1(P_i) = r_1(P_i')] \Rightarrow [r_1(P_{i|\bar{A}}) = r_1(P'_{i|\bar{A}})]$, while in case (2), some preferences share the same top but disagree on the peaks of feasible alternatives, e.g., preference $P_i \in \hat{\mathbb{D}}_{MSP}$ with $r_1(P_i) = (d^1,1)$ can have either $(a^1,1)$ or $(b^1,1)$ as the peak of feasible alternatives. Due to this major difference, constrained RSCFs on $\hat{\mathbb{D}}_{MSP}$ of Example 4 perform differently in both cases. For instance, fix $(a^1,0) \in \bar{A}$, let $\lambda_{(a^1,0)} = 1$, and consider the constrained RSCF $\varphi : [\hat{\mathbb{D}}_{MSP}]^2 \to \Delta(\bar{A})$ constructed in Example 4. In case (1), the constrained RSCF φ still satisfies the original tops-only property regardless whether the peaks of the preferences are feasible or not, while in case (2), it satisfies the *tops-only property w.r.t. feasibility*, i.e., given $P, P' \in [\hat{\mathbb{D}}_{MSP}]^2$, $[r_1(P_{i|\bar{A}}) = r_1(P'_{i|\bar{A}})$ for all $i \in I] \Rightarrow [\varphi(P) = \varphi(P)]$, but fails the original tops-only property, e.g., given $P_i, P_i', P_j \in \hat{\mathbb{D}}_{MSP}$ with $r_1(P_i) = r_1(P_i') = (d^1,1) \notin \bar{A}$, $r_1(P_{i|\bar{A}}) = (a^1,1) \neq (b^1,1) = r_1(P'_{i|\bar{A}})$ and $r_1(P_j) = (b^1,0) \in \bar{A}$, we have $\varphi(P_i,P_j) \equiv f^{(a^1,1)}(P_{i|\bar{A}}, P_{j|\bar{A}}) = e_{(a^1,0)} \neq e_{(b^1,0)} = f^{(a^1,1)}(P'_{i|\bar{A}}, P_{j|\bar{A}}) \equiv \varphi(P_i', P_j)$. In fact, the unique feasible peaks condition is required to be imposed on the multidimensional single-peaked domain w.r.t. feasibility, so that we can ensure all constrained RSCFs generated via the extension of mixed multidimensional projections rules satisfy the tops-only property.

We shall continue to restrict attention to the class of minimally rich and connected$^+$ domains. In particular, preferences whose peaks are infeasible have to be considered in our characterization. The tops-only property still plays an important role in our domain characterization analysis. However, under voting under constraints, we are unable to endogenously establish the tops-only property on all unanimous and strategy-proof constrained RSCFs.[40] Therefore, we exogenously impose the tops-onlyness property without any modification on strategy-proof constrained RSCFs.

[39] We note that $\hat{\mathbb{D}}_{MSP|\bar{A}}$ is a linked domain (see Definition 3.3 of Aswal et al., 2003), i.e., $(a^1,1) \sim^+ (d^1,0)$, $(a^1,1) \sim^+ (b^1,1)$, $(d^1,0) \sim^+ (b^1,1)$, $(c^1,0) \sim^+ (d^1,0)$ and $(c^1,0) \sim^+ (b^1,1)$. Moreover, $\hat{\mathbb{D}}_{MSP|\bar{A}}$ satisfies Condition H of Chatterji et al. (2014), i.e., both $(d^1,0)$ and $(b^1,1)$ are hubs. Consequently, Theorem 3 of Chatterji et al. (2014) implies that every unanimous and strategy-proof on $\hat{\mathbb{D}}_{MSP|\bar{A}}$ is a random dictatorship.

[40] We can modify the definition of connectedness$^+$ to accord with feasibility: Both the Interior$^+$ property and the Exterior$^+$ property are established w.r.t. the peaks of feasible alternatives in all preferences. Then, we can endogenously establish the tops-only property w.r.t. feasibility. However, without the tops-only property, our domain implication analysis fails. For instance, the validation of Lemma 23 requires the tops-only property, not the tops-only property w.r.t feasibility. In a tops-only RSCF, even though infeasible alternatives never receive probabilities at any preference profile, preferences whose peaks are infeasible alternatives still play an important role in determining social lotteries.

Now, we present the result.

Theorem **3** *Let domain $\mathbb{D}$ be minimally rich and connected$^+$. If it admits a unanimous, tops-only and strategy-proof constrained RSCF satisfying the compromise property, then it is multi-dimensional single-peaked w.r.t. $\bar{A}$, and satisfies the unique feasible peaks condition.*

Conversely, let domain $\mathbb{D}$ be multidimensional single-peaked w.r.t. $\bar{A}$ that satisfies the unique feasible peaks condition. There exists a unanimous, tops-only and strategy-proof constrained RSCF satisfying the compromise property.

The proof of Theorem 3 is available in Appendix C.

Remark **6** Barberà et al. (1997) considered a model where A is located on a product of lines, the domain is the multidimensional single-peaked domain $\mathbb{D}_{MSP}$ on the product of lines (Barberà et al., 1993), and $\bar{A}$ is an arbitrary subset of A. They characterized the class of unanimous and strategy-proof DSCFs that map to $\bar{A}$: These are feasible generalized median voter schemes satisfying the intersection property.[41] The structure of the feasible set $\bar{A}$ determines the size of the class of feasible generalized median voter schemes satisfying the intersection property. On the one hand, if $\bar{A}$ is factorizable, then all feasible generalized median voter schemes satisfies the intersection property automatically, the feasible set $\bar{A}$ is automatically located on a product of lines $\times_{s\in M}G(\bar{A}^s)$ (see cases (1) and (2) of Example 4), and therefore $\mathbb{D}_{MSP}$ is obviously multidimensional single-peaked w.r.t. $\bar{A}$. Furthermore, one can construct a multidimensional projection rule with a projector of feasible alternative on $\mathbb{D}_{MSP|\bar{A}}$, and then extend it to a feasible generalized median voter scheme on $\mathbb{D}_{MSP}$ satisfying unanimity, anonymity, the tops-only property (see case (1) of Example 4) or the tops-only property w.r.t. feasibility (see case (2) of Example 4) and strategy-proofness. On the other hand, if $\bar{A}$ is not factorizable, in particular see case (3) of Example 4, Section 4 of Aswal et al. (2003) or Theorem 2 of Barberà et al. (2005), every feasible generalized median voter schemes satisfying the intersection property degenerates to a constrained dictatorship. Then, a natural question arises: What structure on $\bar{A}$ is implied by the existence of a "well-behaved" strategy-proof DSCF on $\mathbb{D}_{MSP}$ whose range is $\bar{A}$? Our analysis in this section addresses a more general research question in the framework of RSCFs, and shows that the existence of a unanimous, tops-only and strategy-proof constrained RSCF satisfying the compromise property implies that $\bar{A}$ must be factorizable. Moreover, in contrast to the model of Barberà et al. (1997) where domain $\mathbb{D}_{MSP}$ was the primitive and automatically multidimensional single-peaked w.r.t every factorizable feasible set, our domain characterization analysis (i) takes a more general class of domains as the primitive, connected$^+$ domains, (ii) establishes the factorizability of $\bar{A}$ and induces a product of trees $\times_{s\in M}G(\bar{A}^s)$ embedding $\bar{A}$ endogenously, and (iii) elicits the restriction of multidimensional single-peakedness w.r.t. feasibility. Barberà et al. (1999) considered another model where A is located on a product of lines, $\bar{A}$ is an arbitrary subset of A, and a feasible generalized median voter scheme satisfying the intersection property is fixed. They induced preference restrictions to retrieve strategy-proofness of the primitive generalized median voter scheme. However, the preference

[41] A feasible generalized median voter scheme is a generalized median voter scheme who always chooses a feasible alternative at each preference profile. The formal definition of the intersection property can be found in Definition 9 of Barberà et al. (1997). An alternative formulation of the intersection property can be found in Section 3.3. of Nehring and Puppe (2007).

restrictions elicited by them depend on the specific form of the primitive generalized median voter scheme. Our analysis does not rely on a specific RSCF, but only takes a general strategy-proof constrained RSCF with "well-behavedness" axioms as the primitive. More importantly, our characterization of multidimensional single-peakedness is independent of our primitive RSCF. □

4 Conclusion

We have proposed a class of multidimensional domains, connected^{+} domains. We first prove that multidimensional single-peakedness is the necessary and sufficient condition in the class of minimally rich and connected^{+} domain for the existence of a unanimous and strategy-proof RSCF satisfying the compromise property. Next, we show that our characterization is robust w.r.t. voting under constraints. The results for multidimensional models presented here are in the spirit of earlier results (e.g., Nehring and Puppe, 2007; Chatterji et al., 2013; Chatterji and Massó, 2016; Chatterji et al., 2016) that indicate that some form of single-peakedness is inherent in preference domains that allow the construction of "well-behaved" rules that are strategy-proof.

We suggest that connected^{+} domains may be useful in resolving other open issues; one such issue is the equivalence of strategy-proofness and local strategy-proofness where the latter is formulated by requiring that *only* a manipulation via a preference adjacent or adjacent^{+} to the sincere one is forbidden from being profitable.

The characterization of all well-behaved strategy-proof RSCFs on connected^{+} domains is not attempted in this paper and is left for future work. It would also be of interest to extend the analysis to situations where some of the dimensions include private goods or monetary transfers.

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Appendix

A Proof of Proposition 3

We first provide two general results which will be repeatedly applied in the proof of Proposition 3. Let $\varphi : \mathbb{D}^N \to \Delta(A)$ be a strategy-proof RSCF.

Lemma **9** *Let $P_i \sim P_i'$ and $\{a, b\}$ be the local switching pair in P_i and P_i'. Let $P_j \sim P_j'$ or $P_j \sim^+ P_j'$. Assume that either aP_jb and $aP_j'b$, or bP_ja and $bP_j'a$. We have*

$$\big[\varphi(P_i, P_j, P_{-\{i,j\}}) = \varphi(P_i', P_j, P_{-\{i,j\}})\big] \Rightarrow \big[\varphi(P_i, P_j', P_{-\{i,j\}}) = \varphi(P_i', P_j', P_{-\{i,j\}})\big].$$

Proof: Since $P_j \sim P_j'$ or $P_j \sim^+ P_j'$, and either aP_jb and $aP_j'b$, or bP_ja and $bP_j'a$, we can find an integer $1 \leq t \leq |A|$ such that $B^t(P_j) \equiv \{r_k(P_j)\}_{k=1}^t = \{r_k(P_j')\}_{k=1}^t \equiv B^t(P_j')$, $a \in B^t(P_j)$ and $b \notin B^t(P_j)$. Thus, a and b are referred to be *isolated* in P_j and P_j'. Then, the verification of this lemma follows from Lemma 1 of Chatterji and Zeng (2017). □

Lemma **10** *Let $P_i \sim^+ P_i'$ and $\big\{(x^s, z^{-s}), (y^s, z^{-s})\big\}_{z^{-s} \in A^{-s}}$ be the local switching pairs in P_i and P_i'. Assume that either $(x^s, z^{-s})P_j(y^s, z^{-s})$ and $(x^s, z^{-s})P_j'(y^s, z^{-s})$ for all $z^{-s} \in A^{-s}$, or $(y^s, z^{-s})P_j(x^s, z^{-s})$ and $(y^s, z^{-s})P_j'(x^s, z^{-s})$ for all $z^{-s} \in A^{-s}$. We have*

$$\big[\varphi(P_i, P_j, P_{-\{i,j\}}) = \varphi(P_i', P_j, P_{-\{i,j\}})\big] \Rightarrow \big[\varphi(P_i, P_j', P_{-\{i,j\}}) = \varphi(P_i', P_j', P_{-\{i,j\}})\big].$$

Proof: According to item 2(ii) of Lemma 1, to verify this lemma, it suffices to show that for every $z^{-s} \in A^{-s}$, there exists $a^s \in \{x^s, y^s\}$ such that $\varphi_{(a^s,z^{-s})}(P_i, P_j', P_{-\{i,j\}}) = \varphi_{(a^s,z^{-s})}(P_i', P_j', P_{-\{i,j\}})$.

We assume $(x^s, z^{-s})P_j(y^s, z^{-s})$ and $(x^s, z^{-s})P_j'(y^s, z^{-s})$ for all $z^{-s} \in A^{-s}$. The verification related to the other case is symmetric and we hence omit it. Since $P_j \sim^+ P_j'$, we know that P_j and P_j' are separable preferences. Moreover, we assume $(\bar{x}^\tau, \bar{z}^{-\tau})P_j!(\bar{y}^\tau, \bar{z}^{-\tau})$ and $(\bar{y}^\tau, \bar{z}^{-\tau})P_j'!(\bar{x}^\tau, \bar{z}^{-\tau})$ for all $\bar{z}^{-\tau} \in A^{-\tau}$. We consider two situations: $\tau = s$ and $\tau \neq s$. Assume $\tau = s$. Given $z^{-s} \in A^{-s}$, since $(x^s, z^{-s})P_j(y^s, z^{-s})$ and $(x^s, z^{-s})P_j'(y^s, z^{-s})$, it is true that there exists $a^s \in \{x^s, y^s\}$ such that $a^s \notin \{\bar{x}^s, \bar{y}^s\}$. Therefore, item 2(iii) of Lemma 1 implies $\varphi_{(a^s,z^{-s})}(P_i, P_j, P_{-\{i,j\}}) = \varphi_{(a^s,z^{-s})}(P_i, P_j', P_{-\{i,j\}})$ and $\varphi_{(a^s,z^{-s})}(P_i', P_j, P_{-\{i,j\}}) = \varphi_{(a^s,z^{-s})}(P_i', P_j', P_{-\{i,j\}})$. Since $\varphi(P_i, P_j, P_{-\{i,j\}}) = \varphi(P_i', P_j, P_{-\{i,j\}})$ by the hypothesis, we have $\varphi_{(a^s,z^{-s})}(P_i, P_j', P_{-\{i,j\}}) = \varphi_{(a^s,z^{-s})}(P_i', P_j', P_{-\{i,j\}})$, as required.

Next, assume $\tau \neq s$. Given $z^{-s} \in A^{-s}$, either one of two cases occurs: (i) There exists $a^s \in \{x^s, y^s\}$ such that $(a^s, z^{-s}) \notin (\bar{x}^\tau, A^{-\tau}) \cup (\bar{y}^\tau, A^{-\tau})$, or (ii) $(x^s, z^{-s}), (y^s, z^{-s}) \in (\bar{x}^\tau, A^{-\tau}) \cup (\bar{y}^\tau, A^{-\tau})$. In the first case, item 2(iii) of Lemma 1 implies $\varphi_{(a^s,z^{-s})}(P_i, P_j, P_{-\{i,j\}}) = \varphi_{(a^s,z^{-s})}(P_i, P_j', P_{-\{i,j\}})$ and $\varphi_{(a^s,z^{-s})}(P_i', P_j, P_{-\{i,j\}}) = \varphi_{(a^s,z^{-s})}(P_i', P_j', P_{-\{i,j\}})$.
Since $\varphi(P_i, P_j, P_{-\{i,j\}}) = \varphi(P_i', P_j, P_{-\{i,j\}})$ by the hypothesis, we have $\varphi_{(a^s,z^{-s})}(P_i, P_j', P_{-\{i,j\}}) = \varphi_{(a^s,z^{-s})}(P_i', P_j', P_{-\{i,j\}})$, as required.

If the second case occurs, it must be either $(x^s, z^{-s}) = (x^s, \bar{x}^\tau, z^{-\{s,\tau\}})$ and $(y^s, z^{-s}) = (y^s, \bar{x}^\tau, z^{-\{s,\tau\}})$, or $(x^s, z^{-s}) = (x^s, \bar{y}^\tau, z^{-\{s,\tau\}})$ and $(y^s, z^{-s}) = (y^s, \bar{y}^\tau, z^{-\{s,\tau\}})$. Assume $(x^s, z^{-s}) = (x^s, \bar{x}^\tau, z^{-\{s,\tau\}})$ and $(y^s, z^{-s}) = (y^s, \bar{x}^\tau, z^{-\{s,\tau\}})$. The verification related to the other case is similar and we hence omit it. Then, by item 2(ii) of Lemma 1, we have

$$\begin{aligned}
\sum_{\bar{a}^\tau \in \{\bar{x}^\tau, \bar{y}^\tau\}} \varphi_{(x^s, \bar{a}^\tau, z^{-\{s,\tau\}})}(P_i, P_j, P_{-\{i,j\}}) &= \sum_{\bar{a}^\tau \in \{\bar{x}^\tau, \bar{y}^\tau\}} \varphi_{(x^s, \bar{a}^\tau, z^{-\{s,\tau\}})}(P_i, P_j', P_{-\{i,j\}}) \\
\sum_{\bar{a}^\tau \in \{\bar{x}^\tau, \bar{y}^\tau\}} \varphi_{(x^s, \bar{a}^\tau, z^{-\{s,\tau\}})}(P_i', P_j, P_{-\{i,j\}}) &= \sum_{\bar{a}^\tau \in \{\bar{x}^\tau, \bar{y}^\tau\}} \varphi_{(x^s, \bar{a}^\tau, z^{-\{s,\tau\}})}(P_i', P_j', P_{-\{i,j\}})
\end{aligned}$$

Moreover, since $\varphi(P_i, P_j, P_{-\{i,j\}}) = \varphi(P_i', P_j, P_{-\{i,j\}})$ by the hypothesis, we have

$$\sum_{\bar{a}^\tau \in \{\bar{x}^\tau, \bar{y}^\tau\}} \varphi_{(x^s, \bar{a}^\tau, z^{-\{s,\tau\}})}(P_i, P_j', P_{-\{i,j\}}) = \sum_{\bar{a}^\tau \in \{\bar{x}^\tau, \bar{y}^\tau\}} \varphi_{(x^s, \bar{a}^\tau, z^{-\{s,\tau\}})}(P_i', P_j', P_{-\{i,j\}}).$$

Furthermore, item 2(i) of Lemma 1 implies

$$\begin{aligned} \varphi_{(x^s, \bar{x}^\tau, z^{-\{s,\tau\}})}(P_i, P_j', P_{-\{i,j\}}) &\geq \varphi_{(x^s, \bar{x}^\tau, z^{-\{s,\tau\}})}(P_i', P_j', P_{-\{i,j\}}) \\ \varphi_{(x^s, \bar{y}^\tau, z^{-\{s,\tau\}})}(P_i, P_j', P_{-\{i,j\}}) &\geq \varphi_{(x^s, \bar{y}^\tau, z^{-\{s,\tau\}})}(P_i', P_j', P_{-\{i,j\}}). \end{aligned}$$

Therefore, it must be the case that $\varphi_{(x^s, z^{-s})}(P_i, P_j', P_{-\{i,j\}}) \equiv \varphi_{(x^s, \bar{x}^\tau, z^{-\{s,\tau\}})}(P_i, P_j', P_{-\{i,j\}}) = \varphi_{(x^s, \bar{x}^\tau, z^{-\{s,\tau\}})}(P_i', P_j', P_{-\{i,j\}}) \equiv \varphi_{(x^s, z^{-s})}(P_i', P_j', P_{-\{i,j\}})$, as required. □

Now, we start to prove Proposition 3. Let domain $\mathbb{D}$ be connected$^+$. If $N = 1$, it is evident that unanimity implies the tops-only property.[42] Next, we provide an induction argument on the number of voters.

Induction Hypothesis: Given $N \geq 2$, every unanimous and strategy-proof RSCF with $1 \leq n < N$ voters satisfies the tops-only property.

Given a unanimous and strategy-proof RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$, we show that φ satisfies the tops-only property. According to the Interior$^+$ property, it suffices to show that **fixing** $i \in I$, and given $P_i, P_i' \in \mathbb{D}$ with $r_1(P_i) = r_1(P_i')$ and either $P_i \sim P_i'$ or $P_i \sim^+ P_i'$, we have $\varphi(P_i, P_{-i}) = \varphi(P_i', P_{-i})$ for all $P_{-i} \in \mathbb{D}^{N-1}$.

We first induce an $(N-1)$-voter RSCF. Fixing $j \in I \backslash \{i\}$, let $\phi(P_i, P_{-\{i,j\}}) = \varphi(P_i, P_i, P_{-\{i,j\}})$ for all $P_i \in \mathbb{D}$ and $P_{-\{i,j\}} \in \mathbb{D}^{N-2}$. It is evident that ϕ is a well-defined RSCF satisfying unanimity and strategy-proofness. Hence, induction hypothesis implies that ϕ satisfies the tops-only property. Henceforth, we fix $P_i, P_i' \in \mathbb{D}$ with $r_1(P_i) = r_1(P_i') = x^*$ and either $P_i \sim P_i'$ or $P_i \sim^+ P_i'$, and fix $P_{-\{i,j\}} \in \mathbb{D}^{N-2}$. We show $\varphi(P_i, P_j, P_{-\{i,j\}}) = \varphi(P_i', P_j, P_{-\{i,j\}})$ for all $P_j \in \mathbb{D}$.

The lemma below implies that if $r_1(P_j) = x^*$, then $\varphi(P_i, P_j, P_{-\{i,j\}}) = \varphi(P_i', P_j, P_{-\{i,j\}})$.

Lemma 11 *Given $P_j, P_j' \in \mathbb{D}$ with $r_1(P_j) = r_1(P_j') = x^*$, we have*
(i) $\varphi(P_i, P_j, P_{-\{i,j\}}) = \varphi(P_i', P_j, P_{-\{i,j\}})$, (ii) $\varphi(P_i, P_j, P_{-\{i,j\}}) = \varphi(P_i, P_j', P_{-\{i,j\}})$ *and*
(iii) $\varphi(P_i', P_j, P_{-\{i,j\}}) = \varphi(P_i', P_j', P_{-\{i,j\}})$.

Proof: Given $P_{-\{i,j\}} \in \mathbb{D}^{N-2}$, by strategy-proofness, we have that for every $1 \leq l \leq |A|$,

$$\left.\begin{aligned} &\textstyle\sum_{k=1}^l \varphi_{r_k(P_i)}(P_j, P_j, P_{-\{i,j\}}) \leq \sum_{k=1}^l \varphi_{r_k(P_i)}(P_i, P_j, P_{-\{i,j\}}) \leq \sum_{k=1}^l \varphi_{r_k(P_i)}(P_i, P_i, P_{-\{i,j\}}), \\ &\textstyle\sum_{k=1}^l \varphi_{r_k(P_i')}(P_j, P_j, P_{-\{i,j\}}) \leq \sum_{k=1}^l \varphi_{r_k(P_i')}(P_i', P_j, P_{-\{i,j\}}) \leq \sum_{k=1}^l \varphi_{r_k(P_i')}(P_i', P_i', P_{-\{i,j\}}), \end{aligned}\right\} \quad (1)$$

$$\left.\begin{aligned} &\textstyle\sum_{k=1}^l \varphi_{r_k(P_j)}(P_i, P_i, P_{-\{i,j\}}) \leq \sum_{k=1}^l \varphi_{r_k(P_j)}(P_i, P_j, P_{-\{i,j\}}) \leq \sum_{k=1}^l \varphi_{r_k(P_j)}(P_j, P_j, P_{-\{i,j\}}), \\ &\textstyle\sum_{k=1}^l \varphi_{r_k(P_j')}(P_i, P_i, P_{-\{i,j\}}) \leq \sum_{k=1}^l \varphi_{r_k(P_j')}(P_i, P_j', P_{-\{i,j\}}) \leq \sum_{k=1}^l \varphi_{r_k(P_j')}(P_j', P_j', P_{-\{i,j\}}), \end{aligned}\right\} \quad (2)$$

$$\left.\begin{aligned} &\textstyle\sum_{k=1}^l \varphi_{r_k(P_j)}(P_i', P_i', P_{-\{i,j\}}) \leq \sum_{k=1}^l \varphi_{r_k(P_j)}(P_i', P_j, P_{-\{i,j\}}) \leq \sum_{k=1}^l \varphi_{r_k(P_j)}(P_j, P_j, P_{-\{i,j\}}), \\ &\textstyle\sum_{k=1}^l \varphi_{r_k(P_j')}(P_i', P_i', P_{-\{i,j\}}) \leq \sum_{k=1}^l \varphi_{r_k(P_j')}(P_i', P_j', P_{-\{i,j\}}) \leq \sum_{k=1}^l \varphi_{r_k(P_j')}(P_j', P_j', P_{-\{i,j\}}). \end{aligned}\right\} \quad (3)$$

In Inequalities (1), since $\varphi(P_j, P_j, P_{-\{i,j\}}) \equiv \phi(P_j, P_{-\{i,j\}}) = \phi(P_i, P_{-\{i,j\}}) \equiv \varphi(P_i, P_i, P_{-\{i,j\}})$ and $\varphi(P_j, P_j, P_{-\{i,j\}}) \equiv \phi(P_j, P_{-\{i,j\}}) = \phi(P_i', P_{-\{i,j\}}) \equiv \varphi(P_i', P_i', P_{-\{i,j\}})$, it is true that

[42] We follow Chatterji and Sen (2011) and add the case $N = 1$ *just* to simplify the proof.

$\varphi(P_i, P_j, P_{-\{i,j\}}) = \varphi(P_j, P_j, P_{-\{i,j\}})$ and $\varphi(P_i', P_j, P_{-\{i,j\}}) = \varphi(P_j, P_j, P_{-\{i,j\}})$. Therefore, we have proved the item (i) $\varphi(P_i, P_j, P_{-\{i,j\}}) = \varphi(P_i', P_j, P_{-\{i,j\}})$. Symmetrically, according to Inequalities (2) and (3), we have $\varphi(P_i, P_j, P_{-\{i,j\}}) = \varphi(P_i, P_j', P_{-\{i,j\}})$ and $\varphi(P_i', P_j, P_{-\{i,j\}}) = \varphi(P_i', P_j', P_{-\{i,j\}})$. $\square$

The lemma below considers the situation $P_i \sim P_i'$.

Lemma **12** *Let $P_i \sim P_i'$ and $\{a, b\}$ be the local switching pair in P_i and P_i'. For all $P_j \in \mathbb{D}$, we have $\varphi(P_i, P_j, P_{-\{i,j\}}) = \varphi(P_i', P_j, P_{-\{i,j\}})$.*

Proof: Fix $P_j \in \mathbb{D}$. By Lemma 11, we only need to consider the situation $r_1(P_j) \neq x^*$. It is evident that either aP_jb or bP_ja. We assume aP_jb. The verification related to the case bP_ja is symmetric and we hence omit it. Now, by the Exterior$^+$ property, we have a simple$^+$ path $\{P_j^k\}_{k=1}^t \subseteq \mathbb{D}$ connecting P_i and P_j such that aP_j^kb for all $1 \leq k \leq t$. Note that $\varphi(P_i, P_j^1, P_{-\{i,j\}}) = \varphi(P_i', P_j^1, P_{-\{i,j\}})$ by Lemma 11. Following the simple$^+$ path and repeatedly applying Lemma 9, we have $\varphi(P_i, P_j^t, P_{-\{i,j\}}) = \varphi(P_i', P_j^t, P_{-\{i,j\}})$. Equivalently, $\varphi(P_i, P_j, P_{-\{i,j\}}) = \varphi(P_i', P_j, P_{-\{i,j\}})$. $\square$

Now, to complete the verification, we consider the situation $P_i \sim^+ P_i'$.

Lemma **13** *Let $P_i \sim^+ P_i'$ and $\big\{(x^s, z^{-s}), (y^s, z^{-s})\big\}_{z^{-s} \in A^{-s}}$ be the local switching pairs in P_i and P_i'. For all $P_j \in \mathbb{D}$, we have $\varphi(P_i, P_j, P_{-\{i,j\}}) = \varphi(P_i', P_j, P_{-\{i,j\}})$.*

Proof: By Lemma 11, to verify this lemma, we only need to consider the situation $r_1(P_j) \neq x^*$. Given arbitrary $z^{-s} \in A^{-s}$, we know either $(x^s, z^{-s})P_j(y^s, z^{-s})$ or $(x^s, z^{-s})P_j(y^s, z^{-s})$. Assume $(x^s, z^{-s})P_j(y^s, z^{-s})$. The verification related to the other case is symmetric and we hence omit it. According to the Exterior$^+$ property, we have a simple$^+$ path $\{P_j^k\}_{k=1}^t$ connecting P_i and P_j such that $(a^s, z^{-s})P_j^k(b^s, z^{-s})$, $k = 1, \ldots, t$. Evidently, $\varphi(P_i, P_j^1, P_{-\{i,j\}}) = \varphi(P_i', P_j^1, P_{-\{i,j\}})$ by item (i) of Lemma 11. We introduce a secondary induction hypothesis: Given $1 < k \leq t$, for all $1 \leq k' < k$, we have $\varphi(P_i, P_j^{k'}, P_{-\{i,j\}}) = \varphi(P_i', P_j^{k'}, P_{-\{i,j\}})$.

We show $\varphi(P_i, P_j^k, P_{-\{i,j\}}) = \varphi(P_i', P_j^k, P_{-\{i,j\}})$. First, we know either $P_j^{k-1} \sim^+ P_j^k$ or $P_j^{k-1} \sim P_j^k$. Assume $P_j^{k-1} \sim^+ P_j^k$. Thus, P_j^{k-1} and P_j^k are separable preferences. Since $(x^s, z^{-s})P_j^{k-1}(y^s, z^{-s})$ and $(x^s, z^{-s})P_j^k(y^s, z^{-s})$, separability implies $(x^s, z^{-s})P_j^{k-1}(y^s, z^{-s})$ and $(x^s, z^{-s})P_j^k(y^s, z^{-s})$ for all $z^{-s} \in A^{-s}$. Consequently, by Lemma 10, $\varphi(P_i, P_j^{k-1}, P_{-\{i,j\}}) = \varphi(P_i', P_j^{k-1}, P_{-\{i,j\}})$ implies $\varphi(P_i, P_j^k, P_{-\{i,j\}}) = \varphi(P_i', P_j^k, P_{-\{i,j\}})$, as required.

Next, assume $P_j^{k-1} \sim P_j^k$. By the definition of a simple$^+$ path, it must be the case that $r_1(P_j^{k-1}) = r_1(P_j^k) \equiv z$. Assume $xP_j^{k-1}!y$ and $yP_j^k!x$. If $z = x^* \equiv r_1(P_i) = r_1(P_i')$, then item (i) of Lemma 11 implies $\varphi(P_i, P_j^k, P_{-\{i,j\}}) = \varphi(P_i', P_j^k, P_{-\{i,j\}})$. Now, assume $z \neq x^*$. Evidently, we know either xP_iy or yP_ix. Assume xP_iy. The verification related to the other case is symmetric and we hence omit it. Thus, the Exterior$^+$ property implies that there exists a simple$^+$ path $\{P_i^k\}_{k=1}^q \subseteq \mathbb{D}$ connecting P_j^{k-1} and P_i such that xP_i^ky, $k = 1, \ldots, q$. Item (ii) of Lemma 11 first implies $\varphi(P_i^1, P_j^{k-1}, P_{-\{i,j\}}) = \varphi(P_i^1, P_j^k, P_{-\{i,j\}})$. Following the simple$^+$ path $\{P_i^k\}_{k=1}^q$ and repeatedly applying Lemma 9, we have $\varphi(P_i, P_j^{k-1}, P_{-\{i,j\}}) = \varphi(P_i, P_j^k, P_{-\{i,j\}})$.[43]

[43]To apply Lemma 9 here, we need to make a notational change on the expression of Lemma 9 by switching voters i and j: Let $P_j \sim P_j'$ and $\{a, b\}$ be the local switching pair in P_j and P_j'. Let $P_i \sim P_i'$ or $P_i \sim^+ P_i'$. Assume that either aP_ib and $aP_i'b$, or bP_ia and $bP_i'a$. We have $\big[\varphi(P_i, P_j, P_{-\{i,j\}}) = \varphi(P_i, P_j', P_{-\{i,j\}})\big] \Rightarrow \big[\varphi(P_i', P_j, P_{-\{i,j\}}) = \varphi(P_i', P_j', P_{-\{i,j\}})\big]$.

Analogously, with the initial equality of item (iii) of Lemma 11, since either xP'_iy or yP'_ix, we can apply a similar argument and show $\varphi(P'_i, P_j^{k-1}, P_{-\{i,j\}}) = \varphi(P'_i, P_j^k, P_{-\{i,j\}})$. Last, since $\varphi(P_i, P_j^{k-1}, P_{-\{i,j\}}) = \varphi(P'_i, P_j^{k-1}, P_{-\{i,j\}})$ by the secondary induction hypothesis, we have $\varphi(P_i, P_j^k, P_{-\{i,j\}}) = \varphi(P'_i, P_j^k, P_{-\{i,j\}})$, as required. This completes the verification of the secondary induction hypothesis. Therefore, $\varphi(P_i, P_j, P_{-\{i,j\}}) = \varphi(P'_i, P_j, P_{-\{i,j\}})$. $\square$

Finally, by Lemmas 12 and 13, we have $\varphi(P_i, P_j, P_{-\{i,j\}}) = \varphi(P'_i, P_j, P_{-\{i,j\}})$ for all $P_j \in \mathbb{D}$. This completes the verification of the induction hypothesis and hence proves Proposition 3.

B Proof of Theorem 2

First, by the verification of the sufficiency part of Theorem 1, we know that all multidimensional projection rules are unanimous, anonymous and strategy-proof on the multidimensional single-peaked domain. Therefore, we omit the verification of the sufficiency part of Theorem 2, but focus on the necessity part.

Let $\mathbb{D}$ be a minimally rich and connected$^+$ domain. Let $\bar{f} : \mathbb{D}^N \to A$ be a unanimous, anonymous and strategy-proof DSCF where N is an even integer. First, Proposition 3 implies that DSCF $\bar{f}$ satisfies the tops-only property. By a similar argument in the beginning proof of Theorem 1, we can also induce a two-voter unanimous, anonymous, tops-only and strategy-proof DSCF $f : \mathbb{D}^2 \to A$. Let $I = \{i, j\}$. Note that we establish all Lemmas 2, 6, 7 and 8 without referring to any RSCFs. Therefore, to complete the verification, we only need to use DSCF f to show Lemma 4, establish a counterpart result of Lemma 3, and prove the result of Lemma 5.

Lemma **14** *Given $s \in M$ and $x^{-s} \in A^{-s}$, $G_{\sim^+}\big((A^s, x^{-s})\big)$ is a tree.*

Proof: Suppose that $G_{\sim^+}\big((A^s, x^{-s})\big)$ is not a tree. Thus, there exists a cycle $\{x_k\}_{k=1}^t \subseteq (A^s, x^{-s})$, $t \geq 3$, such that $x_k \sim^+ x_{k+1}$, $k = 1, \ldots, t$, where $x_{t+1} = x_1$. Given distinct $1 \leq k, k' \leq t$, we have a clockwise adjacency$^+$ path and a counter clockwise adjacency$^+$ path which connect x_k and $x_{k'}$. By the proof of Lemma 3, the social outcome $f(x_k, x_{k'})$ must belong to both paths. Therefore, $f(x_k, x_{k'}) \in \{x_k, x_{k'}\}$. We start with $f(x_1, x_2)$, and assume $f(x_1, x_2) = x_1$. The verification related to the case $f(x_1, x_2) = x_2$ is symmetric and we hence omit it. Along the clockwise adjacency$^+$ path from x_2 to x_t, by a repeated application of item 2(iii) of Lemma 1, we know $x_1 = f(x_1, x_2) = \cdots = f(x_1, x_{t-1}) = f(x_1, x_t)$. Next, since $x_2 \sim^+ x_1$, item 2(ii) of Lemma 1 implies $f(x_2, x_t) \in \{x_1, x_2\}$. Furthermore, since $f(x_2, x_t) \in \{x_2, x_t\}$, we have $f(x_2, x_t) = x_2$. Last, since $x_t \sim^+ x_1$, item 2(iii) of Lemma 1 implies $f(x_2, x_1) = f(x_2, x_t) = x_2 \neq f(x_1, x_2)$ which contradicts anonymity of f. Therefore, $G_{\sim^+}\big((A^s, x^{-s})\big)$ is a tree. $\square$

Given $s \in M$, $x^{-s} \in A^{-s}$ and $a, b \in (A^s, x^{-s})$, let $\langle a, b\rangle^{(A^s, x^{-s})}$ denote the unique graph path in $G_{\sim^+}\big((A^s, x^{-s})\big)$ connecting a and b. Since $G_{\sim^+}\big((A^s, x^{-s})\big)$ is a tree, the notion of projection (recall the proof of sufficiency of Theorem 1) is well-defined. Thus, given $x, a, b \in (A^s, x^{-s})$, let $\pi\big(x, \langle a, b\rangle^{(A^s, x^{-s})}\big)$ denote the projection of x on the graph path $\langle a, b\rangle^{(A^s, x^{-s})}$.

Lemma **15** *Given $s \in M$ and $x^{-s} \in A^{-s}$, there exists $\bar{a} \in (A^s, x^{-s})$ such that for all $a, b \in (A^s, x^{-s})$, $f(a, b) = \pi\big(\bar{a}, \langle a, b\rangle^{(A^s, x^{-s})}\big)$. Moreover, given an adjacent$^+$ path $\{x_k\}_{k=1}^q \subseteq (A^s, x^{-s})$ and $P_i \in \mathbb{D}^{x_1}$, we have $x_k P_i x_{k+1}$, $k = 1, \ldots, q-1$.*

Proof: According to the proof of Lemma 3, we know $f(a,b) \in (A^s, x^{-s})$ for all $a, b \in (A^s, x^{-s})$. Then, the first part of this lemma follows exactly from the proof of the necessity part of the Theorem of Chatterji et al. (2013).

Next, suppose $x_{k+1} P_i x_k$ for some $1 \leq k < q$. It is evident $1 < k < q$. Pick an arbitrary $P_i' \in \mathbb{D}^{x_{k+1}}$ by minimal richness. By the no-detour property, we have a simple$^+$ path $\{P_i^l\}_{l=1}^q \subseteq \mathbb{D}^{(A^s, x^{-s})}$ connecting P_i and P_i' such that $x_{k+1} P_i^l x_k$, $l = 1, \ldots, q$. Evidently, $r_1(P_i^l) \neq x_k$ for all $1 \leq l \leq q$. Then, by an argument similar to the proof of Lemma 2, we can elicit an adjacency$^+$ path connecting x_1 and x_{k+1} from $\{P_i^l\}_{l=1}^q$. Consequently, we have two distinct adjacent$^+$ paths connecting x_1 and x_{k+1}: One includes x_k (see the hypothesis of the lemma), while the other excludes x_k. This contradicts Lemma 14. Therefore, $x_k P_i x_{k+1}$, $k = 1, \ldots, q-1$. $\Box$

Before proceeding further with the proof, we note that the order of Lemmas 3 and 4 is opposite to the order of Lemmas 14 and 15. This difference arises mainly from the difference between the random setting and the deterministic one. In the random setting, the preference restriction in Lemma 3 is simply induced from the compromise property of the unanimous and strategy-proof RSCF, and Lemma 4 is proved by the RSCF characterization result in Lemma 3. In the deterministic case, Lemma 14 (which is identical to Lemma 4) is proved using mainly the anonymity of the unanimous and strategy-proof DSCF, and the same preference restriction in Lemma 15 (which is the counterpart of Lemma 3) is elicited from the result of Lemma 14 and the richness condition of connectedness$^+$.

To prove the result of Lemma 5 by using a DSCF f, we first provide an intermediate step which will be repeatedly applied in the subsequent verification.

Lemma **16** *If $f(x,y) = y$, $f(y,z) = z$ and $y \sim^+ z$, then $f(x,z) = z$.*

Proof: Since $z \sim^+ y$ and $f(x,y) = y$, strategy-proofness implies $f(x,z) \in \{y,z\}$. If $f(x,z) = y$, strategy-proofness implies $f(y,z) = y$ which contradicts the hypothesis $f(y,z) = z$. Therefore, $f(x,z) = z$. $\Box$

Now, we are ready to prove the result of Lemma 5. We fix the following four alternatives: $a = (x^s, x^\tau, z^{-\{s,\tau\}})$, $b = (y^s, y^\tau, z^{-\{s,\tau\}})$, $c = (x^s, y^\tau, z^{-\{s,\tau\}})$ and $d = (y^s, x^\tau, z^{-\{s,\tau\}})$ where $x^s \neq y^s$ and $x^\tau \neq y^\tau$. We assume $a \sim^+ c$ and $a \sim^+ d$. Recall Figure 3. Let $\{x_k\}_{k=1}^p \subseteq (y^s, A^\tau, z^{-\{s,\tau\}})$ denote the adjacency$^+$ path connecting b and d. Let $\{y_k\}_{k=1}^q \subseteq (A^s, y^\tau, z^{-\{s,\tau\}})$ denote the adjacency$^+$ path connecting b and c. Furthermore, by Lemma 15, let $\bar{x} \in (y^s, A^\tau, z^{-\{s,\tau\}})$ and $\bar{y} \in (A^s, y^\tau, z^{-\{s,\tau\}})$ be such that

- $f(x,y) = \pi\big(\bar{x}, \langle x, y\rangle^{(A^s, y^\tau, z^{-\{s,\tau\}})}\big)$ for all $x, y \in (y^s, A^\tau, z^{-\{s,\tau\}})$;
- $f(x', y') = \pi\big(\bar{y}, \langle x', y'\rangle^{(y^s, A^\tau, z^{-\{s,\tau\}})}\big)$ for all $x', y' \in (A^s, y^\tau, z^{-\{s,\tau\}})$.

By the first paragraph of the proof of Lemma 3, since $a \sim^+ c$ and $a \sim^+ d$, we have $f(a,c) = f(c,a) \in \{c,a\}$ and $f(a,d) = f(d,a) \in \{a,d\}$. Therefore, there are four situations:

Situation 1. $f(a,c) = f(c,a) = c$ and $f(a,d) = f(d,a) = a$ (see the first diagram of Figure 6).

Situation 2. $f(a,c) = f(c,a) = c$ and $f(a,d) = f(d,a) = d$ (see the second diagram of Figure 6).

Situation 3. $f(a,c) = f(c,a) = a$ and $f(a,d) = f(d,a) = d$ (see the third diagram of Figure 6).

Situation 4. $f(a,c) = f(c,a) = a$ and $f(a,d) = f(d,a) = a$ (see the fourth diagram of Figure 6).

Situation 1 *Situation* 2 *Situation* 3 *Situation* 4

Figure 6: Four situations[44]

Note that Situations 1 and 3 are analogous. Thus, we only consider Situations 1, 2 and 4. In Lemmas 17, 18 and 19, we show that in each situation, $b \sim^{+} c$ and $b \sim^{+} d$.

Lemma **17** *In Situation 1, $b \sim^{+} c$ and $b \sim^{+} d$.*

Proof: Since $f(d,a) = a$, $f(a,c) = c$ and $c \sim^{+} a$, Lemma 16 implies $f(d,c) = c$. We first show $b \sim^{+} d$. Suppose not, i.e., $p > 2$. Thus, $x_{p-1} \equiv (y^s, x^{\tau}_{p-1}, z^{-\{s,t\}})$ and $x^{\tau}_{p-1} \notin \{x^{\tau}, y^{\tau}\}$.

Claim 1: $f(x_p, x_{p-1}) = f(x_{p-1}, x_p) = x_p$.

Since $x_p \sim^{+} x_{p-1}$, we have $P_i \in \mathbb{D}^{x_p}$ and $P'_i \in \mathbb{D}^{x_{p-1}}$ such that $P_i \sim^{+} P'_i$. Since $c^{\tau} = y^{\tau} \notin \{x^{\tau}_p, x^{\tau}_{p-1}\}$, item 2(iii) of Lemma 1 implies $f_c(x_{p-1}, c) = f_c(P'_i, c) = f_c(P_i, c) = f_c(x_p, c) \equiv f_c(d, c) = 1$. Thus, $f(x_{p-1}, c) = c$. Next, since $a \sim^{+} c$, strategy-proofness implies $f(x_{p-1}, a) \in \{c, a\}$. Suppose $f(x_{p-1}, a) = c$. Since $x_p \sim^{+} x_{p-1}$ and $c^{\tau} \equiv y^{\tau} \notin \{x^{\tau}_p, x^{\tau}_{p-1}\}$, by a similar argument right above, we know $f_c(x_p, a) = f_c(x_{p-1}, a) = 1$. Hence, $f(d,a) \equiv f(x_p, a) = c$ which contradicts the hypothesis $f(d,a) = a$. Therefore, $f(x_{p-1}, a) = a$. Furthermore, since $x_p \sim^{+} a$, strategy-proofness implies $f(x_{p-1}, x_p) \in \{x_p, a\}$. Last, since $x_p \sim^{+} x_{p-1}$, by the first paragraph of the proof of Lemma 3, we have $f(x_{p-1}, x_p) \in \{x_p, x_{p-1}\}$. Therefore, $f(x_p, x_{p-1}) = f(x_{p-1}, x_p) = x_p$. This completes the verification of the claim.

Now, we know $\pi\big(\bar{x}, \langle x_p, x_{p-1}\rangle^{(A^s, y^{\tau}, z^{-\{s,\tau\}})}\big) = f(x_p, x_{p-1}) = x_p$. Thus, $x_p \in \langle \bar{x}, x_{p-1}\rangle^{(A^s, y^{\tau}, z^{-\{s,\tau\}})}$, and hence $x_p \in \langle \bar{x}, x_1\rangle^{(A^s, y^{\tau}, z^{-\{s,\tau\}})}$. Therefore, $f(b,d) \equiv f(x_1, x_p) = \pi\big(\bar{x}, \langle x_1, x_p\rangle^{(A^s, y^{\tau}, z^{-\{s,\tau\}})}\big) = x_p \equiv d$. Furthermore, since $f(d,a) = a$ and $d \sim^{+} a$ by the hypothesis of Situation 1, Lemma 16 implies $f(b,a) = a$. By connectedness^{+}, we have a separable preference $\hat{P}_i \in \mathbb{D}^b$. Since $(y^s, y^{\tau}, z^{-\{s,\tau\}}) \equiv b = r_1(\hat{P}_i)$, separability implies $c \equiv (x^s, y^{\tau}, z^{-\{s,\tau\}}) \hat{P}_i (x^s, x^{\tau}, z^{-\{s,\tau\}}) \equiv a$. Consequently, by Situation 1, $f(c,a) = c\hat{P}_i a = f(b,a) = f(\hat{P}_i, a)$. Then, voter i will manipulate at $(\hat{P}_i, a)$ via $P'_i \in \mathbb{D}^c$. Therefore, $p = 2$. Equivalently, $b \sim^{+} d$.

Claim 2: $f(b,d) = f(d,b) = b$.

Since $b \sim^{+} d$, strategy-proofness implies $f(b,d) \in \{b,d\}$. Suppose $f(b,d) = d$. Since $f(d,a) = a$ and $d \sim^{+} a$ by the hypothesis, Lemma 16 implies $f(b,a) = a$. We adopt the preference $\hat{P}_i$ specified right above. Since $f(c,a) = c\hat{P}_i a = f(b,a) = f(\hat{P}_i, a)$, voter i will manipulate at $(\hat{P}_i, a)$ via $P'_i \in \mathbb{D}^c$. Therefore, $f(b,d) = f(d,b) = b$. This completes the verification of the claim.

Last, we show $b \sim^{+} c$. Suppose not, i.e., $q > 2$.

Claim 3: $f(y_1, y_2) = f(y_2, y_1) = y_1$.

Since $y_1 \sim^{+} y_2$, strategy-proofness implies $f(y_1, y_2) \in \{y_1, y_2\}$. Suppose $f(y_1, y_2) = y_2$. According to $y_2 \equiv (y^s_2, y^{\tau}, z^{-\{s,\tau\}})$, $y_1 \equiv (y^s_1, y^{\tau}, z^{-\{s,\tau\}})$ and $d \equiv (y^s, x^{\tau}, z^{-\{s,\tau\}}) = (y^s_1, x^{\tau}, z^{-\{s,\tau\}})$, we induce another alternative $x^* = (y^s_2, x^{\tau}, z^{-\{s,\tau\}})$. Now, note that $y_1 \sim^{+} y_2$, $y_1 = b \sim^{+} d$;

[44] Taking the first diagram of Figure 6 as an example, "$a \to c$" represents that $a \sim^{+} c$ and $f(a,c) = f(c,a) = c$.

$f(y_2, y_1) = f(y_1, y_2) = y_2$ (by the hypothesis above) and $f(y_1, d) = f(d, y_1) = y_1$ (by Claim 2). The analogy of Situation 1 occurs on alternatives $\{y_2, x^*, y_1, d\}$. See Figure 7 below.

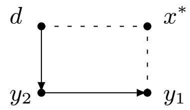

Figure 7: The analogy of Situation 1 on alternatives $\{y_2, x^*, y_1, d\}$

Then, by a similar argument in the verification of Claims 1 and 2, we have $x^* \sim^+ d$ and $f(x^*, d) = f(d, x^*) = x^*$. Note that $x^*, d, a \in (A^s, x^\tau, z^{-\{s,\tau\}})$, $x^* \sim^+ d$ and $d \sim^+ a$. Since $f(a, d) = a$ by Situation 1, $d \sim^+ x^*$ and $a \notin (d^s, A^{-s}) \cup (x^{*s}, A^{-s})$, item 2(iii) of Lemma 1 implies $f_a(a, x^*) = f_a(a, d) = 1$. Thus, $f(a, x^*) = a$. Then, by $d \sim^+ a$, strategy-proofness implies $f(d, x^*) \in \{a, d\}$ which contradicts the result $f(x^*, d) = f(d, x^*) = x^*$. Therefore, $f(y_1, y_2) = y_1$. This completes the verification of the claim.

Now, we know $\pi\big(\bar{y}, \langle y_1, y_2\rangle^{(A^s, y^\tau, z^{-\{s,\tau\}})}\big) = f(y_1, y_2) = y_1$. Thus, $y_1 \in \langle \bar{y}, y_2\rangle^{(A^s, y^\tau, z^{-\{s,\tau\}})}$ and hence, $y_k \in \langle \bar{y}, y_{k'}\rangle^{(A^s, y^\tau, z^{-\{s,\tau\}})}$ for all $1 \leq k < k' \leq q$. Therefore, $f(y_{k+1}, y_k) = f(y_k, y_{k+1}) = \pi\big(\bar{y}, \langle y_k, y_{k+1}\rangle^{(A^s, y^\tau, z^{-\{s,\tau\}})}\big) = y_k$ for all $1 \leq k \leq q-1$. Since $f(a, y_q) \equiv f(a, c) = c$ by Situation 1, $f(y_{k+1}, y_k) = y_k$ and $y_{k+1} \sim^+ y_k$ for all $1 \leq k \leq q-1$, by a repeated application of Lemma 16, we have $f(a, y_k) = y_k$ for all $k = q-1, \ldots, 1$. Thus, $f(b, a) = f(a, b) \equiv f(a, y_1) = y_1 \equiv b$. Last, since $d \sim^+ b$, we have $\bar{P}_i \in \mathbb{D}^d$ with $r_2(\bar{P}_i) = b$. Consequently, $f(b, a) = b\bar{P}_i a = f(d, a) = f(\bar{P}_i, a)$. Then, voter i will manipulate at $(\bar{P}_i, a)$ via $P'_i \in \mathbb{D}^b$. Therefore, $q = 2$. Equivalently, $b \sim^+ c$. $\square$

Lemma **18** *In Situation 2, $b \sim^+ c$ and $b \sim^+ d$.*

Proof: We first provide two mutually exclusive claims.

Claim 1: If $p > 2$, then $f(a, b) = f(b, a) = d$.

Since $x_p \sim^+ x_{p-1}$, by the first paragraph of the proof of Lemma 3, we have $f(x_p, x_{p-1}) = f(x_{p-1}, x_p) \in \{x_p, x_{p-1}\}$. Suppose $f(x_p, x_{p-1}) = x_{p-1}$. According to $x_{p-1} \equiv (y^s, x^\tau_{p-1}, z^{-\{s,\tau\}})$, $x_p \equiv (y^s, x^\tau_p, z^{-\{s,\tau\}})$ and $a \equiv (x^s, x^\tau, z^{-\{s,\tau\}}) = (x^s, x^\tau_p, z^{-\{s,\tau\}})$, we induce another alternative $x^* = (x^s, x^\tau_{p-1}, z^{-\{s,\tau\}})$. Thus, the analogy of Situation 1 occurs to $\{x_p, x^*, x_{p-1}, a\}$. See Figure 8 below.

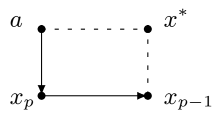

Figure 8: The analogy of Situation 1 on $\{x_p, x^*, x_{p-1}, a\}$

Hence $x^* \sim^+ a$ and $x^* \sim^+ x_{p-1}$ by Lemma 17.

Furthermore, we claim $f(x^*, a) = f(a, x^*) = x^*$. Since $x^* \sim^+ a$, by the first paragraph of the proof of Lemma 3, we have $f(x^*, a) = f(a, x^*) \in \{a, x^*\}$. Suppose $f(x^*, a) = f(a, x^*) = a$. Since $f(x^*, a) = a$, $f(a, x_p) = x_p$ and $a \sim^+ x_p$, Lemma 16 first implies $f(x^*, x_p) = x_p$. Next, since $f(x^*, x_p) = x_p$, $f(x_p, x_{p-1}) = x_{p-1}$ and $x_p \sim^+ x_{p-1}$, Lemma 16 further implies $f(x^*, x_{p-1}) = x_{p-1}$. However, on the other hand, since $f(x^*, x_{p-1}) = x_{p-1}$ and $f(x_{p-1}, x_p) = x_{p-1}$, by strategy-proofness, $x_p \sim^+ x_{p-1}$ implies $f(x^*, x_p) \in \{x_{p-1}, x_p\}$, and $x_{p-1} \sim^+ x^*$ implies $f(x^*, x_p) \in \{x_{p-1}, x^*\}$. Therefore, $f(x^*, x_p) = x_{p-1}$. Contradiction! Hence, $f(x^*, a) = f(a, x^*) = x^*$.

Note that $c, a, x^* \in (x^s, A^\tau, z^{-\{s,\tau\}})$, $c \sim^+ a$ and $a \sim^+ x^*$. Since $f(c,a) = c$ by Situation 2, $a \sim^+ x^*$ and $c^\tau \notin \{a^\tau, x^{*\tau}\}$, item 2(iii) of Lemma 1 implies $f_c(c, x^*) = f_c(c,a) = 1$. Thus, $f(c,x^*) = c$. Furthermore, since $a \sim^+ c$, strategy-proofness implies $f(a,x^*) \in \{a,c\}$ which contradicts $f(x^*,a) = f(a,x^*) = x^*$. Therefore, we have $f(x_p, x_{p-1}) = f(x_{p-1}, x_p) = x_p$.

Now, we know $\pi\big(\bar{x}, \langle x_p, x_{p-1}\rangle^{(y^s, A^\tau, z^{-\{s,\tau\}})}\big) = f(x_p, x_{p-1}) = x_p$. Thus, $x_p \in \langle \bar{x}, x_{p-1}\rangle^{(y^s, A^\tau, z^{-\{s,\tau\}})}$, and hence $x_p \in \langle \bar{x}, x_1\rangle^{(y^s, A^\tau, z^{-\{s,\tau\}})}$. Therefore, $f(b,d) \equiv f(x_1, x_p) = \pi\big(\bar{x}, \langle x_1, x_p\rangle^{(y^s, A^\tau, z^{-\{s,\tau\}})}\big) = x_p \equiv d$. Furthermore, since $a \sim^+ d$, strategy-proofness implies $f(b,a) \in \{a,d\}$. Suppose $f(b,a) = a$. By connectedness$^+$, we have a separable preference $\hat{P}_i \in \mathbb{D}^b$. Since $(y^s, y^\tau, z^{-\{s,\tau\}}) \equiv b = r_1(\hat{P}_i)$, separability implies $d \equiv (y^s, x^\tau, z^{-\{s,\tau\}})\hat{P}_i(x^s, x^\tau, z^{-\{s,\tau\}}) \equiv a$. Consequently, by Situation 2, $f(d,a) = d\hat{P}_i a = f(b,a) = f(\hat{P}_i, a)$. Then, voter i will manipulate at $(\hat{P}_i, a)$ via $P'_i \in \mathbb{D}^d$. Therefore, $f(a,b) = f(b,a) = b$. This completes the verification of the claim.

Claim 2: If $q > 2$, then $f(a,b) = f(b,a) = c$.

The verification of this claim is symmetric to the verification of Claim 1.

Evidently, Claims 1 and 2 are mutually exclusive. Therefore, it must be either $p = 2$ or $q = 2$. Thus, either $b \sim^+ d$ or $b \sim^+ c$. Suppose $p > 2$. Then, $q = 2$, and equivalently, $b \sim^+ c$. The verification related to the case $p = 2$ and $q > 2$ is symmetric, and we hence omit it. Thus, Claim 1 implies $f(b,a) = d$. However, on the other hand, since $f(c,a) = c$ by Situation 2 and $c \sim^+ b$, strategy-proofness implies $f(b,a) \in \{b,c\}$. Contradiction! Therefore, $p = 2$. Equivalently, $b \sim^+ d$. Symmetrically, we have $b \sim^+ c$. $\square$

Lemma **19** *In Situation 4, $b \sim^+ c$ and $b \sim^+ d$.*

Proof: We first show $b \sim^+ d$. Suppose not, i.e., $p > 2$. According to the adjacency$^+$ path $\{x_k\}_{k=1}^p \subseteq (y^s, A^\tau, z^{-\{s,\tau\}})$, by replacing the element y^s in each x_k by x^s, we construct another sequence $\{z_k\}_{k=1}^p = \{(x^s, x_k^{-s})\}_{k=1}^p \equiv \big\{(x^s, x_k^\tau, z^{-\{s,\tau\}})\big\}_{k=1}^p \subseteq (x^s, A^\tau, z^{-\{s,\tau\}})$. Note that $\{z_k\}_{k=1}^p$ is simply a sequence of alternatives, not necessarily an adjacency$^+$ path. We will show that $\{z_k\}_{k=1}^p$ is an adjacency$^+$ path. Note that $z_p \equiv (x^s, x_p^\tau, z^{-\{s,\tau\}}) = (x^s, x^\tau, z^{-\{s,\tau\}}) \equiv a$ and $z_1 \equiv (x^s, x_1^\tau, z^{-\{s,\tau\}}) = (x^s, y^\tau, z^{-\{s,\tau\}}) \equiv c$.

We first consider x_p, z_{p-1}, z_p and x_{p-1}. Note that $x_p \sim^+ x_{p-1}$, $x_p \sim^+ z_p$ and $f(x_p, z_p) = f(z_p, x_p) = z_p$ by Situation 4. Since $x_p \sim^+ x_{p-1}$, by the first paragraph of the proof of Lemma 3, we have $f(x_p, x_{p-1}) = f(x_{p-1}, x_p) \in \{x_p, x_{p-1}\}$. If $f(x_p, x_{p-1}) = f(x_{p-1}, x_p) = x_p$, then the analog of Situation 1 occurs to x_p, z_{p-1}, z_p and x_{p-1}. If $f(x_p, x_{p-1}) = f(x_{p-1}, x_p) = x_{p-1}$, then the analog of Situation 2 occurs to $\{x_p, z_{p-1}, z_p, x_{p-1}\}$. See Figure 9 below.

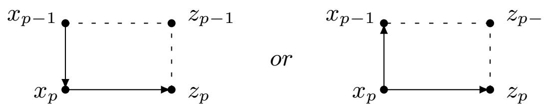

Figure 9: The analogy of Situation 1 or 2 on $\{x_p, z_{p-1}, z_p, x_{p-1}\}$

Then, by Lemma 17 or 18, we have $z_{p-1} \sim^+ z_p$ and $z_{p-1} \sim^+ x_{p-1}$.

Furthermore, we claim $f(z_{p-1}, x_{p-1}) = f(x_{p-1}, z_{p-1}) = z_{p-1}$. If $f(x_p, x_{p-1}) = f(x_{p-1}, x_p) = x_p$, then by an argument similar to the second paragraph of the verification of Claim 1 of Lemma 18, we have $f(z_{p-1}, x_{p-1}) = f(x_{p-1}, z_{p-1}) = z_{p-1}$. Next, assume $f(x_p, x_{p-1}) = f(x_{p-1}, x_p) =$

x_{p-1}, and we show $f(z_{p-1}, x_{p-1}) = f(x_{p-1}, z_{p-1}) = z_{p-1}$. Since $z_{p-1} \sim^+ x_{p-1}$, by the first paragraph of the proof of Lemma 3, we have $f(z_{p-1}, x_{p-1}) = f(x_{p-1}, z_{p-1}) \in \{x_{p-1}, z_{p-1}\}$. Suppose $f(z_{p-1}, x_{p-1}) = f(x_{p-1}, z_{p-1}) = x_{p-1}$. Since $f(x_p, x_{p-1}) = x_{p-1}$ and $z_{p-1} \sim^+ x_{p-1}$, strategy-proofness implies $f(x_p, z_{p-1}) \in \{x_{p-1}, z_{p-1}\}$. Meanwhile, since $f(x_{p-1}, z_{p-1}) = x_{p-1}$ and $x_p \sim^+ x_{p-1}$, strategy-proofness implies $f(x_p, z_{p-1}) \in \{x_p, x_{p-1}\}$. Thus, $f(x_p, z_{p-1}) = x_{p-1}$. However, on the other hand, since $f(x_p, z_p) = z_p$ by Situation 4 and $z_{p-1} \sim^+ z_p$, strategy-proofness implies $f(x_p, z_{p-1}) \in \{z_p, z_{p-1}\}$. Contradiction! Therefore, $f(z_{p-1}, x_{p-1}) = f(x_{p-1}, z_{p-1}) = z_{p-1}$.

Moving from from z_{p-1} to z_1 according to the sequence $\{z_k\}_{k=1}^{p-1}$, and applying the argument above to $\{x_k, z_{k-1}, z_k, x_{k-1}\}$ from $k = p-1$ to $k = 2$ step by step, we have $z_{k-1} \sim^+ z_k$, $z_{k-1} \sim^+ x_{k-1}$ and $f(z_{k-1}, x_{k-1}) = f(x_{k-1}, z_{k-1}) = z_{k-1}$, $k = p-1, \ldots, 2$. Consequently, since $c \sim^+ a$ (equivalently, $z_1 \sim^+ z_p$), we have a cycle in $G\big((x^s, A^\tau, z^{-\{s,\tau\}})\big)$ which contradicts Lemma 14. Therefore, $p = 2$. Equivalently, $b \sim^+ d$. Symmetrically, we have $b \sim^+ c$. $\square$

Now, by Lemmas 17 - 19, we know $b \sim^+ c$ and $b \sim^+ d$. This proves the result of Lemma 5, as required. This completes the verification of the necessity part of Theorem 2.

C Proof of Theorem 3

We first show the necessity part of Theorem 3. Let $\phi : \mathbb{D}^N \to \Delta(\bar{A})$ be a constrained RSCF satisfying unanimity, tops-onlyness, strategy-proofness and the compromise property. Then, similar to the verification of the necessity part of Theorem 1, we induce a two-voter constrained RSCF $\varphi : \mathbb{D}^2 \to \Delta(\bar{A})$ satisfying unanimity, tops-onlyness, strategy-proofness and the compromise property.

Lemma **20** *Domain $\mathbb{D}$ satisfies the unique feasible peaks condition.*

Proof: Given distinct $P_i, P_i' \in \mathbb{D}$ with $r_1(P_i) = r_1(P_i') \notin \bar{A}$, suppose that $r_1(P_{i|\bar{A}}) \equiv x \neq y \equiv r_1(P'_{i|\bar{A}})$. By unanimity, we have $\varphi_x(P_i, P_i) = 1$ and $\varphi_y(P_i', P_i') = 1$, which contradict the tops-only property. $\square$

Lemma **21** *Given $s \in M$, $a^s, b^s \in A^s$ and $x^{-s} \in A^{-s}$, if $(a^s, x^{-s}), (b^s, x^{-s}) \in \bar{A}$, there exists a unique adjacent$^+$ path $\{x_k\}_{k=1}^q \subseteq (A^s, x^{-s})$ connecting (a^s, x^{-s}) and (b^s, x^{-s}), i.e., $x_1 = (a^s, x^{-s})$, $x_q = (b^s, x^{-s})$ and $x_k \sim^+ x_{k+1}$, $k = 1, \ldots, q-1$. Moreover, $\{x_k\}_{k=1}^q \subseteq \bar{A}$.*

Proof: First, by Lemma 2, we have an adjacent$^+$ path $\{x_k\}_{k=1}^q \subseteq (A^s, x^{-s})$ connecting (a^s, x^{-s}) and (b^s, x^{-s}). Suppose that $x_k \notin \bar{A}$ for some $1 < k < q$. Thus, we elicit an adjacent$^+$ subpath $\{x_k\}_{k=\underline{k}}^{\overline{k}}$ such that $\overline{k} - \underline{k} \geq 2$, $x_{\underline{k}}, x_{\overline{k}} \in \bar{A}$ and $x_{\underline{k}+1}, \ldots, x_{\overline{k}-1} \notin \bar{A}$.

Since $x_{\underline{k}} \sim^+ x_{\underline{k}+1}$, we have $P_i \in \mathbb{D}^{x_{\underline{k}}}$ and $P_j \in \mathbb{D}^{x_{\underline{k}+1}}$ with $P_i \sim^+ P_j$. Since $x_{\underline{k}} \in \bar{A}$ and $x_{\underline{k}+1} \notin \bar{A}$, tops-onlyness and unanimity imply $\varphi_{x_{\underline{k}}}(x_{\underline{k}}, x_{\underline{k}+1}) = \varphi_{x_{\underline{k}}}(P_i, P_j) = 1$. Moving from $x_{\underline{k}+1}$ up to $x_{\overline{k}}$ along the subpath, by a repeated application of item 2(iii) of Lemma 1, we have $1 = \varphi_{x_{\underline{k}}}(x_{\underline{k}}, x_{\underline{k}+1}) = \cdots = \varphi_{x_{\underline{k}}}(x_{\underline{k}}, x_{\overline{k}-1}) = \varphi_{x_{\underline{k}}}(x_{\underline{k}}, x_{\overline{k}})$. Conversely, from $x_{\overline{k}}$ down to $x_{\underline{k}}$ along the subpath, by a symmetric argument, we have $1 = \varphi_{x_{\overline{k}}}(x_{\overline{k}-1}, x_{\overline{k}}) = \cdots = \varphi_{x_{\overline{k}}}(x_{\underline{k}+1}, x_{\overline{k}}) = \varphi_{x_{\overline{k}}}(x_{\underline{k}}, x_{\overline{k}})$. Contradiction! Thus, $\{x_k\}_{k=1}^q \subseteq \bar{A}$.

The verification also implies that *every* adjacent$^+$ path connecting (a^s, x^{-s}) and (b^s, x^{-s}) in (A^s, x^{-s}) consists of all feasible alternatives. Since Lemma 4 remains valid when all alternatives on an adjacent$^+$ path in question are feasible, it is true that the adjacent$^+$ path $\{x_k\}_{k=1}^q$ above must be unique. $\square$

Thus, given $s \in M$ and $x^{-s} \in A^{-s}$ with $(A^s, x^{-s}) \cap \bar{A} \neq \emptyset$, we assert that all feasible alternatives of $(A^s, x^{-s}) \cap \bar{A}$ are located on a tree, denoted $G\left((A^s, x^{-s}) \cap \bar{A}\right)$, where two feasible alternatives form an edge if and only if they are adjacent$^+$.

Lemma **22** *Fix $s \in M$ and $x^{-s} \in A^{-s}$ with $(A^s, x^{-s}) \cap \bar{A} \neq \emptyset$. Given $a \in (A^s, x^{-s})$, there exists $\bar{a} \in (A^s, x^{-s}) \cap \bar{A}$ such that $\bar{a} = r_1\big(P_{i|(A^s,x^{-s})\cap\bar{A}}\big)$ for all $P_i \in \mathbb{D}^a$.*[45]

Proof: The lemma holds evidently if $(A^s, x^{-s}) \cap \bar{A}$ is a singleton set. Thus, we assume $\left|(A^s, x^{-s}) \cap \bar{A}\right| \geq 2$. Next, if $a \in (A^s, x^{-s}) \cap \bar{A}$, then it is evident that $\bar{a} = a$. Thus, we assume $a \in (A^s, x^{-s}) \backslash \bar{A}$. Pick an arbitrary $b \in (A^s, x^{-s}) \cap \bar{A}$. By Lemma 2, we have an adjacent$^+$ path $\{x_k\}_{k=1}^q \subseteq (A^s, x^{-s})$ connecting b and a. Since $x_1 \in \bar{A}$ and $x_q \notin \bar{A}$, there exists there exists $1 \leq \bar{k} < q$ such that $x_{\bar{k}} \in \bar{A}$ and $x_{\bar{k}+1}, \ldots, x_q \notin \bar{A}$. Moreover, by Lemma 21, we know that there exists a unique adjacent$^+$ in $(A^s, x^{-s}) \cap \bar{A}$ connecting x_1 and $x_{\bar{k}}$. Thus, we can assume w.l.o.g. that $x_1, x_2, \ldots, x_{\bar{k}} \in (A^s, x^{-s}) \cap \bar{A}$. We show $x_{\bar{k}} = r_1\big(P_{i|(A^s,x^{-s})\cap\bar{A}}\big)$ for all $P_i \in \mathbb{D}^a$.

Suppose not, i.e., there exists $P_i \in \mathbb{D}^a$ and $x \in (A^s, x^{-s}) \cap \bar{A}$ such that $x P_i x_{\bar{k}}$. Picking arbitrary $P_i' \in \mathbb{D}^x$, by the no-detour property, we have a simple$^+$ path $\{P_i^k\}_{k=1}^t$ connecting P_i and P_i' such that $r_1(P_i^k) \in (A^s, x^{-s})$ and $x P_i^k x_{\bar{k}}$ for all $k = 1, \ldots, t$. Thus, $x_{\bar{k}}$ is never the peak of any preference of the simple$^+$ path. Then, by sorting all preferences of $\{P_i^k\}_{k=1}^t$ according to the peaks of preferences and removing those repetitions of top alternatives, we induce an adjacent$^+$ path $\{y_k\}_{k=1}^p \subseteq (A^s, x^{-s})$ connecting a and x. It is evident that $x_{\bar{k}} \notin \{y_k\}_{k=1}^p$. Consequently, combining $\{x_{\bar{k}}, x_{\bar{k}+1}, \ldots, x_q \equiv a\}$ and $\{a \equiv y_1, \ldots, y_p \equiv x\}$ and removing repetitions, we can construct an adjacent$^+$ path $\{z_k\}_{k=1}^l \subseteq (A^s, x^{-s})$ connecting $x_{\bar{k}} \in \bar{A}$ and $x \in \bar{A}$. Last, since $x_{\bar{k}+1}, \ldots, x_q \notin \bar{A}$ and $x_{\bar{k}} \notin \{y_k\}_{k=1}^p$, the adjacent$^+$ path $\{z_k\}_{k=1}^l$ must include at least one invalid alternative, which contradicts Lemma 21. $\square$

Lemma **23** *Given $s \in M$ and $x^{-s} \in A^{-s}$, let the adjacent$^+$ path $\{x_k\}_{k=1}^q \subseteq (A^s, x^{-s})$ be such that $x_1, \ldots, x_{\bar{k}} \in \bar{A}$ and $x_{\bar{k}+1}, \ldots, x_q \notin \bar{A}$, where $1 \leq \bar{k} \leq q$. There exist $0 \leq \alpha_1 < \cdots < \alpha_{\bar{k}-1} \leq 1$ such that $\varphi(x_1, x_q) = \alpha_1 e_{x_1} + \sum_{k=2}^{\bar{k}-1}(\alpha_k - \alpha_{k-1}) e_{x_k} + (1-\alpha_{\bar{k}-1}) e_{x_{\bar{k}}}$. Moreover, for every $P_i \in \mathbb{D}^{x_1}$, $x_k P_i x_{k+1}$, $k = 1, \ldots, \bar{k}-1$.*

Proof: The verification is similar to Lemma 3. We omit the detailed proof. $\square$

To establish the next lemma which is similar to Lemma 5, we fix the following four alternatives: $a = (x^s, x^\tau, z^{-\{s,\tau\}})$, $b = (y^s, y^\tau, z^{-\{s,\tau\}})$, $c = (x^s, y^\tau, z^{-\{s,\tau\}})$ and $d = (y^s, x^\tau, z^{-\{s,\tau\}})$ where $x^s \neq y^s$ and $x^\tau \neq y^\tau$.

Lemma **24** *If $a, c, d \in \bar{A}$, $a \sim^+ c$ and $a \sim^+ d$, then $b \in \bar{A}$, $b \sim^+ c$ and $b \sim^+ d$.*

Proof: This lemma follows exactly from Lemma 5 if $b \in \bar{A}$. Therefore, in the rest of verification, we show $b \in \bar{A}$. Since $b, d \in (y^s, A^\tau, z^{-\{s,\tau\}})$ and $b, c \in (A^s, y^\tau, z^{-\{s,\tau\}})$, by Lemma 2, we

[45] The notation $P_{i|(A^s,x^{-s})\cap\bar{A}}$ is the induced preference over $(A^s, x^{-s}) \cap \bar{A}$ according to preference P_i.

have an adjacent$^+$ path $\{x_k\}_{k=1}^p \subseteq (y^s, A^\tau, z^{-\{s,\tau\}})$ connecting b and d, and an adjacent$^+$ path $\{y_k\}_{k=1}^q \subseteq (A^s, y^\tau, z^{-\{s,\tau\}})$ connecting b and c. Recall Figure 3.

Suppose $b \notin \bar{A}$. Since $b, c \in (A^s, y^\tau, z^{-\{s,\tau\}})$ and $c \in \bar{A}$, by Lemma 22 we have $\bar{b} \in (A^s, y^\tau, z^{-\{s,\tau\}})$ such that $\bar{b} = r_1\big(P_{i|(A^s,y^\tau,z^{-\{s,\tau\}})\cap\bar{A}}\big)$ for all $P_i \in \mathbb{D}^b$. Furthermore, by the proof of Lemma 22, it must be the case $\bar{b} = y_{\bar{k}}$ for some $1 < \bar{k} \leq q$, $y_1, \ldots, y_{\bar{k}-1} \notin \bar{A}$ and $y_{\bar{k}}, \ldots, y_q \in \bar{A}$. Then, Lemma 23 implies $\sum_{k=\bar{k}}^q \varphi_{y_k}(c, b) = 1$. Thus, since $b \notin \bar{A}$ and $d \notin \{y_{\bar{k}}, \ldots, y_q\}$, we have $\varphi_b(c, b) = 0$ and $\varphi_d(c, b) = 0$. Furthermore, since $a \sim^+ c$, we have $P_i \in \mathbb{D}^a$ and $P_i' \in \mathbb{D}^c$ with $P_i \sim^+ P_i'$. Note that d and b form a local switching pair in P_i and P_i'. Therefore, tops-onlyness and item 2(ii) of Lemma 1 imply $\varphi_d(a,b)+\varphi_b(a,b) = \varphi_d(P_i,b)+\varphi_b(P_i,b) = \varphi_d(P_i',b)+\varphi_b(P_i',b) = \varphi_d(c,b) + \varphi_b(c,b) = 0$. Thus, $\varphi_d(a,b) = 0$.

We will induce a contradiction by showing $\varphi_d(a,b) > 0$. Consider the adjacent$^+$ path $\{x_k\}_{k=1}^p$. First, since $a \sim^+ d$ and $d \sim^+ x_{p-1}$, we have $P_i \in \mathbb{D}^a$ and $P_j \in \mathbb{D}^{x_{p-1}}$ such that $r_2(P_i) = r_2(P_j) = d \in \bar{A}$. Then, tops-onlyness and the compromise property imply $\varphi_d(a, x_{p-1}) = \varphi_d(P_i, P_j) > 0$. Moving from x_{p-1} to $x_1 \equiv b$ along the adjacent$^+$ path $\{x_k\}_{k=1}^p$, by a repeated application of item 2(iii) of Lemma 1, we have $\varphi_d(a,b) > 0$. Contradiction! Therefore, $b \in \bar{A}$, as required. $\square$

We introduce a new notion. Given $c, d \in \bar{A}$, let $c^s \neq d^s$ for every $s \in S \subseteq M$ and $c^{-S} = d^{-S} \equiv z^{-S}$ where $|S| \geq 1$. We say that c and d **formulate a feasible box** if the following two conditions are satisfied.

(i) For each $s \in S$, there exists a sequence $\{x_k^s\}_{k=1}^{q(s)} \subseteq A^s$ where $q(s) \geq 2$, $x_1^s = c^s$ and $x_{q(s)}^s = d^s$ such that $B(c,d) \equiv \left(\times_{s\in S}\{x_k^s\}_{k=1}^{q(s)}, z^{-S}\right) \subseteq \bar{A}$.

(ii) For all $x, y \in B(c,d)$, we have

$$[x^s = x_k^s, y^s = x_{k+1}^s \text{ and } x^{-s} = y^{-s} \text{ for some } s \in S \text{ and } 1 \leq k < q(s)] \Rightarrow [x \sim^+ y].$$

Lemma **25** *Every pair of distinct feasible alternatives formulate a feasible box.*

Proof: Evidently, Lemma 21 implies that every two similar feasible alternatives always formulate a feasible box. Next, we provide an induction argument to complete the verification.

Induction hypothesis: Given an integer $2 < l \leq m$, for all $c, d \in \bar{A}$ which disagree on at least one component and at most $l-1$ components, i.e., $c^s \neq d^s$ for every $s \in S \subseteq M$ and $c^{-S} = d^{-S} \equiv z^{-S}$ where $1 \leq |S| < l$, we know that c and d formulate a feasible box.

Given $c, d \in \bar{A}$, let $c^s \neq d^s$ for every $s \in S \subseteq M$ and $c^{-S} = d^{-S} \equiv z^{-S}$ where $|S| = l$. We show that c and d formulate a feasible box. For notational convenience, let $S = \{1, 2, \ldots, l\}$.

Claim 1: If there exists $s \in S$ such that $a \equiv (c^1, \ldots, c^{s-1}, d^s, c^{s+1}, \ldots, c^l, z^{-S}) \in \bar{A}$, then c and d formulate a feasible box.

Assume w.l.o.g. that $s = 1$. Thus, $a \equiv (d^1, c^2, \ldots, c^l, z^{-S})$, a and d disagree on $l-1$ components, and induction hypothesis implies that a and d formulate a feasible box $B(a,d)$. Specifically,

(i) For each $\tau \in \{2, \ldots, l\}$, there exists a sequence $\{x_k^\tau\}_{k=1}^{q(s)} \subseteq A^\tau$ such that $q(s) \geq 2$, $x_1^\tau = c^\tau$ and $x_{q(\tau)}^\tau = d^\tau$ such that $B(a,d) \equiv \left(d^1, \{x_k^2\}_{k=1}^{q(2)}, \ldots, \{x_k^l\}_{k=1}^{q(l)}, z^{-S}\right) \subseteq \bar{A}$.

(ii) For all $x, y \in B(a,d)$, we have

$$\big[x^s = x^s_k, y^s = x^s_{k+1} \text{ and } x^{-s} = y^{-s} \text{ for some } s \in \{2,\ldots,l\} \text{ and } 1 \leq k < q(s)\big] \Rightarrow [x \sim^+ y].$$

Next, since $a, c \in (A^1, c^2, \ldots, c^l, z^{-S}) \cap \bar{A}$, by Lemma 21, we have a unique adjacent$^+$ path $\{x_k\}_{k=1}^q \equiv \big\{(x^1_k, c^2, \ldots, c^l, z^{-S})\big\}_{k=1}^q \subseteq \bar{A}$ connecting a and c.

Pick an arbitrary adjacent$^+$ path $\{y_k\}_{k=1}^p \subseteq B(a,d)$ connecting a and d. Note that $(d^1, c^2, \ldots, c^l, z^{-S}) \equiv a = x_1 \equiv (x^s_1)_{s\in M}$. Thus, we can rewrite $B(a,d) \equiv \Big(x^1_1, \{x^2_k\}_{k=1}^{q(2)}, \ldots, \{x^l_k\}_{k=1}^{q(l)}, z^{-S}\Big)$. Note that all alternatives of $\{y_k\}_{k=1}^p$ have d^1 in component 1, and $d^1 \equiv x^1_1 \neq x^1_2$. We replace d^1 by x^1_2 in every alternative of $\{y_k\}_{k=1}^p$, and hence construct another sequence $\{z_k\}_{k=1}^p \equiv \big\{(x^1_2, y^{-1}_k)\big\}_{k=1}^p$. We will show that $\{z_k\}_{k=1}^p \subseteq \bar{A}$ and $\{z_k\}_{k=1}^p$ is an adjacent$^+$ path. Since $(d^1, c^2, \ldots, c^l, z^{-S}) \equiv a = y_1 \equiv (y^1_1, y^{-1}_1)$, we know $z_1 \equiv (x^1_2, y^{-1}_1) = (x^1_2, c^2, \ldots, c^l, z^{-S}) \equiv x_2 \in \bar{A}$. Note that $z_p \equiv (x^1_2, y^{-1}_p) \equiv (x^1_2, d^2, \ldots, d^l, z^{-S})$. Given $y_1 = x_1 \sim^+ x_2 = z_1$, along the adjacent$^+$ path $\{y_k\}_{k=1}^p$, by a repeated application of Lemma 24, we have $\{z_k\}_{k=1}^p \subseteq \bar{A}$, $z_k \sim^+ z_{k+1}$, $k = 1, \ldots, p-1$, and $z_k \sim^+ y_k$, $k = 1, \ldots, p$. Since we choose the adjacent$^+$ path $\{y_k\}_{k=1}^p$ arbitrarily, it is true that $z_1 \equiv (x^1_2, c^2, \ldots, c^m, z^{-S})$ and $z_p \equiv (x^1_2, d^2, \ldots, d^m, z^{-S})$ formulate a feasible box $B(z_1, z_p) = \Big(x^1_2, \{x^2_k\}_{k=1}^{q(2)}, \ldots, \{x^l_k\}_{k=1}^{q(l)}, z^{-S}\Big)$, and moreover, for all $y \in B(a,d) \equiv B(y_1, y_p)$ and $z \in B(z_1, z_p)$, $[y^{-1} = z^{-1}] \Rightarrow [y \sim^+ z]$.

Along the adjacent$^+$ path $\{x_k\}_{k=1}^q$, moving from x_2 to $x_q \equiv c$, by repeating argument above, we know that the following two statements hold:

(i) Given $k = 1, \ldots, q$, $(x^1_k, c^2, \ldots, c^l, z^{-S})$ and $(x^1_k, d^2, \ldots, d^l, z^{-S})$ formulate a feasible box $B\big((x^1_k, c^2, \ldots, c^l, z^{-S}), (x^1_k, d^2, \ldots, d^l, z^{-S})\big)$, and

(ii) Given $k = 1, \ldots, q-1$, $y \in B\big((x^1_k, c^2, \ldots, c^l, z^{-S}), (x^1_k, d^2, \ldots, d^l, z^{-S})\big)$ and $z \in B\big((x^1_{k+1}, c^2, \ldots, c^l, z^{-S}), (x^1_{k+1}, d^2, \ldots, d^l, z^{-S})\big)$, we have $[y^{-1} = z^{-1}] \Rightarrow [y \sim^+ z]$.

Consequently, c and d formulate a feasible box $B(c,d) = \Big(\{x^1_k\}_{k=1}^q, \{x^2_k\}_{k=1}^{q(2)}, \ldots, \{x^l_k\}_{k=1}^{q(l)}, z^{-S}\Big)$. This completes the verification of Claim 1.

Claim 2: If there exists $s \in S$ such that $a \equiv (d^1, \ldots, d^{s-1}, c^s, d^{s+1}, \ldots, d^l, z^{-S}) \in \bar{A}$, then c and d formulate a feasible box.

The verification of this claim is symmetric to the verification of Claim 1.

Claim 3: There exists $s \in S$ such that either $(d^1, \ldots, d^{s-1}, c^s, d^{s+1}, \ldots, d^l, z^{-S}) \in \bar{A}$ or $(c^1, \ldots, c^{s-1}, d^s, c^{s+1}, \ldots, c^l, z^{-S}) \in \bar{A}$.

Suppose that it is not true. Thus, we know $a \equiv (c^1, d^2, \ldots, d^l, z^{-S}) \notin \bar{A}$ and $b \equiv (d^1, c^2, \ldots, c^l, z^{-S}) \notin \bar{A}$. Note that $b, c \in (A^1, c^2, \ldots, c^l, z^{-S})$ and $c \in \bar{A}$. By Lemma 2, let $\{x_k\}_{k=1}^q \subseteq (A^1, c^2, \ldots, c^l, z^{-S})$ be an adjacent$^+$ path connecting b and c. By Lemma 22 and its proof, let $\bar{b} = x_{\bar{k}}$ where $1 < \bar{k} \leq q$ be such that $x_1, \ldots, x_{\bar{k}-1} \notin \bar{A}$, $x_{\bar{k}}, \ldots, x_q \in \bar{A}$ and $\bar{b} = r_1\big(P_{i|(A^1,c^2,\ldots,c^l,z^{-S})\cap\bar{A}}\big)$ for all $P_i \in \mathbb{D}^b$. Thus, Lemma 23 implies $\sum_{k=\bar{k}}^q \varphi_{x_k}(c,b) = 1$.

Since c and a disagree on components $2, \ldots, m$ and agree on the rest of components, we can induce the following alternatives $y_{q(s)} \equiv (c^1, d^2, \ldots, d^s, c^{s+1}, \ldots, c^l, z^{-S})$, $s = 1, \ldots, m$. Evidently, $y_{q(1)} = c$ and $y_{q(l)} = a$. Moreover, for each $1 \leq s < l$, by Lemma 2, we have an adjacent$^+$ path in $(c^1, d^2, \ldots, d^s, A^{s+1}, c^{s+2}, \ldots, c^l, z^{-S})$ which connects $y_{q(s)}$ and $y_{q(s+1)}$.

Combining all these $l-1$ adjacent$^+$ paths, we have an adjacent$^+$ path to connect c and a: $\{y_k\}_{k=1}^p \equiv \{y_{q(1)},\dots,y_{q(2)},\dots,y_{q(s)},\dots,y_{q(s+1)},\dots,y_{q(l)}\}$.

Initially, since $\sum_{k=\bar{k}}^q \varphi_{x_k}(y_1,b) \equiv \sum_{k=\bar{k}}^q \varphi_{x_k}(c,b) = 1$ and $x_k^1 \neq d^1$ for all $k=\bar{k},\dots,q$, it is true that $\varphi_{(d^1,y^{-1})}(y_1,b)=0$ for all $y^{-1}\in A^{-1}$.

Next, we provide an additional induction argument.

The Secondary Induction Hypothesis: Given $1<k\le p$, for all $1\le k'<k$, we have $\varphi_{(d^1,y^{-1})}(y_{k'},b)=0$ for all $y^{-1}\in A^{-1}$.

We show $\varphi_{(d^1,y^{-1})}(y_k,b)=0$ for all $y^{-1}\in A^{-1}$. Since $y_{k-1}\sim^+ y_k$, we know $y_{k-1}^s\neq y_k^s$ and $y_{k-1}^{-s}=y_k^{-s}$ for some $s\in\{2,\dots,l\}$. Given $y^{-1}\in A^{-1}$ and $s\in\{2,\dots,l\}$, we know either $y^s\in\{y_{k-1}^s,y_k^s\}$ or $y^s\notin\{y_{k-1}^s,y_k^s\}$. If $y^s\in\{y_{k-1}^s,y_k^s\}$, let $\underline{y}^s\in\{y_{k-1}^s,y_k^s\}\backslash\{y^s\}$, and we construct $\underline{y}^{-1}\equiv(\underline{y}^s,y^{-\{1,s\}})$. By item 2(ii) of Lemma 1 and the induction hypothesis, we have $\varphi_{(d^1,y^{-1})}(y_k,b)+\varphi_{(d^1,\underline{y}^{-1})}(y_k,b)=\varphi_{(d^1,y^{-1})}(y_{k-1},b)+\varphi_{(d^1,\underline{y}^{-1})}(y_{k-1},b)=0$. Thus, $\varphi_{(d^1,y^{-1})}(y_k,b)=0$. If $y^s\notin\{y_{k-1}^s,y_k^s\}$, then item 2(iii) of Lemma 1 and the induction hypothesis imply $\varphi_{(d^1,y^{-1})}(y_k,b)=\varphi_{(d^1,y^{-1})}(y_{k-1},b)=0$. This completes the verification of the secondary induction hypothesis. Therefore,$\varphi_{(d^1,y^{-1})}(a,b)=0$ for all $y^{-1}\in A^{-1}$. Thus, $\sum_{y^{-1}\in A^{-1}}\varphi_{(d^1,y^{-1})}(a,b)=0$.

We are going to derive a contradiction by showing $\sum_{y^{-1}\in A^{-1}}\varphi_{(d^1,y^{-1})}(a,b)>0$. Since $a,d\in(A^1,d^2,\dots,d^l,z^{-S})$, we have an adjacent$^+$ path $\{\bar{x}_k\}_{k=1}^{\bar{q}}\subseteq(A^1,d^2,\dots,d^l,z^{-S})$ connecting d and a. Similar to the adjacent$^+$ path $\{y_k\}_{k=1}^p$ which connects c and a, we can also construct an adjacent$^+$ path $\{z_k\}_{k=1}^{\bar{p}}\subseteq(d^1,A^2,\dots,A^l,z^{-S})$ connecting d and b.

Start from profile $(\bar{x}_2,z_2)$. Since $\bar{x}_2\sim^+\bar{x}_1\equiv d$ and $z_2\sim^+ z_1\equiv d$, we have $P_i\in\mathbb{D}^{\bar{x}_2}$ and $P_j\in\mathbb{D}^{z_2}$ with $r_2(P_i)=r_2(P_j)=d\in\bar{A}$. Then, tops-onlyness and the compromise property imply $\varphi_d(\bar{x}_2,z_2)=\varphi_d(P_i,P_j)>0$. Moving from $\bar{x}_2$ to $\bar{x}_{\bar{q}}\equiv a$ along the adjacent$^+$ path $\{\bar{x}_k\}_{k=1}^{\bar{q}}$, by a repeated application of item 2(iii) of Lemma 1, we have $\varphi_d(a,z_2)\equiv\varphi_d(\bar{x}_{\bar{q}},z_2)=\cdots=\varphi_d(\bar{x}_2,z_2)>0$. Therefore, $\sum_{y^{-1}\in A^{-1}}\varphi_{(d^1,y^{-1})}(a,z_2)>0$.

Move from z_2 to $z_{\bar{p}}\equiv b$ along the adjacent$^+$ path $\{z_k\}_{k=1}^{\bar{p}}$. Given $3\le k\le\bar{p}$, we know $z_k^s\neq z_{k-1}^s$ and $z_k^{-s}=z_{k-1}^{-s}$ for some $s\in\{2,\dots,l\}$. Thus, by items 2(ii) and 2(iii) of Lemma 1, we have

$$\begin{aligned}
& \sum_{y^{-1}\in A^{-1}}\varphi_{(d^1,y^{-1})}(a,z_k)\\
\equiv\ & \sum_{y^{-\{1,s\}}\in A^{-\{1,s\}}}\Big[\sum_{y^s\in\{z_k^s,z_{k-1}^s\}}\varphi_{(d^1,y^s,y^{-\{1,s\}})}(a,z_k)\Big]+\sum_{y^{-1}\in A^{-1}:y^s\notin\{z_k^s,z_{k-1}^s\}}\varphi_{(d^1,y^{-1})}(a,z_k)\\
=\ & \sum_{y^{-\{1,s\}}\in A^{-\{1,s\}}}\Big[\sum_{y^s\in\{z_k^s,z_{k-1}^s\}}\varphi_{(d^1,y^s,y^{-\{1,s\}})}(a,z_{k-1})\Big]+\sum_{y^{-1}\in A^{-1}:y^s\notin\{z_k^s,z_{k-1}^s\}}\varphi_{(d^1,y^{-1})}(a,z_{k-1})\\
\equiv\ & \sum_{y^{-1}\in A^{-1}}\varphi_{(d^1,y^{-1})}(a,z_{k-1}).
\end{aligned}$$

Consequently, we eventually have $\sum_{y^{-1}\in A^{-1}}\varphi_{(d^1,y^{-1})}(a,b)\equiv\sum_{y^{-1}\in A^{-1}}\varphi_{(d^1,y^{-1})}(a,z_{\bar{p}})=\cdots=\sum_{y^{-1}\in A^{-1}}\varphi_{(d^1,y^{-1})}(a,z_2)>0$. Contradiction!

Therefore, either $a\in\bar{A}$ or $b\in\bar{A}$. Then, Claim 1 or 2 implies that c and d formulate a feasible box. This completes the verification of the induction hypothesis and Claim 3, and hence, proves the lemma. □

Now, given an arbitrary $a\in\bar{A}$, by Lemma 25, we assert that for every $s\in M$ and $x^{-s}\in A^{-s}$ with $(A^s,x^{-s})\cap\bar{A}\neq\emptyset$, $(a^s,x^{-s})\in\bar{A}$. Next, we claim that for every $s\in A$ and $x^{-s}\in A^{-s}$ with

$(A^s, x^{-s}) \cap \bar{A} \neq \emptyset$, $\left|(A^s, x^{-s}) \cap \bar{A}\right| \geq 2$. Suppose not, i.e., there exist $s \in A$ and $x^{-s} \in A^{-s}$ such that $\left|(A^s, x^{-s}) \cap \bar{A}\right| = 1$. Then, for all $x^{-s} \in A^{-s}$ with $(A^s, x^{-s}) \cap \bar{A} \neq \emptyset$, $\left|(A^s, x^{-s}) \cap \bar{A}\right| = 1$. Consequently, all feasible alternatives agree on component s which contradicts Assumption 1. Thus, we have that for each $s \in M$, there exists $x^{-s} \in A^{-s}$ such that $(A^s, x^{-s}) \cap \bar{A} \neq \emptyset$, and moreover, $\left|(A^s, x^{-s}) \cap \bar{A}\right| \geq 2$, as claimed above.

Given $s \in M$, we pick $x^{-s} \in A^{-s}$ with $(A^s, x^{-s}) \cap \bar{A} \neq \emptyset$. By Lemma 21, we induce a tree $G_{\sim^+}\left((A^s, x^{-s}) \cap \bar{A}\right)$. We next claim that there exists another $y^{-s} \in A^{-s}$ such that $(A^s, y^{-s}) \cap \bar{A} \neq \emptyset$. Otherwise, for all $y^{-s} \in A^{-s} \backslash \{x^{-s}\}$, $(A^s, y^{-s}) \cap \bar{A} = \emptyset$ implies that all feasible alternatives agree on every component other than s which contradicts Assumption 1. Thus, we pick another $y^{-s} \in A^{-s}$ with $(A^s, y^{-s}) \cap \bar{A} \neq \emptyset$, and induce a tree $G_{\sim^+}\left((A^s, y^{-s}) \cap \bar{A}\right)$. By Lemma 25, it must be the case that $G_{\sim^+}\left((A^s, x^{-s}) \cap \bar{A}\right)$ and $G_{\sim^+}\left((A^s, y^{-s}) \cap \bar{A}\right)$ coincide in following sense: For all $a^s, b^s \in A^s$,

$$\left[(a^s, x^{-s}), (b^s, x^{-s}) \in \bar{A} \text{ and } (a^s, x^{-s}) \sim^+ (b^s, x^{-s})\right]$$
$$\Leftrightarrow \left[(a^s, y^{-s}), (b^s, y^{-s}) \in \bar{A} \text{ and } (a^s, y^{-s}) \sim^+ (b^s, y^{-s})\right].$$

Therefore, $\bar{A}$ must be factorizable, i.e., $\bar{A} = \times_{s \in M} \bar{A}^s$ where $\bar{A}^s \subseteq A^s$, $\left|\bar{A}^s\right| \geq 2$ for every $s \in M$, and $\bar{A}^s$ is located on a tree $G(\bar{A}^s)$ where $a^s, b^s \in A^s$ form an edge if and only if $(a^s, x^{-s}), (b^s, x^{-s}) \in \bar{A}$ and $(a^s, x^{-s}) \sim^+ (b^s, x^{-s})$ for some $x^{-s} \in A^{-s}$. Thus, we have a product of trees $\times_{s \in M} G(\bar{A}^s)$, and the first condition of Definition 7 is satisfied.

Applying the same verifications of Lemmas 7 and 8, we know that for all $P_i \in \mathbb{D}$, if $r_1(P_i) \in \bar{A}$, then $P_{i|\bar{A}}$ is multidimensional single-peaked on $\times_{s \in M} G(\bar{A}^s)$. Therefore, in the rest of the proof, we focus on the preferences whose peaks are invalid alternatives.

Lemma 26 *Given $a \in A \backslash \bar{A}$ and $P_i \in \mathbb{D}^a$, if $P_{i|\bar{A}}$ is multidimensional single-peaked on $\times_{s \in M} G(\bar{A}^s)$, then every preference of $\mathbb{D}^a$ is multidimensional single-peaked on $\times_{s \in M} G(\bar{A}^s)$.*

Proof: Let $\bar{a} \in \bar{A}$ be such that $r_1(P_{i|\bar{A}}) = \bar{a}$. Evidently, Lemma 20 implies $r_1(\bar{P}_{i|\bar{A}}) = \bar{a}$ for all $\bar{P}_i \in \mathbb{D}^a$. Since $\bar{a} \in \bar{A}$, factorizability implies $\bar{a}^s \in \bar{A}^s$ for all $s \in M$.

Claim 1: Given $s \in M$ and $x^s \in \bar{A}^s$, $\varphi\left(\bar{P}_i, (x^s, \bar{a}^{-s})\right) = \varphi\left(\bar{a}, (x^s, \bar{a}^{-s})\right)$ for all $\bar{P}_i \in \mathbb{D}^a$.

Let $\langle \bar{a}^s, x^s \rangle = \{x^s_k\}_{k=1}^t$ be the adjacent$^+$ path connecting $\bar{a}^s$ and x^s in $G(\bar{A}^s)$. First, unanimity implies $\varphi_{\bar{a}}(P_i, \bar{a}) = 1$. Thus, $\sum_{k=1}^1 \varphi_{(x^s_k, \bar{a}^{-s})}\left(P_i, (x^s_1, \bar{a}^{-s})\right) = 1$. Next, we provide an induction hypothesis: Given $1 < k \leq t$, for all $1 \leq k' < k$, $\sum_{\nu=1}^{k'} \varphi_{(x^s_\nu, \bar{a}^{-s})}\left(P_i, (x^s_{k'}, \bar{a}^{-s})\right) = 1$.

We show $\sum_{\nu=1}^{k} \varphi_{(x^s_\nu, \bar{a}^{-s})}\left(P_i, (x^s_{k}, \bar{a}^{-s})\right) = 1$. Since $(x^s_k, \bar{a}^{-s}) \sim^+ (x^s_{k-1}, \bar{a}^{-s})$, we have

$$\begin{aligned}
& \sum_{\nu=1}^{k} \varphi_{(x^s_\nu, \bar{a}^{-s})}\left(P_i, (x^s_k, \bar{a}^{-s})\right) \\
\equiv\ & \sum_{\nu=k-1}^{k} \varphi_{(x^s_\nu, \bar{a}^{-s})}\left(P_i, (x^s_k, \bar{a}^{-s})\right) + \sum_{\nu=1}^{k-2} \varphi_{(x^s_\nu, \bar{a}^{-s})}\left(P_i, (x^s_k, \bar{a}^{-s})\right) \\
=\ & \sum_{\nu=k-1}^{k} \varphi_{(x^s_\nu, \bar{a}^{-s})}\left(P_i, (x^s_{k-1}, \bar{a}^{-s})\right) + \sum_{\nu=1}^{k-2} \varphi_{(x^s_\nu, \bar{a}^{-s})}\left(P_i, (x^s_{k-1}, \bar{a}^{-s})\right) \quad \text{by items 2(ii) and 2(iii) of Lemma 1} \\
\equiv\ & \sum_{\nu=1}^{k-1} \varphi_{(x^s_\nu, \bar{a}^{-s})}\left(P_i, (x^s_{k-1}, \bar{a}^{-s})\right) = 1, \quad \text{by the induction hypothesis.}
\end{aligned}$$

This completes the verification of the induction hypothesis. Thus, $\sum_{z \in \langle \bar{a}, (x^s, \bar{a}^{-s}) \rangle} \varphi_z\left(P_i, (x^s, \bar{a}^{-s})\right) = 1$. Next, pick arbitrary $P'_i \in \mathbb{D}^{\bar{a}}$ by minimal richness. Since $\bar{a} \in \bar{A}$, we know that P'_i is multidimensional single-peaked w.r.t. $\bar{A}$. Consequently, the induced preferences $P_{i|\langle \bar{a}, (x^s, \bar{a}^{-s}) \rangle}$ and

$P'_{i|\langle \bar{a},(x^s,\bar{a}^{-s})\rangle}$ are single-peaked on the line $\langle \bar{a},(x^s,\bar{a}^{-s})\rangle$, and hence, $P_{i|\langle \bar{a},(x^s,\bar{a}^{-s})\rangle}$ and $P'_{i|\langle \bar{a},(x^s,\bar{a}^{-s})\rangle}$ are identical. Furthermore, since $\sum_{z\in\langle \bar{a},(x^s,\bar{a}^{-s})\rangle} \varphi_z\big(P_i,(x_t^s,\bar{a}^{-s})\big) = 1$ and $\sum_{z\in\langle \bar{a},(x^s,\bar{a}^{-s})\rangle} \varphi_z\big(P'_i,(x_t^s,\bar{a}^{-s})\big) = 1$, we have that for all $\bar{P}_i \in \mathbb{D}^a$, $\varphi\big(\bar{P}_i,(x_t^s,\bar{a}^{-s})\big) = \varphi\big(P_i,(x_t^s,\bar{a}^{-s})\big) = \varphi\big(P'_i,(x_t^s,\bar{a}^{-s})\big) = \varphi\big(\bar{a},(x_t^s,\bar{a}^{-s})\big)$ where the first and third equalities are implied by the tops-only property while the second equality is implied by strategy-proofness. This completes the verification of the claim.

Claim 2: For every separable preference $P'_i \in \mathbb{D}^a$, the induced preference $P'_{i|\bar{A}}$ is multidimensional single-peaked on $\times_{s\in M} G(\bar{A}^s)$.[46]

Given a separable preference $P'_i \in \mathbb{D}^a$, to show that $P'_{i|\bar{A}}$ is multidimensional single-peaked on $\times_{s\in M} G(\bar{A}^s)$, it suffices to show that $[P'_i]^s_{|\bar{A}}$ is single-peaked on $G(\bar{A}^s)$ for every $s \in M$.

Given $s \in M$, since $\varphi\left(P'_i,(x^s,\bar{a}^{-s})\right) = \varphi\left(\bar{a},(x^s,\bar{a}^{-s})\right)$ for all $x^s \in \bar{A}^s$ by Claim 1, the proof of Lemma 7 implies that $[P'_i]^s_{|\bar{A}}$ is single-peaked on $G(\bar{A}^s)$. Then, by separability, $P'_{i|\bar{A}}$ is multidimensional single-peaked on $\times_{s\in M} G(\bar{A}^s)$. This completes the verification of the claim.

Claim 3: For every $\bar{P}_i \in \mathbb{D}^a$, the induced preference $\bar{P}_{i|\bar{A}}$ is multidimensional single-peaked on $\times_{s\in M} G(\bar{A}^s)$.

Given $\bar{P}_i \in \mathbb{D}^a$, suppose that $\bar{P}_{i|\bar{A}}$ is not multidimensional single-peaked on $\times_{s\in M} G(\bar{A}^s)$. Thus, there exist distinct $x, y \in \bar{A}$ such that $x \in \langle \bar{a}, y\rangle$ and $y\bar{P}_i x$. Pick arbitrary $P'_i \in \mathbb{D}^y$. Evidently, $a \neq y$. Since $y\bar{P}_i x$ and $yP'_i x$, by the Exterior$^+$ property, we have a simple$^+$ path $\{P_i^k\}_{k=1}^q$ connecting $\bar{P}_i$ and P'_i such that $yP_i^k x$ for all $k = 1, \ldots, q$. Since $a \neq y$, there exists $1 \leq k^* < q$ such that $r_1(P_i^{k^*}) = a$ and $r_1(P_i^{k^*+1}) \neq a$. Thus, $P_i^{k^*} \sim^+ P_i^{k^*+1}$ and $P_i^{k^*}$ is a separable preference. However, since $P^{k^*}_{i|\bar{A}}$ is multidimensional single-peaked on $\times_{s\in M} G(\bar{A}^s)$ by Claim 2, $xP_i^{k^*} y$. Contradiction! This completes the verification of the claim and the lemma. $\square$

Lemma **27** *Given $P_i, P'_i \in \mathbb{D}$ with $P_i \sim^+ P'_i$, if $P_{i|\bar{A}}$ is multidimensional single-peaked on $\times_{s\in M} G(\bar{A}^s)$, then $P'_{i|\bar{A}}$ is multidimensional single-peaked on $\times_{s\in M} G(\bar{A}^s)$.*

Proof: If $r_1(P_i) = r_1(P'_i)$, then Lemma 26 implies that $P'_{i|\bar{A}}$ is multidimensional single-peaked on $\times_{s\in M} G(\bar{A}^s)$. If $r_1(P'_i) \in \bar{A}$, it is also evident that $P'_{i|\bar{A}}$ is multidimensional single-peaked on $\times_{s\in M} G(\bar{A}^s)$. Hence, we assume $r_1(P_i) \equiv a \neq b \equiv r_1(P'_i)$ and $b \notin \bar{A}$. Since $P_i \sim^+ P'_i$, we know $a^s \neq b^s$ and $a^{-s} = b^{-s} \equiv z^{-s}$ for some $s \in M$, and both P_i and P'_i are separable preferences. Let $\bar{a} \equiv r_1(P_{i|\bar{A}})$ and $\bar{b} = r_1(P'_{i|\bar{A}})$.

Claim 1: If $\bar{a} \neq \bar{b}$, then $P'_{i|\bar{A}}$ is multidimensional single-peaked on $\times_{s\in M} G(\bar{A}^s)$.

Since $\bar{a}P_i\bar{b}$, $\bar{b}P'_i\bar{a}$ and $P_i \sim^+ P'_i$, it must be the case that $\bar{a}P_i!\bar{b}$ and $\bar{b}P'_i!\bar{a}$, and hence $\bar{a}^s = a^s$, $\bar{b}^s = b^s$ and $\bar{a}^{-s} = \bar{b}^{-s} \equiv \bar{z}^{-s}$. Since $\bar{a}, \bar{b} \in \bar{A}$, $a^s = \bar{a}^s \in \bar{A}^s$, $b^s = \bar{b}^s \in \bar{A}^s$ and $\bar{z}^{-s} \in \bar{A}^{-s}$.

Next, we claim $\langle a^s, b^s\rangle = \{a^s, b^s\}$. Suppose not, i.e., there exists $c^s \in \langle a^s, b^s\rangle\backslash\{a^s, b^s\}$. Thus, $c^s \in \bar{A}^s$, $\bar{c} \equiv (c^s, \bar{z}^{-s}) \in \bar{A}$ and $(c^s, \bar{z}^{-s}) \in \langle (a^s, \bar{z}^{-s}), (b^s, \bar{z}^{-s})\rangle \equiv \langle \bar{a}, \bar{b}\rangle$. Furthermore, since P_i is multidimensional single-peaked w.r.t. $\bar{A}$, we know $\bar{a}P_i\bar{c}$ and $\bar{c}P_i\bar{b}$. Contradiction! Therefore, $\langle a^s, b^s\rangle = \{a^s, b^s\}$.

Suppose that $P'_{i|\bar{A}}$ is multidimensional single-peaked on $\times_{s\in M} G(\bar{A}^s)$, i.e., there exist distinct $x, y \in \bar{A}$ such that $x \in \langle \bar{b}, y\rangle$ and $yP'_i x$. Since $x \in \langle \bar{b}, y\rangle$, $x^s \in \langle \bar{b}^s, y^s\rangle \equiv \langle b^s, y^s\rangle$ and $x^{-s} \in$

[46] Since every alternative is adjacent+ to some other alternative which is implied by the Exterior$^+$ property, it is true that for each alternative, there exists a separable preference whose peak is this alternative.

$\langle \bar{b}^{-s}, y^{-s}\rangle = \langle \bar{z}^{-s}, y^{-s}\rangle$. Since $\langle a^s, b^s\rangle = \{a^s, b^s\}$, $x \in \langle b^s, y^s\rangle$ implies $x^s \in \langle a^s, y^s\rangle$. Thus, $x = (x^s, x^{-s}) \in \langle (a^s, \bar{z}^{-s}), (y^s, y^{-s})\rangle = \langle \bar{a}, y\rangle$, and xP_iy by multidimensional single-peakedness w.r.t. $\bar{A}$, Thus, x and y form a local switching pair of P_i and P'_i, and hence $x^s = a^s$, $y^s = b^s$ and $x^{-s} = y^{-s}$. Consequently, $x^s \in \langle b^s, y^s\rangle = \langle b^s, b^s\rangle = \{b^s\}$ which contradicts $x^s = a^s$. This completes the verification of the claim.

Claim 2: If $\bar{a} = \bar{b}$, then $P'_{i|\bar{A}}$ is multidimensional single-peaked on $\times_{s\in M}G(\bar{A}^s)$.

Note that if either $a^s \notin \bar{A}^s$ or $b^s \notin \bar{A}^s$, then $P_i \sim^+ P'_i$ implies $P_{i|\bar{A}} = P'_{i|\bar{A}}$. Then, $P'_{i|\bar{A}}$ is multidimensional single-peaked on $\times_{s\in M}G(\bar{A}^s)$. Next, we show either $a^s \notin \bar{A}^s$ or $b^s \notin \bar{A}^s$. Suppose not, i.e., $a^s, b^s \in \bar{A}^s$. Since $\bar{a} = \bar{b} \in \bar{A}$, it is evident that $\bar{z}^{-s} \equiv \bar{a}^{-s} = \bar{b}^{-s} \in \bar{A}^{-s}$. Thus, $(a^s, \bar{z}^{-s}), (b^s, \bar{z}^{-s}) \in \bar{A}$. Recall that both P_i and P'_i are separable preferences. Since $r_1(P_i) = a \equiv (a^s, z^{-s})$ and $r_1(P'_i) = b \equiv (b^s, z^{-s})$, separability implies that either $(a^s, \bar{z}^{-s})P_i(\bar{a}^s, \bar{z}^{-s}) \equiv \bar{a}$ or $a^s = \bar{a}^s$, and $(b^s, \bar{z}^{-s})P'_i(\bar{b}^s, \bar{z}^{-s}) \equiv \bar{b}$ or $b^s = \bar{b}^s$. Furthermore, since $\bar{a} = \bar{b}$ and $a^s \neq b^s$, it must be either $(a^s, \bar{z}^{-s})P_i\bar{a}$ or $(b^s, \bar{z}^{-s})P'_i\bar{b}$ which contradicts $\bar{a} = r_1(P_{i|\bar{A}})$ and $\bar{b} = r_1(P'_{i|\bar{A}})$. This completes the verification of the claim and proves the lemma. □

Now, we show that for every preference whose peak is an invalid alternative, the induced preference over $\bar{A}$ is multidimensional single-peaked on $\times_{s\in M}G(\bar{A}^s)$. Given an arbitrary $P_i \in \mathbb{D}$, let $r_1(P_i) = a \notin \bar{A}$ and $r_1(P_{i|\bar{a}}) = \bar{a}$. Pick an arbitrary $P'_i \in \mathbb{D}^{\bar{a}}$ by minimal richness. Since $\bar{a} \in \bar{A}$, $P'_{i|\bar{A}}$ is multidimensional single-peaked on $\times_{s\in M}G(\bar{A}^s)$. We have a simple$^+$ path $\{P_i^k\}_{k=1}^q$ connecting P'_i and P_i. We first consider preference P_i^2. If $P_i^2 \sim P_i^1$, then $r_1(P_i^2) = r_1(P_i^1) = \bar{a}$ and hence $P^2_{i|\bar{A}}$ is is multidimensional single-peaked on $\times_{s\in M}G(\bar{A}^s)$. If $P_i^2 \sim^+ P_i^1$, Lemma 27 implies that $P^2_{i|\bar{A}}$ is is multidimensional single-peaked on $\times_{s\in M}G(\bar{A}^s)$. Therefore, we assert that $P^2_{i|\bar{A}}$ is is multidimensional single-peaked on $\times_{s\in M}G(\bar{A}^s)$. Next, we provide an induction hypothesis: Given $2 < k \leq q$, for all $2 \leq k' < k$, $P^{k'}_{i|\bar{A}}$ is multidimensional single-peaked on $\times_{s\in M}G(\bar{A}^s)$. We show that $P^k_{i|\bar{A}}$ is multidimensional single-peaked on $\times_{s\in M}G(\bar{A}^s)$. If $P_i^k \sim P_i^{k-1}$, then it is true that $r_1(P_i^k) = r_1(P_i^{k-1})$. Then, Lemma 26 implies that $P^k_{i|\bar{A}}$ is multidimensional single-peaked on $\times_{s\in M}G(\bar{A}^s)$. If $P_i^k \sim^+ P_i^{k-1}$, Lemma 27 implies that $P^k_{i|\bar{A}}$ is multidimensional single-peaked on $\times_{s\in M}G(\bar{A}^s)$. This completes the verification of the induction hypothesis. Therefore, P_i is multidimensional single-peaked on $\times_{s\in M}G(\bar{A}^s)$. This completes the verification of the necessity part of the theorem.

We now turn to the sufficiency part of Theorem 3. Let $\bar{A} = \times_{s\in M}\bar{A}^s$ where $\bar{A}^s \subseteq A^s$ and $|\bar{A}^s| \geq 2$ for each $s \in M$. Let $G(\bar{A}^s)$ be a tree for each $s \in M$. Thus, $\bar{A}$ is located on a product of trees $\times_{s\in M}G(\bar{A}^s)$. Let $\mathbb{D}$ be multidimensional single-peaked w.r.t. $\bar{A}$. Let $\mathbb{D}_{|\bar{A}} = \{P_{i|\bar{A}} | P_i \in \mathbb{D}\}$. Thus, the induced domain $\mathbb{D}|_{\bar{A}}$ is multidimensional single-peaked on $\times_{s\in M}G(\bar{A}^s)$. According the proof of the sufficiency part of Theorem 1, we can construct a mixed multidimensional projection rule $\phi : [\mathbb{D}_{|\bar{A}}]^N \to \Delta(\bar{A})$ which is unanimous, tops-only and strategy-proof and satisfies the compromise property. Next, we define a new function $\varphi : \mathbb{D}^N \to \Delta(\bar{A})$: For all $(P_1, \ldots, P_N) \in \mathbb{D}^N$, $\varphi(P_1, \ldots, P_N) = \phi(P_{1|\bar{A}}, \ldots, P_{N|\bar{A}})$. It is evident that φ is a unanimous and strategy-proof constrained RSCF. We first claim that φ satisfies the tops-only property. Given $P, P' \in \mathbb{D}^N$ with $r_1(P_i) = r_1(P'_i)$ for all $i \in I$, we know $r_1(P_{i|\bar{A}}) = r_1(P'_{i|\bar{A}})$ for all $i \in I$ by the second condition of Definition 7. Consequently, by the construction of φ and tops-onlyness of ϕ, $\varphi(P) \equiv \phi(P_{1|\bar{A}}, \ldots, P_{N|\bar{A}}) = \phi(P'_{1|\bar{A}}, \ldots, P'_{N|\bar{A}}) \equiv \varphi(P')$. Thus, φ satisfies the tops-only property. Last, we show that φ satisfies the compromise property. Given $\hat{I} \subseteq I$ with $|\hat{I}| = \frac{N}{2}$ if N is even, and $|\hat{I}| = \frac{N+1}{2}$ if N is odd, fix $P_i, P_j \in \mathbb{D}$ with $r_1(P_i) \equiv x \neq y \equiv r_1(P_j)$ and $r_2(P_i) = r_2(P_j) \equiv a \in \bar{A}$. If either $x \notin \bar{A}$ or $y \notin \bar{A}$, we know $f_a^a\left(\frac{P_i|_{\bar{A}}}{\hat{I}}, \frac{P_j|_{\bar{A}}}{I \backslash \hat{I}}\right) = a$. If $x, y \in \bar{A}$, recall the verification of the sufficiency part of Theorem 1, it is true that $a \in \langle x, y\rangle \backslash \{x, y\}$. Then, $f_a^a\left(\frac{P_i|_{\bar{A}}}{\hat{I}}, \frac{P_j|_{\bar{A}}}{I \backslash \hat{I}}\right) = a$. Consequently, no matter x and y are feasible or not, we have $\varphi_a\left(\frac{P_i}{\hat{I}}, \frac{P_j}{I \backslash \hat{I}}\right) \equiv \phi_a\left(\frac{P_i|_{\bar{A}}}{\hat{I}}, \frac{P_j|_{\bar{A}}}{I \backslash \hat{I}}\right) = \sum_{z \in \bar{A}} \lambda_z f_a^z\left(\frac{P_i|_{\bar{A}}}{\hat{I}}, \frac{P_j|_{\bar{A}}}{I \backslash \hat{I}}\right) \geq \lambda_a > 0$. Therefore, φ satisfies the compromise property. This completes the verification of the sufficiency part of Theorem 1.

D Supplementary material

D.1 An example related to the no-detour property

Let $A \equiv A^1 \times A^2$, $A^1 = \{0,1,2\}$ and $A^2 = \{0,1\}$. We highlight two separable preferences: $P_i : (0,0) \rightharpoonup (1,0) \rightharpoonup (0,1) \rightharpoonup (1,1) \rightharpoonup (2,0) \rightharpoonup (2,1)$ and $P_i' : (1,0) \rightharpoonup (0,0) \rightharpoonup (1,1) \rightharpoonup (0,1) \rightharpoonup (2,0) \rightharpoonup (2,1)$. Note that $(1,1)P_i(2,1)$, $(1,1)P_i'(2,1)$ and $P_i \sim^+ P_i'$. Thus, $\{P_i, P_i'\}$ formulates a simple$^+$ path connecting P_i and P_i' where $(1,1)$ is always ranked above $(2,1)$, and satisfies the no-detour property, i.e., $r_1(P_i) = (0,0) \in (A^1, 0)$ and $r_1(P_i) = (1,0) \in (A^1, 0)$. We can construct another simple$^+$ path $\{P_i^k\}_{k=1}^8 \subseteq \mathbb{D}_S$ in Table 2 below connecting P_i and P_i' where $(1,1)$ is always ranked above $(2,1)$. However, this simple$^+$ path violates the no-detour property: It starts from P_i, first takes a detour to preference P_i^3 with peak $(0,1) \notin (A^1, 0)$, then diverges further to preference P_i^5 with peak $(1,1) \notin (A^1, 0)$, and finally goes back to preference P_i'.

$P_i \equiv P_i^1$	$\sim$	P_i^2	$\sim^+$	P_i^3	$\sim$	P_i^4	$\sim^+$	P_i^5	$\sim$	P_i^6	$\sim^+$	P_i^7	$\sim$	$P_i^8 \equiv P_i'$
$(0,0)$		$(0,0)$		$(0,1)$		$(0,1)$		$(1,1)$		$(1,1)$		$(1,0)$		$(1,0)$
$(1,0)$		$(0,1)$		$(0,0)$		$(1,1)$		$(0,1)$		$(1,0)$		$(1,1)$		$(0,0)$
$(0,1)$		$(1,0)$		$(1,1)$		$(0,0)$		$(1,0)$		$(0,1)$		$(0,0)$		$(1,1)$
$(1,1)$		$(1,1)$		$(1,0)$		$(1,0)$		$(0,0)$		$(0,0)$		$(0,1)$		$(0,1)$
$(2,0)$		$(2,0)$		$(2,1)$		$(2,1)$		$(2,1)$		$(2,1)$		$(2,0)$		$(2,0)$
$(2,1)$		$(2,1)$		$(2,0)$		$(2,0)$		$(2,0)$		$(2,0)$		$(2,1)$		$(2,1)$

Table 2: A simple$^+$ path violating the no-detour property

D.2 Related verification in Example 3

Since each $f \in \{f^a\}_{a \notin \{(1,1,0),(1,1,1)\}} \cup \{f^{(j,i,i)}, f^{(i,j,j)}\}$ is unanimous, RSCF φ must be unanimous. To show strategy-proofness of φ, it suffices to show that each f is strategy-proof.

As a generalized median voter scheme of Barberà et al. (1993), we first know that each f above is strategy-proof on $\mathbb{D}_{MSP}$. Moreover, note that f also satisfies the tops-only property. Therefore, (i) for all $P_i \in \mathbb{D}_{MSP}$ and $b \in A$, voter i never manipulates at (P_i, b) via any $P_i' \in \mathbb{D}$, and (ii) for all $P_j \in \mathbb{D}_{MSP}$ and $b \in A$, voter j never manipulates at (b, P_j) via any $P_j' \in \mathbb{D}$. Hence, the verification only concerns the following two possible manipulations:

(i) Given (P_i^*, P_j), voter i considers to deviate to $P_i' \in \mathbb{D}$.

(ii) Given (P_i, P_j^*) where P_j^* is identical to preference P_i^* in Example 3, voter j considers to deviate to $P_j' \in \mathbb{D}$.

We first identify an important multidimensional single-peaked preference below:

$$\hat{P}_i: \quad (0,0,0) \rightharpoonup (0,0,1) \rightharpoonup (1,0,0) \rightharpoonup (1,0,1) \rightharpoonup (0,1,0) \rightharpoonup (0,1,1) \rightharpoonup (1,1,0) \rightharpoonup (1,1,1).$$

Note that $P_i^* \sim \hat{P}_i$, $(1,1,1)P_i^*!(1,1,0)$ and $(1,1,0)\hat{P}_i!(1,1,1)$.

Claim 1: Given $f \in \{f^a\}_{a \notin \{(1,1,0),(1,1,1)\}} \cup \{f^{(j,i,i)}, f^{(i,j,j)}\}$, the first and second possible manipulations are never profitable.

Suppose not, there exists $P_i' \in \mathbb{D}$ such that $f(P_i', P_j)P_i^* f(P_i^*, P_j)$. Assume $f(P_i', P_j) = x$ and $f(P_i^*, P_j) = y$. Evidently, $P_i' \neq P_i^*$ and hence $P_i' \in \mathbb{D}_{MSP}$. Moreover, it is evidently that $r_1(P_j) \neq (0,0,0) = r_1(P_i^*)$, and hence $P_j \in \mathbb{D}_{MSP}$. Otherwise, $f^a(P_i^*, P_j) = (0,0,0) = r_1(P_i^*)$ by unanimity, and consequently, there exist no $P_i' \in \mathbb{D}$ such that $f(P_i', P_j)P_i^* f(P_i^*, P_j)$. By the tops-only property, we know $f(\hat{P}_i, P_j) = y$. Since voter i cannot manipulate at $(\hat{P}_i, P_j)$ via P_i', it is true that $y\hat{P}_i x$. Now, since $P_i^* \sim \hat{P}_i$, $xP_i^* y$ and $y\hat{P}_i x$ imply $x = (1,1,1)$ and $y = (1,1,0)$. Thus, $f(P_i^*, P_j) = (1,1,0)$.

First, assume $f \in \{f^a\}_{a \notin \{(1,1,0),(1,1,1)\}}$. Assume $r_1(P_j) = b \equiv (b^1, b^2, b^3)$. According to the definition of the projection rule f^a, where $a \equiv (a^1, a^2, a^3)$, $f^a\big((0,0,0), (b^1, b^2, b^3)\big) = f^a(P_i^*, P_j) = (1,1,1)$ implies $med(0, b^1, a^1) = 1$, $med(0, b^2, a^2) = 1$ and $med(0, b^3, a^3) = 0$. Consequently, $a^1 = 1$ and $a^2 = 1$, and hence $a \in \{(1,1,0),(1,1,1)\}$. Contradiction!

Next, assume $f \in \{f^{(j,i,i)}, f^{(i,j,j)}\}$. If $f = f^{(j,i,i)}$, the definition of $f^{(j,i,i)}$ implies $f(P_i^*, P_j)^2 = r_1(P_i^*)^2 = 0$, and hence $f(P_i^*, P_j) \neq (1,1,0)$. Contradiction! If $f = f^{(i,j,j)}$, the definition of $f^{(i,j,j)}$ implies $f(P_i^*, P_j)^1 = r_1(P_i^*)^1 = 0$, and hence $f(P_i^*, P_j) \neq (1,1,0)$. Contradiction! Therefore, the first possible manipulation cannot occur. By a symmetric argument, the second possible manipulation cannot occur either. This completes the verification of the claim.

Since each $f \in \{f^a\}_{a \notin \{(1,1,0),(1,1,1)\}} \cup \{f^{(j,i,i)}, f^{(i,j,j)}\}$ is strategy-proof, as a mixture of these eight DSCFs, it is true that φ is strategy-proof.

Last, we show that RSCF φ satisfies the compromise property. Given $P_i, P_j \in \mathbb{D}$, assume $r_1(P_i) \equiv x \neq y \equiv r_1(P_j)$ and $r_2(P_i) = r_2(P_j) \equiv z$. We consider the following three cases: (i) $z = (1,1,0)$, (ii) $z = (1,1,1)$, and (iii) $z \notin \{(1,1,0),(1,1,1)\}$.

In the first case, since $z = (1,1,0)$, it is true that either $\{x,y\} = \{(1,0,0),(0,1,0)\}$, or $\{x,y\} = \{(0,1,0),(1,1,1)\}$, or $\{x,y\} = \{(1,0,0),(1,1,1)\}$. If $\{x,y\} = \{(0,1,0),(1,0,0)\}$, we know $f^{(j,i,i)}\big((0,1,0),(1,0,0)\big) = (1,1,0) = z$ and $f^{(i,j,j)}\big((1,0,0),(0,1,0)\big) = (1,1,0) = z$. If $\{x,y\} = \{(0,1,0),(1,1,1)\}$, we know $f^{(1,0,0)}(x,y) = (1,1,0) = z$. If $\{x,y\} = \{(1,0,0),(1,1,1)\}$, we know $f^{(0,1,0)}(x,y) = (1,1,0) = z$. Therefore, there always exists $f \in \{f^a\}_{a \notin \{(1,1,0),(1,1,1)\}} \cup \{f^{(j,i,i)}, f^{(i,j,j)}\}$ such that $f(P_i, P_j) = z$, and hence, $\varphi_z(P_i, P_j) > 0$.

In the second case, since $z = (1,1,1)$, it is true that either $\{x,y\} = \{(0,1,1),(1,0,1)\}$, or $\{x,y\} = \{(0,1,1),(1,1,0)\}$, or $\{x,y\} = \{(1,1,0),(1,0,1)\}$. If $\{x,y\} = \{(0,1,1),(1,0,1)\}$, we know $f^{(j,i,i)}\big((0,1,1),(1,0,1)\big) = (1,1,1) = z$ and $f^{(i,j,j)}\big((1,0,1),(0,1,1)\big) = (1,1,1) = z$. If $\{x,y\} = \{(0,1,1),(1,1,0)\}$, we know $f^{(1,0,1)}(x,y) = (1,1,1) = z$. If $\{x,y\} = \{(1,1,0),(1,0,1)\}$, we know $f^{(0,1,1)}(x,y) = (1,1,1) = z$. Therefore, there always exists $f \in \{f^a\}_{a \notin \{(1,1,0),(1,1,1)\}} \cup \{f^{(j,i,i)}, f^{(i,j,j)}\}$ such that $f(P_i, P_j) = z$, and hence, $\varphi_z(P_i, P_j) > 0$.

In the third case, since $z \notin \{(1,1,0),(1,1,1)\}$, we have $f^z(x,y) = z$, and hence $\varphi_z(P_i, P_j) > 0$. In conclusion, RSCF φ satisfies the compromise property.

D.3 The separable domain $\mathbb{D}_S$ is a connected$^+$ domain

We specify two facts which together show that the separable domain $\mathbb{D}_S$ is a connected$^+$ domain. The detailed proof of these facts can be found in the working paper of Chatterji and Zeng (2015).

Fact **1** *Given $P_j, P'_j \in \mathbb{D}_S$ with $[P_j]^s \neq [P'_j]^s$ for some $s \in M$, and $x, y \in A$ with xP_jy and xP'_jy, there exist $t \geq 1$ pair(s) of separable preferences $\{\bar{P}^k_j, \hat{P}^k_j\}$, $k = 1, \ldots, t$, such that*

(i) *$[P_j]^s = [\bar{P}^1_j]^s$ and $[P'_j]^s = [\hat{P}^t_j]^s$ for all $s \in M$;*

(ii) *for each $1 \leq k \leq t$, $\bar{P}^k_j \sim^+ \hat{P}^k_j$, $x\bar{P}^k_jy$ and $x\hat{P}^k_jy$;*

(iii) *for each $1 \leq k \leq t-1$, $[\hat{P}^k_j]^s = [\bar{P}^{k+1}_j]^s$ for all $s \in M$.*

In particular, if $r_1(P_j)$ and $r_1(P'_j)$ are similar, say $r_1(P_j) = (a^s, z^{-s})$ and $r_1(P'_j) = (b^s, z^{-s})$, then $r_1(\bar{P}^k_j), r_1(\hat{P}^k_j) \in \big\{(a^s, z^{-s}), (b^s, z^{-s})\big\}$, $k = 1, \ldots, t$.

Fact **2** *Given two distinct $P_j, P'_j \in \mathbb{D}_S$ with $[P_j]^s = [P'_j]^s$ for all $s \in M$, and $x, y \in A$ with xP_jy and xP'_jy, there exists a simple path $\{P^k_j\}^q_{k=1} \subseteq \mathbb{D}_S$ connecting P_j and P'_j such that xP^k_jy, $k = 1, \ldots, q$.*

We provide the following example to illustrate how both facts are used to show connectedness$^+$ of the separable domain.

Example **5** Let $A \equiv A^1 \times A^2$, $A^1 = \{0, 1, 2\}$ and $A^2 = \{0, 1\}$. Fix two particular separable preferences:

$P_i:\ (0,0) \rightharpoonup (0,1) \rightharpoonup (1,0) \rightharpoonup (1,1) \rightharpoonup (2,0) \rightharpoonup (2,1)$, and $P'_i:\ (2,1) \rightharpoonup (0,1) \rightharpoonup (2,0) \rightharpoonup (1,1) \rightharpoonup (0,0) \rightharpoonup (1,0)$.

Note that $(0,1)P_i(1,1)$ and $(0,1)P'_i(1,1)$.

First, we construct the following transitions of marginal preferences to reconcile the differences of marginal preferences of P_i and P'_i:

$$\big([P_i]^1; [P_i]^2\big) \equiv (0 \rightharpoonup 1 \rightharpoonup 2; 0 \rightharpoonup 1) \xRightarrow{\textcircled{1}} (0 \rightharpoonup 1 \rightharpoonup 2; 1 \rightharpoonup 0) \xRightarrow{\textcircled{2}} (0 \rightharpoonup 2 \rightharpoonup 1; 1 \rightharpoonup 0) \xRightarrow{\textcircled{3}} (2 \rightharpoonup 0 \rightharpoonup 1; 1 \rightharpoonup 0) \equiv \big([P'_i]^1; [P'_i]^2\big).$$

For each transition, we identify a pair of adjacent$^+$ preferences ranking $(0,1)$ above $(1,1)$ to illustrate Fact 1.

	$\xRightarrow{\textcircled{1}}$			$\xRightarrow{\textcircled{2}}$			$\xRightarrow{\textcircled{3}}$	
$\bar{P}^1_i$	$\sim^+$	$\hat{P}^1_i$	$\bar{P}^2_i$	$\sim^+$	$\hat{P}^2_i$	$\bar{P}^3_i$	$\sim^+$	$\hat{P}^3_i$
$(0,0)$		$(0,1)$	$(0,1)$		$(0,1)$	$(0,1)$		$(2,1)$
$(0,1)$		$(0,0)$	$(0,0)$		$(0,0)$	$(2,1)$		$(0,1)$
$(1,0)$		$(1,1)$	$(1,1)$		$(2,1)$	$(1,1)$		$(1,1)$
$(1,1)$		$(1,0)$	$(2,1)$		$(1,1)$	$(0,0)$		$(2,0)$
$(2,0)$		$(2,1)$	$(1,0)$		$(2,0)$	$(2,0)$		$(0,0)$
$(2,1)$		$(2,0)$	$(2,0)$		$(1,0)$	$(1,0)$		$(1,0)$

Table 3: Three pairs of adjacent$^+$ preferences

Next, we make the following two observations to illustrate Fact 2.

(i) $P_i = \bar{P}_i^1$, $\hat{P}_i^1 \sim \bar{P}_i^2$ and $\hat{P}_i^3 \sim P_i'$.

(ii) Preferences $\hat{P}_i^2$ and $\bar{P}_i^3$ share the same marginal preferences, i.e., $\big([\hat{P}_i^2]^1, [\hat{P}_i^2]^2\big) \equiv (0 \rightharpoondown 2 \rightharpoondown 1; 1 \rightharpoondown 0) \equiv \big([\bar{P}_i^3]^1, [\bar{P}_i^3]^2\big)$. We identify the another separable preference $\tilde{P}_i$: $(0,1) \rightharpoondown (2,1) \rightharpoondown (0,0) \rightharpoondown (1,1) \rightharpoondown (2,0) \rightharpoondown (1,0)$ which admits the same marginal preferences. Thus, we have a simple path $\{\hat{P}_i^2, \tilde{P}_i, \bar{P}_i^3\}$ connecting $\hat{P}_i^2$ and $\bar{P}_i^3$ such that $(0,1)$ is always ranked above $(1,1)$.[47]

Eventually, we construct the appropriate simple$^+$ path connecting P_i and P_i' below which meets the requirement of the Exterior$^+$ property: $P_i \equiv \bar{P}_i^1 \sim^+ \hat{P}_i^1 \sim \bar{P}_i^2 \sim^+ \hat{P}_i^2 \sim \tilde{P}_i \sim \bar{P}_i^3 \sim^+ \hat{P}_i^3 \sim P_i'$. Furthermore, note that the simple$^+$ subpaths $\{\bar{P}_i^1, \hat{P}_i^1, \bar{P}_i^2, \hat{P}_i^2, \tilde{P}_i, \bar{P}_i^3\}$ and $\{\hat{P}_i^1, \bar{P}_i^2, \hat{P}_i^2, \tilde{P}_i, \bar{P}_i^3, \hat{P}_i^3, P_i'\}$ satisfy the no-detour property.

Last, take the simple$^+$ subpath $\{\hat{P}_i^1, \bar{P}_i^2, \hat{P}_i^2, \tilde{P}_i, \bar{P}_i^3\}$ as an example where every preference has the peak $(0,1)$. It illustrates that the requirement of the Interior$^+$ property can be verified in the separable domain as well. $\square$

D.4 The top-separable domain $\mathbb{D}_{TS}$ is a connected$^+$ domain

To verify that $\mathbb{D}_{TS}$ is a connected$^+$ domain, we first provide two facts.

FACT **3** *Given $P_j \in \mathbb{D}_{TS} \backslash \mathbb{D}_S$ and $x, y \in A$ with xP_jy, there exists $\bar{P}_j \in \mathbb{D}_S$ such that $r_1(\bar{P}_j) = r_1(P_j)$ and $x\bar{P}_jy$.*

Proof: Assume $r_1(P_j) = a \equiv (a^s)_{s\in M}$. Assume $x^s \neq y^s$ for all $s \in S \subseteq M$ and $x^{-S} = y^{-S}$. Evidently, $S \neq \emptyset$. There exist two cases for x and y: (i) $x^s \neq a^s$ and $y^s \neq a^s$ for some $s \in S$, and (ii) $x^s = a^s$ or $y^s = a^s$ for every $s \in S$. Furthermore, in the second case, due top-separability, there must exist $s \in S$ such that $x^s = a^s$ and $y^s \neq a^s$. Now, in both cases, we can first identify a marginal preference $[\bar{P}_j]^s$ on A^s such that $x^s[\bar{P}_j]^s y^s$. Next, we construct a lexicographically separable preference $\bar{P}_j$ such that $r_1(\bar{P}_j) = a$ and component s is lexicographically dominant. Consequently, $\bar{P}_j \in \mathbb{D}_S$ and $x\bar{P}_jy$. $\square$

FACT **4** *Given $P_j \in \mathbb{D}_{TS}$ and $\bar{P}_j \in \mathbb{D}_S$, assume $r_1(P_j) = r_1(\bar{P}_j) \equiv a$. Given $b, c \in A$, assume $bP_j!c$ and $c\bar{P}_jb$. There exists $\hat{P}_j \in \mathbb{D}_{TS}^a$ such that $\hat{P}_j \sim P_j$ and $c\hat{P}_j!b$.*

Proof: Since $r_1(P_j) = r_1(\bar{P}_j) \equiv a$, $bP_j!c$ and $c\bar{P}_jb$, it is evident that $a \notin \{b, c\}$. We first construct preference $\hat{P}_j$ by locally switching b and c in P_j. Thus, $r_1(\hat{P}_j) = a$, $\hat{P}_j \sim P_j$ and $c\hat{P}_j!b$. We show $\hat{P}_j \in \mathbb{D}_{TS}$. Suppose that $\hat{P}_j \notin \mathbb{D}_{TS}$. Since $P_j \in \mathbb{D}_{TS}$ and $\hat{P}_j \sim P_j$, $\hat{P}_j \notin \mathbb{D}_{TS}$ implies that $b^s = a^s \neq c^s$ and $b^{-s} = c^{-s}$ for some $s \in M$. Consequently, since $\bar{P}_j \in \mathbb{D}_S$, we have $b\bar{P}_jc$. Contradiction! $\square$

Now, we consider two distinct preferences $P_j, P_j' \in \mathbb{D}_{TS}$ with either $r_1(P_j) = r_1(P_j')$ or $r_1(P_j) \neq r_1(P_j')$, and two alternatives $x, y \in A$ such that xP_jy and $xP_j'y$. We construct an **appropriate** simple$^+$ path $\{P_j^k\}_{k=1}^q \subseteq \mathbb{D}_{TS}$ connecting P_j and P_j' such that xP_j^ky, $k = 1, \ldots, q$,

[47] Note that the existence of preference $\tilde{P}_i$ also illustrates the importance of the co-existence of adjacency and adjacency$^+$. If we forbid the presence of adjacency, then preference $\bar{P}_i^3$ cannot be obtained via the transition of preference $\hat{P}_i^2$, and consequently, the adjacency$^+$ between $\bar{P}_3$ and $\hat{P}_3$ disappears.

and $[r_1(P_j) = r_1(P'_j) \equiv a] \Rightarrow [r_1(P^k_j) = a$ for all $k = 1, \ldots, q]$. There are three cases: (i) $P_j \in \mathbb{D}_S$ and $P'_j \in \mathbb{D}_S$, (ii) $P_j \in \mathbb{D}_{TS} \backslash \mathbb{D}_S$ and $P'_j \in \mathbb{D}_S$ (or symmetrically, $P_j \in \mathbb{D}_S$ and $P'_j \in \mathbb{D}_{TS} \backslash \mathbb{D}_S$), and (iii) $P_j, P'_j \in \mathbb{D}_{TS} \backslash \mathbb{D}_S$.

The first case is covered by Section D.3. In the second case, we first identify $\bar{P}_j \in \mathbb{D}_S$ with $r_1(\bar{P}_j) = r_1(P_j)$ and $x\bar{P}_j y$ by Fact 3. Between $\bar{P}_j$ and P'_j, by case (i), we have an appropriate simple$^+$ path, while Between P_j and $\bar{P}_j$, we can construct another appropriate simple path by repeatedly applying Fact 4. Combining these two paths, we have an appropriate simple$^+$ path connecting P_j and P'_j. In the third case, by Fact 3, we first identify $\bar{P}_j \in \mathbb{D}_S$ with $r_1(\bar{P}_j) = r_1(P_j)$ and $x\bar{P}_j y$, and $\bar{P}'_j \in \mathbb{D}_S$ with $r_1(\bar{P}'_j) = r_1(P'_j)$ and $x\bar{P}'_j y$. Similar to the second case, between P_i and $\bar{P}_j$, between $\bar{P}_j$ and $\bar{P}'_j$, and between $\bar{P}'_j$ and P'_j, we have an appropriate simple path, an appropriate simple$^+$ path and an appropriate simple path respectively. Combining these three paths, we have an appropriate simple$^+$ path connecting P_j and P'_j. Furthermore, by the construction of the appropriate simple$^+$ path and verification in Section D.3, we can prove that the top-separable domain satisfies the no-detour property. Therefore, the top-separable domain $\mathbb{D}_{TS}$ is a connected$^+$ domain.

D.5 The multidimensional single-peaked domain $\mathbb{D}_{MSP}$ is a connected$^+$ domain

According to the proof of Proposition 2 of Chatterji and Zeng (2017), we first provide two facts.

Fact 5 *Given $P_j, P'_j \in \mathbb{D}_{MSP}$ with $r_1(P_j) \neq r_1(P'_j)$, and $x, y \in A$ with $xP_j y$ and $xP'_j y$, there exist $t \geq 1$ pair(s) of multidimensional single-peaked preferences $\{\bar{P}^k_j, \hat{P}^k_j\}^t_{k=1}$, such that*

(i) *$r_1(P_j) = r_1(\bar{P}^1_j)$ and $r_1(P'_j) = r_1(\hat{P}^t_j)$;*

(ii) *for each $1 \leq k \leq t$, $r_1(\bar{P}^k_j)$ and $r_1(\hat{P}^k_j)$ are similar; $\bar{P}^k_j$ and $\hat{P}^k_j$ are $|A^{-s}|$-adjacent for some $s \in M$; $x\bar{P}^k_j y$ and $x\hat{P}^k_j y$;*

(iii) *for each $1 \leq k \leq t-1$, $r_1(\hat{P}^k_j) = r_1(\bar{P}^{k+1}_j)$.*

In particular, if $r_1(P_j)$ and $r_1(P'_j)$ are similar, say $r_1(P_j) = (a^s, z^{-s})$ and $r_1(P'_j) = (b^s, z^{-s})$, then $r_1(\bar{P}^k_j), r_1(\hat{P}^k_j) \in \langle (a^s, z^{-s}), (b^s, z^{-s}) \rangle$, $k = 1, \ldots, t$.

Fact 6 *Given two distinct $P_j, P'_j \in \mathbb{D}_{MSP}$ with $r_1(P_j) = r_1(P'_j) \equiv a$, and $x, y \in A$ with $xP_j y$ and $xP'_j y$, there exists a simple path $\{P^k_j\}^q_{k=1} \subseteq \mathbb{D}^a_{MSP}$ connecting P_j and P'_j such that $xP^k_j y$, $k = 1, \ldots, q$.*

Note that preferences $\bar{P}^k_j$ and $\hat{P}^k_j$ in Fact 5 are $|A^{-s}|$-adjacent, not necessarily adjacent$^+$. Fact 7 below shows that such two $|A^{-s}|$-adjacent preferences in Fact 3 can be replaced by two appropriate adjacent$^+$ preferences.

Fact 7 *Given two distinct $P_j, P'_j \in \mathbb{D}_{MSP}$ with $r_1(P_j) \neq r_1(P'_j)$ and $P_j \sim^{|A^{-s}|} P'_j$ for some $s \in M$, and $x, y \in A$ with $xP_j y$ and $xP'_j y$, there exist $\bar{P}_j, \bar{P}'_j \in \mathbb{D}_{MSP} \cap \mathbb{D}_S$ such that*

(i) *$r_1(P_j) = r_1(\bar{P}_j)$ and $r_1(P'_j) = r_1(\bar{P}'_j)$,*

(ii) *$\bar{P}_j \sim^+ \bar{P}'_j$,*

(iii) *$x\bar{P}_j y$ and $x\bar{P}'_j y$.*

Proof: Assume $r_1(P_j) = a \neq b = r_1(P'_j)$. Since $P_j \sim^{|A^{-s}|} P'_j$, it is true $a^s \neq b^s$ and $a^{-s} = b^{-s} \equiv z^{-s}$. Since xP_jy and xP'_jy, it is true that $y \notin \langle a, x\rangle$ and $y \notin \langle b, x\rangle$. We consider two cases: (i) There exists $\tau \neq s$ such that $y^\tau \notin \langle z^\tau, x^\tau\rangle$, and (ii) $y^\tau \in \langle z^\tau, x^\tau\rangle$ for all $\tau \neq s$. In the first case, it is evident that $y^\tau \notin \langle a^\tau, x^\tau\rangle$ and $y^\tau \notin \langle b^\tau, x^\tau\rangle$. In the second case, it must be true that $y^s \notin \langle a^s, x^s\rangle$ and $y^s \notin \langle b^s, x^s\rangle$. In conclusion, there exists $\tau \in M$ such that $y^\tau \notin \langle a^\tau, x^\tau\rangle$ and $y^\tau \notin \langle b^\tau, x^\tau\rangle$.

On each component set $\omega \in M$, according to the tree $G(A^\omega)$, we construct two particular marginal preferences $[\bar{P}_j]^\omega$ and $[\bar{P}'_j]^\omega$ such that the following conditions are satisfied:

(i) $[\bar{P}_j]^\omega$ and $[\bar{P}'_j]^\omega$ are single-peaked on $G(A^\omega)$ for all $\omega \in M$.

(ii) $r_1([\bar{P}_j]^\omega) = a^\omega$ and $r_1([\bar{P}'_j]^\omega) = b^\omega$ for all $\omega \in M$.

(iii) $x^\tau[\bar{P}_j]^\tau y^\tau$ and $x^\tau[\bar{P}'_j]^\tau y^\tau$.

(iv) $[\bar{P}_j]^s \sim [\bar{P}_j]^s$ and $[\bar{P}_j]^\omega = [\bar{P}_j]^\omega$ for all $\omega \neq s$.

Last, according to all marginal preferences $[\bar{P}_j]^\omega$ and $[\bar{P}'_j]^\omega$, $\omega \in M$, we assemble two multidimensional single-peaked and lexicographically separable preferences $\bar{P}_j, \bar{P}'_j \in \mathbb{D}_{MSP} \cap \mathbb{D}_{LS}$ where both $\bar{P}_j$ and $\bar{P}'_j$ share the same lexicographic order over components, and component τ lexicographically dominates all other components. Thus, $r_1(\bar{P}_j) = a$, $r_1(\bar{P}_j) = b$, $\bar{P}_j \sim^+ \bar{P}_j$, $x\bar{P}_jy$ and $x\bar{P}'_jy$. This completes the verification of Fact 7. □

Last, after updating Fact 5 by Fact 7, similar to the verification in Section D.3, we assert that the multidimensional single-peaked domain $\mathbb{D}_{MSP}$ is a connected$^+$ domain by a repeated application of Facts 5 and 6.

D.6 The intersection of the separable domain and the multidimensional single-peaked domain $\mathbb{D}_S \cap \mathbb{D}_{MSP}$ is a connected$^+$ domain

The verification here is similar to that in Section D.3. We replace the separable domain $\mathbb{D}_S$ in Facts 1 and 2 by the intersection $\mathbb{D}_S \cap \mathbb{D}_{MSP}$, and establish the following two facts.

Fact **8** *Given two preferences $P_j, P'_j \in \mathbb{D}_S \cap \mathbb{D}_{MSP}$ with $[P_j]^s \neq [P'_j]^s$ for some $s \in M$, and $x, y \in A$ with xP_jy and xP'_jy, there exist $t \geq 1$ pair(s) of separable and multidimensional single-peaked preferences $\{\bar{P}^k_j, \hat{P}^k_j\}k = 14$ such that*

(i) *$[P_j]^s = [\bar{P}^1_j]^s$ and $[P'_j]^s = [\hat{P}^t_j]^s$ for all $s \in M$;*

(ii) *for each $1 \leq k \leq t$, $\bar{P}^k_j \sim^+ \hat{P}^k_j$, $x\bar{P}^k_jy$ and $x\hat{P}^k_jy$;*

(iii) *for each $1 \leq k \leq t-1$, $[\hat{P}^k_j]^s = [\bar{P}^{k+1}_j]^s$ for all $s \in M$.*

In particular, if $r_1(P_j)$ and $r_1(P'_j)$ are similar, say $r_1(P_j) = (a^s, z^{-s})$ and $r_1(P'_j) = (b^s, z^{-s})$, then $r_1(\bar{P}^k_j), r_1(\hat{P}^k_j) \in \big\langle (a^s, z^{-s}), (b^s, z^{-s})\big\rangle$, $k = 1, \ldots, t$.

Fact **9** *Given two distinct $P_j, P'_j \in \mathbb{D}_S \cap \mathbb{D}_{MSP}$ with $[P_j]^s = [P'_j]^s$ for all $s \in M$, and $x, y \in A$ with xP_jy and xP'_jy, there exists a simple path $\{P^k_j\}^q_{k=1} \subseteq \mathbb{D}_S \cap \mathbb{D}_{MSP}$ connecting P_j and P'_j such that xP^k_jy, $k = 1, \ldots, q$.*

D.7 The union of the separable domain and multidimensional single-peaked domains $\mathbb{D}_S \cup [\cup_{k=1}^t \mathbb{D}_{MSP_k}]$ is a connected$^+$ domain

We first provide a fact which will be repeatedly applied.

Fact 10 *Given $P_j' \in \mathbb{D}_{MSP_k}$ and $x, y \in A$ with $xP_j'y$, there exists $\bar{P}_j \in \mathbb{D}_S \cap \mathbb{D}_{MSP_k}$ such that $r_1(\bar{P}_j) = r_1(P_j')$ and $x\bar{P}_j y$.*

Proof: Assume $r_1(P_j') = (a^s)_{s\in M}$ and $\mathbb{D}_{MSP_k}$ is the multidimensional single-peaked domain on product graph of trees $\times_{s\in M} G(A^s)$. Since $xP_j'y$, it is true that $y \notin \langle a, x\rangle$. Thus, there must exist $s \in M$ such that $y^s \notin \langle a^s, x^s\rangle$. Consequently, we can construct a marginal preference $[\bar{P}_j]^s$ over all elements in A^s which is single-peaked on $G(A^s)$, $r_1([\bar{P}_j]^s) = a^s$ and $x^s[\bar{P}_j]^s y^s$. For every $\tau \in M\backslash\{s\}$, pick arbitrary marginal preference $[\bar{P}_j]^\tau$ which is single-peaked on $G(A^\tau)$ and $r_1([\bar{P}_j]^\tau) = a^\tau$. Last, we construct a lexicographically separable preference $\bar{P}_j$ by using all these marginal preferences and make component s lexicographically dominant. Consequently, $x\bar{P}_j y$. Since each marginal preference is single-peaked, it is true that $\bar{P}_j$ is multidimensional single-peaked on $\times_{s\in M} G(A^s)$. $\square$

To verify that the union $\mathbb{D}_S \cup [\cup_{k=1}^t \mathbb{D}_{MSP_k}]$ is a connected$^+$ domain, we fix two distinct preferences $P_j, P_j' \in \mathbb{D}_S \cup [\cup_{k=1}^t \mathbb{D}_{MSP_k}]$ with either $r_1(P_j) = r_1(P_j')$ or $r_1(P_j) \neq r_1(P_j')$, and $x, y \in A$ with xP_jy and $xP_j'y$, and construct an **appropriate** simple$^+$ path $\{P_j^k\}_{k=1}^q \subseteq \mathbb{D}_S \cup [\cup_{k=1}^t \mathbb{D}_{MSP_k}]$ connecting P_j and P_j' such that $xP_j^k y$, $k = 1, \ldots, q$, and $[r_1(P_j) = r_1(P_j') \equiv a] \Rightarrow [r_1(P_j^k) = a$ for all $k = 1, \ldots, q]$.

If $P_j, P_j' \in \mathbb{D}_S$ or $P_j, P_j' \in \mathbb{D}_{MSP_k}$ for some $k \geq t$, the verification follows from Section D.3 or Section D.5. Thus, we consider the following two cases:

(i) $P_j \in \mathbb{D}_S\backslash\mathbb{D}_{MSP_k}$ and $P_j' \in \mathbb{D}_{MSP_k}\backslash\mathbb{D}_S$ for some $k \geq 1$.

(ii) $P_j \in \mathbb{D}_{MSP_k}\backslash\mathbb{D}_S$ and $P_j' \in \mathbb{D}_{MSP_{k'}}\backslash\mathbb{D}_S$ for some $k \neq k'$.

In case (i), by Fact 10, we identify another preference $\bar{P}_j' \in \mathbb{D}_S \cap \mathbb{D}_{MSP_k}$ such that $r_1(P_j') = r_1(\bar{P}_j')$ and $x\bar{P}_j'y$. First, since $P_j, \bar{P}_j' \in \mathbb{D}_S$, by the verification in Section D.3, we have an appropriate simple$^+$ path in $\mathbb{D}_S$ connecting P_j and $\bar{P}_j'$. Second, since $\bar{P}_j', P_j' \in \mathbb{D}_{MSP_k}$, by the verification in Section D.5, we have an appropriate simple$^+$ path in $\mathbb{D}_{MSP_k}$ connecting $\bar{P}_j'$ and P_j'. Combining these two simple$^+$ paths, we have an appropriate simple$^+$ path in $\mathbb{D}_S \cup [\cup_{k=1}^t \mathbb{D}_{MSP_k}]$ connecting P_j and P_j'.

In case (ii), by Fact 10, we identify two other preferences $\bar{P}_j \in \mathbb{D}_S \cap \mathbb{D}_{MSP_k}$ and $\bar{P}_j' \in \mathbb{D}_S \cap \mathbb{D}_{MSP_{k'}}$ such that $r_1(P_j) = r_1(\bar{P}_j)$, $x\bar{P}_j y$, $r_1(P_j') = r_1(\bar{P}_j')$ and $x\bar{P}_j'y$. First, since $\bar{P}_j, \bar{P}_j' \in \mathbb{D}_S$, by the verification in Section D.3, we have an appropriate simple$^+$ path in $\mathbb{D}_S$ connecting $\bar{P}_j$ and $\bar{P}_j'$. Second, since $P_j, \bar{P}_j \in \mathbb{D}_{MSP_k}$ and $\bar{P}_j', P_j' \in \mathbb{D}_{MSP_{k'}}$, by the verification in Section D.5, we have an appropriate simple$^+$ paths in $\mathbb{D}_{MSP_k}$ connecting P_j and $\bar{P}_j$, and another appropriate simple$^+$ paths in $\mathbb{D}_{MSP_{k'}}$ connecting $\bar{P}_j'$ and P_j'. Combining these three simple$^+$ paths, we have an appropriate simple$^+$ path in $\mathbb{D}_S \cup [\cup_{k=1}^t \mathbb{D}_{MSP_k}]$ connecting P_j and P_j'.

Furthermore, by the construction of the appropriate simple$^+$ path above and verifications in Sections D.3 and D.5, we can prove that the union $\mathbb{D}_S \cup [\cup_{k=1}^t \mathbb{D}_{MSP_k}]$ satisfies the no-detour property. Therefore, the union $\mathbb{D}_S \cup [\cup_{k=1}^t \mathbb{D}_{MSP_k}]$ is a connected$^+$ domain.