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ASPHERICITY OF SYMMETRIC PRESENTATIONS

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Abstract

Using the notion of relative presentation due to Bogley and Pride, we give a new proof of a theorem of Prishchepov on the asphericity of certain symmetric presentations of groups. Then we obtain further results and applications to topology of low-dimensional manifolds.

1. Relative presentations

This section is devoted to recall some definitions and results on the asphericity of relative presentations according to [**2**].

A *relative presentation* is a triple $P = \langle H, X : R\rangle$ such that:

- H is a group,
- $X = \{x_1, x_2, \dots\}$ is a set of elements,
- R is a set of words in the alphabet $H \cup X \cup X^{-1}$ of the form

$$x_1^{\epsilon_1} h_1 x_2^{\epsilon_2} h_2 \cdots x_n^{\epsilon_n} h_n$$

  where $x_i \in X$, $\epsilon_i = \pm 1$ and $h_i \in H$.

We always assume that R contains no proper powers, and that the words are *cyclically reduced* in the following sense: if $h_i = 1$ and $x_i = x_{i+1}$ (subscripts mod n), then $\epsilon_i = \epsilon_{i+1}$. The elements of $X \cup X^{-1}$ are also called X-*symbols*. Let $F(X)$ denote the free group on the set X. Then the *group* $G(P)$ *defined* by the relative presentation P is the quotient of the free product $H * F(X)$ by the normal closure of R.

Let R^* be the set of all cyclic permutations of words from $R \cup R^{-1}$ which begin with X-symbols. Let us consider the bar operator $\bar{\ }$ on R^* defined as follows. For any word $w \in R^*$, we write it in the form $w = uh$, where $h \in H$ and u begins and ends with X-symbols. Then we set $\overline{w} = u^{-1}h^{-1} \in R^*$. Note that $\overline{\overline{w}} = w$, and $\overline{w} = w$ if and only if w has the form $uh_1u^{-1}h_2$, where u begins and ends with X-symbols and h_1, h_2 are elements of order 2 in H. The relative presentation $P = \langle H, X : R\rangle$ is

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slender if $\{w\}^* \cap R = \{w\}$, for any $w \in R$. The relative presentation P is *orientable* if it is slender and no element of R has a cyclic permutation fixed under the bar operator (i.e., no element of R is a cyclic permutation of its inverse).

A *picture* $\mathbf{P}$ is a finite collection of pairwise disjoint discs $\{\Delta_1, \dots, \Delta_m\}$ in the interior of a disc D^2, together with a finite collection of pairwise disjoint simple arcs $\{\alpha_1, \dots, \alpha_n\}$ properly embedded in the closure of $D^2 \setminus \bigcup_{i=1}^m \Delta_i$. For any $i = 1, \dots, m$, the *corners* of Δ_i are the closures of the connected components of $\partial\Delta_i \setminus \bigcup_{j=1}^n \alpha_j$. The *regions* of $\mathbf{P}$ are the closures of the connected components of $D^2 \setminus (\bigcup_{i=1}^m \Delta_i \cup \bigcup_{j=1}^n \alpha_j)$. An *inner region* of $\mathbf{P}$ is a simply connected region of $\mathbf{P}$ which does not meet ∂D^2. The picture $\mathbf{P}$ is *connected* if $\bigcup_{i=1}^m \Delta_i \cup \bigcup_{j=1}^n \alpha_j$ is connected, and is *spherical* if $m \geq 1$ and $(\bigcup_{j=1}^n \alpha_j) \cap \partial D^2 = \emptyset$.

A picture $\mathbf{P}$ is said to be *labelled* if:

- Each arc is equipped with a normal orientation, indicated by a short arrow meeting the arc transversely, and labelled by an X-symbol.
- Each corner of $\mathbf{P}$ is oriented anticlockwise with respect to the disk Δ_i in whose boundary it is contained, and labelled by an element of the group H.

Let c be a corner of a disc Δ_i in the labelled picture $\mathbf{P}$. Then we denote by $w(c)$ the word obtained by reading in anticlockwise order the labels on the arcs and corners meeting $\partial\Delta_i$ beginning with the label on the arc which follows c. A label x on an arc gives the generator x or x^{-1} if its normal orientation agrees or not with the reading sense.

A connected spherical labelled picture $\mathbf{P}$ is said to be a *picture over the relative presentation* $P = \langle H, X : R \rangle$ if the following conditions are satisfied:

- For any corner c of $\mathbf{P}$, the word $w(c)$ belongs to R^*.
- If $h_1, h_2, \dots, h_{\gamma(i)}$ is the sequence of the corner labels encountered in a clockwise traversal of the boundary of an inner region of $\mathbf{P}$, then $h_1 h_2 \cdots h_{\gamma(i)} = 1$ in H.

Remark. An ordinary group presentation can be considered as the particular case of a relative presentation $P = \langle H, X : R \rangle$ for which $H = 1$ (hence, there are no labels at corners of a picture over P).

A *dipole* in a picture $\mathbf{P}$ over a relative presentation P consists of a pair of corners c and c' with an arc α connecting the beginning of one corner with the end of the other such that c and c' belong to the same region of $\mathbf{P}$ and $w(c') = \overline{w(c)}$.

A relative presentation P is said to be (*combinatorially*) *aspherical* if every nonempty connected spherical picture $\mathbf{P}$ over P contains a dipole.

To complete the section, we illustrate a connection between the notion of aspherical relative presentation and the concept of topological asphericity.

Let $P=\langle H, X : R\rangle$ be a relative presentation for a group G. If $K(H,1)$ is a Eilenberg-MacLane space for the group H, then consider the wedge

$$K' = K(H,1) \vee (\vee_{x\in X}\mathbb{S}^1_x).$$

For each $w \in R$, let $\phi_w : \mathbb{S}^1_w \to K'$ be an attaching map which represents the word $w \in H * F(X) \cong \pi_1(K')$. Then the *canonical complex* $K(P)$ associated to P is the CW-complex

$$K(P) = K' \cup (\bigcup_{w\in R} D^2_w)$$

where D^2_w is a 2-cell attached to K' via ϕ_w. By construction, we have the isomorphism $G \cong \pi_1(K(P))$.

Theorem 1. *If $P = \langle H, X : R\rangle$ is an orientable (combinatorially) aspherical relative presentation for a group G, then the canonical complex $K(P)$ is topologically aspherical, that is, $K(P) = K(G,1)$.*

2. A family of symmetric presentations

Prishchepov [**17**] considered a family of symmetric presentations of groups depending on a finite number of positive integers:

$$\begin{aligned} P(r,n,k,s,q) = \langle x_1,\ldots,x_n : \prod_{j=1}^{r} x_{i+(j-1)q} \\ = \prod_{j=1}^{s} x_{i+k-1+(j-1)q} \quad (i=1,\ldots,n)\rangle \end{aligned}$$

where the subscripts are taken modulo n, $r \geq 2$, and $1 \leq q < n$. He gave arithmetic conditions on the parameters (r,n,k,s,q) which imply the asphericity of the presentations $P(r,n,k,s,q)$ (see Section 3). Further results on the groups defined by these presentations and their generalizations can be found in [**8**].

The family $P(r,n,k,s,q)$ is very interesting from a topological point of view, and contains many classes of symmetric presentations, previously considered by several authors.

- The presentations $P(r,n,r+1,1,1)$ define the *Fibonacci groups* $F(r,n)$, $r\geq 2$ and $n\geq 3$ (see for example [**14**]). The group $F(2,2m)$, $m\geq 2$, is the fundamental group of the m-fold cyclic covering of the 3-sphere branched over the figure-eight knot, as proved in [**11**]. The groups $F(n-1,n)$, $n\geq 3$, are fundamental groups of Seifert fibered 3-manifolds [**5**].
- The presentations $P(r,n,2,r-1,2)$ define the *generalized Sieradski groups* $S(r,n)$, $r\geq 2$, $n\geq 2$, introduced and geometrically studied in [**6**]. The group $S(r,n)$ is the fundamental group of the n-fold cyclic covering of the 3-sphere branched over the torus knot of type $(2r-1,2)$, as shown in the quoted paper.
- The presentations $P(r,n,k+r,1,1)$ and $P(r,n,r+1,s,1)$ define the groups $F(r,n,k)$ and $H(r,n,k)$, respectively, for any $r\geq 2$, $n\geq 3$, and $k,s\geq 1$. These groups were introduced in [**4**] as natural generalizations of the Fibonacci groups $F(r,n)$. A lot of topological and algebraic results on these classes of groups can be found in the quoted paper and in [**18**].
- The presentations $P(2,n,2,1,t)$ define the groups $H(n,t)$ studied in [**16**] and [**10**]. The group $H(n,t)$ has infinite abelianization if and only if $n\equiv 0 \pmod 6$ and $t\equiv 2 \pmod 6$. The group $H(n,t)$ is perfect if and only if either $t=1$ or n is coprime to 6 and $t\equiv 2 \pmod 6$.

The following theorem, due to Gilbert and Howie, gives arithmetic conditions for the asphericity of groups $H(n,t)$.

Theorem 2. *Suppose that* $(n,t)\notin\{(8,3),(9,4),(9,7)\}$. *Then the group* $H(n,t)$ *is aspherical, except for the values of* (n,t) *listed below:*

(1) $(n,0)$, *for* $n\geq 2$,
(2) $(n,2)$, *for* $n\geq 3$,
(3) $(n,n-1)$, *for* $n\geq 3$,
(4) $(2t-1,t)$, *for* $t\geq 3$,
(5) $(2t-2,t)$, *for* $t\geq 3$, *and*
(6) $(n,t)=(6,3),(7,3),(7,5),(9,3)$, *or* $(9,6)$.

- The presentations $P(2,n,k+1,1,m)$ define the groups $G_n(m,k)$, introduced in [**7**], and successively studied in [**1**]. They are natural generalizations of the Gilbert-Howie groups as $G_n(m,1)=H(n,m)$. The group $G_n(m,k)$ is said to be *strongly irreducible* if the parameters satisfy the following conditions: $0<m<k<n$, $\gcd(n,m,k)=1$, $\gcd(n,k)>1$, and $\gcd(n,k-m)>1$; otherwise, $G_n(m,k)$ is proved to be cyclic, a non-trivial free product, or a Gilbert-Howie group.

The following theorem, due to Bardakov and Vesnin, gives arithmetic conditions for the asphericity of strongly irreducible groups $G_n(m,k)$.

Theorem 3. *Let $G_n(m,k)$ be a strongly irreducible group. Then $G_n(m,k)$ is aspherical if all the following conditions are not satisfied:*

(1) *There exists an integer $\ell \geq 1$ such that n divides $\ell(2k-m)$ and*

$$\frac{1}{\ell} + \frac{\gcd(n,k)}{n} + \frac{\gcd(n,k-m)}{n} > 1.$$

(2) $n = k + m$.

(3) $n = 2(k-m)$ *and* $\gcd(n,k) \leq \frac{n}{2}$.

(4) $n = 2k$ *and* $\gcd(n,k-m) < \frac{n}{2}$.

3. Asphericity

The following theorem, due to Prishchepov, gives arithmetic conditions for the asphericity of the presentations $P(r,n,k,s,q)$.

Theorem 4. *Let $P(r,n,k,s,q)$ be the symmetric presentation defined in Section 2, where either $r > 2s > 0$ or $s > 2r > 0$. Let $A = k-1$, $B = k-1-(r-s)q$, and suppose that one of conditions (i), (ii) and (iii) holds:*

(i) *n does not divide any of* $3A$, $4A$, $5A$, $2B$, $B \pm A$, $B \pm 2A$, $B+3A$, $2B+A$.

(ii) *n does not divide any of* $3B$, $4B$, $5B$, $2A$, $A \pm B$, $A \pm 2B$, $A+3B$, $2A+B$.

(iii) *n does not divide any of* $2A$, $3A$, $2B$, $3B$, $A \pm B$, $2B+A$, $2A+B$.

Then the presentation $P(r,n,k,s,q)$ is aspherical. In this case, the group defined by $P(r,n,k,s,q)$ is torsion-free and infinite.

We now give a new proof of Theorem 4 by using the concept of relative presentation. We shall proceed as follows. Extending a symmetrically presented group by a finite cyclic group which cyclically permutes the set of generators and the set of relators, one obtains a group defined by a one-relator relative presentation over the finite cyclic group in question. The theory of aspherical relative group presentations, as developed by Bogley and Pride [**2**], applies to this set-up, there being an equivalence between relative asphericity of the relative presentation and asphericity of the original symmetric presentation. Let θ denote the automorphism of $P(r,n,k,s,q)$ which permutes cyclically the generators, i.e., $\theta(x_i) = x_{i+1}$ (subscripts mod n). Let us consider the split extension of $P(r,n,k,s,q)$ by $\mathbb{Z}_n = \langle \theta : \theta^n = 1 \rangle$. If we substitute relations $\theta^{-i} x_1 \theta^i = x_{i+1}$ into those of $P(r,n,k,s,q)$ and set $y^{-1} = x_1 \theta^{-q}$, then the split extension is generated by θ and y and has a presentation

$$Q(r,n,k,s,q) = \langle \theta, y : \theta^n = 1, \quad y^s\theta^{k-1} = \theta^{k-1-(r-s)q}y^r \rangle.$$

We can regard $Q(r,n,k,s,q)$ as a relative presentation in the sense of Bogley and Pride, that is,

$$Q(r,n,k,s,q) = \langle H, y : y^s\theta^A = \theta^B y^r \rangle$$

where $H = \langle \theta : \theta^n = 1 \rangle$, $A = k-1$ and $B = k-1-(r-s)q$.

Lemma 5. *If the relative presentation $Q(r,n,k,s,q)$ is aspherical, then the ordinary presentation $P(r,n,k,s,q)$ is aspherical.*

Proof: Let $\mathbf{P}$ be a spherical picture over the ordinary presentation $P(r,n,k,s,q)$. Then $\mathbf{P}$ contains discs Δ_i corresponding to relations

$$\left(\prod_{j=1}^{r} x_{i+(j-1)q} \right) \left(\prod_{j=1}^{s} x^{-1}_{i+k-1+(s-j)q} \right) = 1$$

as shown in Figure 1.

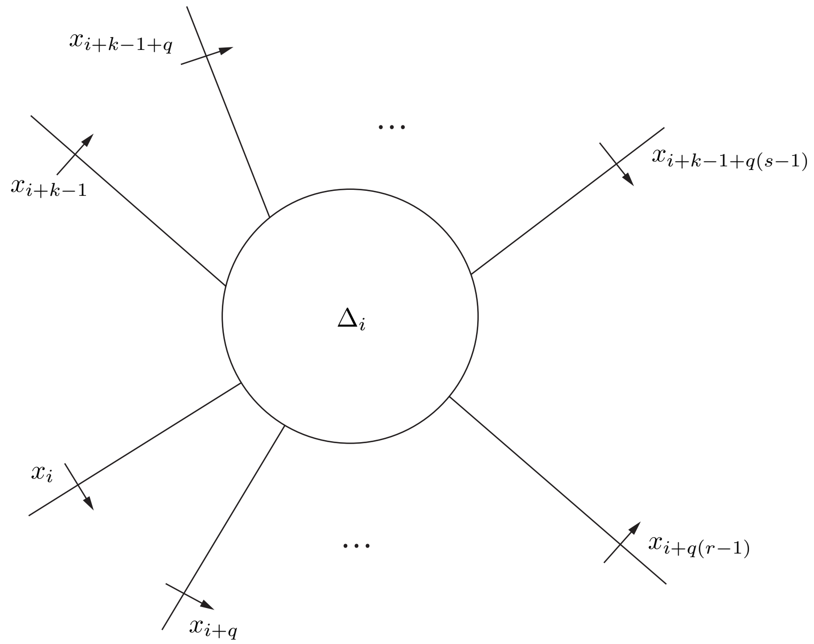

Figure 1. An inner disc in a spherical picture $\mathbf{P}$ over $P(r,n,k,s,q)$.

Here we have no labels at the corners since we regard the ordinary presentation $P(r,n,k,s,q)$ as a relative presentation with $H=1$. Let us replace each inner disc Δ_i by a picture Σ_i over $Q(r,n,k,s,q)$ considered as an ordinary presentation (see Figure 2). Here we have replaced arcs labelled by x_{i+jq} (and similarly for $x^{-1}_{i+k-1+jq}$) by sequences of arcs using relations

$$x_{i+jq} = \theta^{-(i+jq-1)} x_1 \theta^{i+jq-1} = \theta^{-(i+jq-1)} y^{-1} \theta^{i+(j+1)q-1}.$$

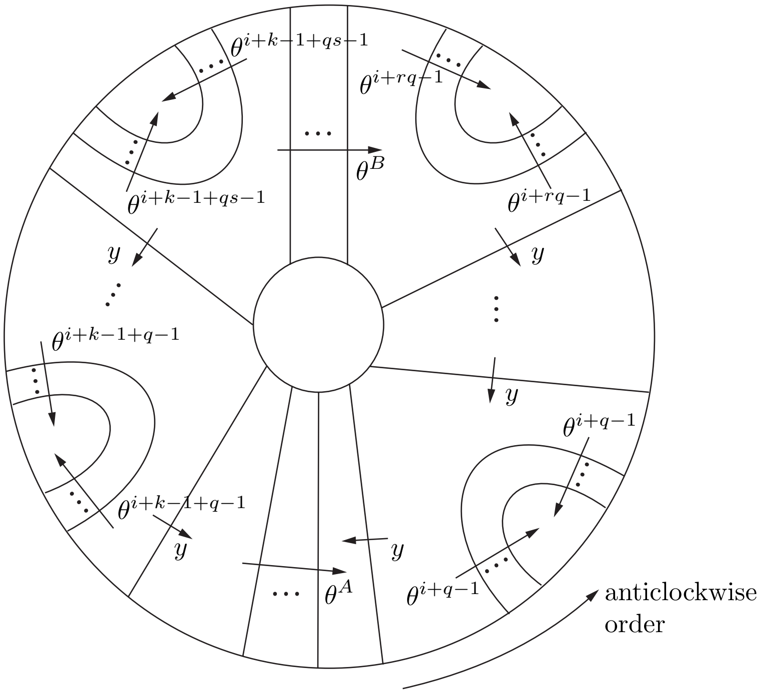

Figure 2. The picture Σ_i over the ordinary presentation $Q(r,n,k,s,q)$.

Along the boundary of Σ_i we get the relation

$$\Big(\prod_{j=1}^{r} y^{-1}\theta^{i+jq-1}\theta^{-(i+jq-1)}\Big)\theta^{-B}$$

$$\Big(\prod_{j=1}^{s} \theta^{i+k-1+(s+1-j)q-1}\theta^{-(i+k-1+(s+1-j)q-1)}y\Big)\theta^{A} = 1$$

which is equivalent to the i-th relation of $P(r,n,k,s,q)$. Along the boundary of the interior disc in Σ_i we get the relation

$$y^{-r}\theta^{-B}y^s\theta^A = 1$$

which is a relation of $Q(r,n,k,s,q)$. The arcs of Σ_i having both ends on $\partial\Sigma_i$ can be made into floating circles. These circles can be removed from the resulting picture. Furthermore, we will replace all other arcs with θ-labels by corner labels on the disc as shown in Figure 3. We get again the relation $y^{-r}\theta^{-B}y^s\theta^A = 1$.

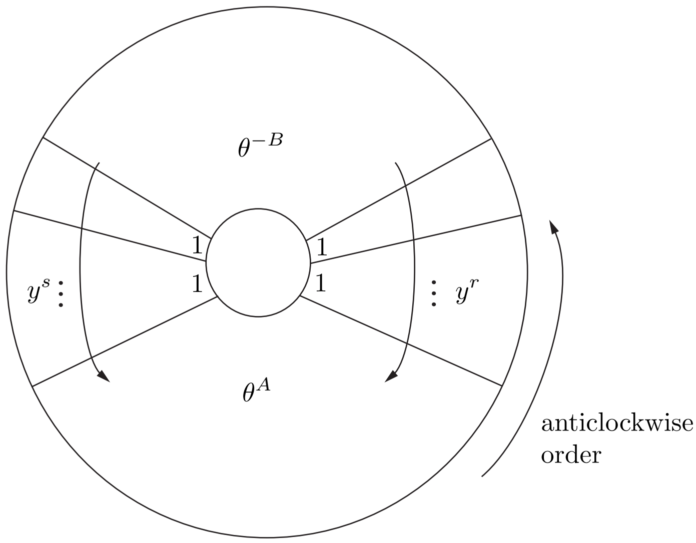

Figure 3. A picture **Q** over the relative presentation $Q(r,n,k,s,q)$.

Repeating the same construction for each disc Δ_i of **P** yields a picture **Q** over the relative presentation $Q(r,n,k,s,q)$. By the assumption of asphericity for $Q(r,n,k,s,q)$, the picture **Q** must contain a dipole, i.e., a pair of opposite oriented discs connected by an arc which define pairwise inverse words (see Figure 4).

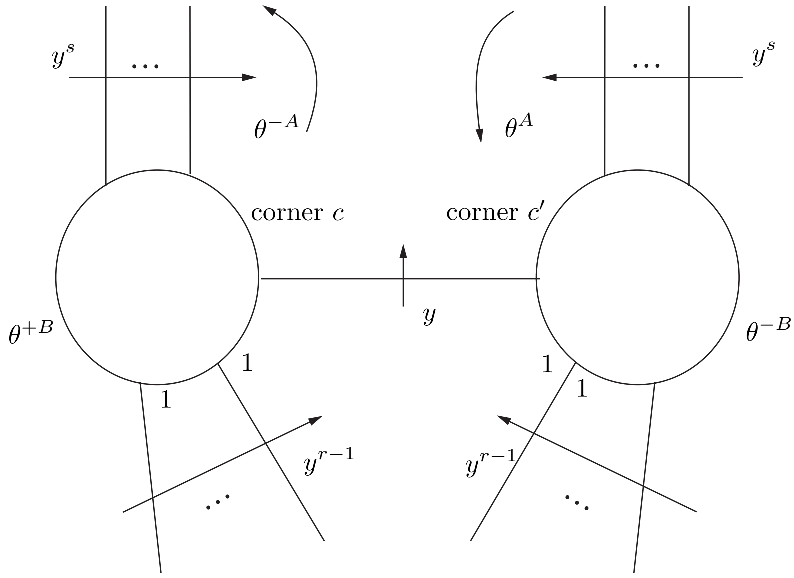

Figure 4. A dipole in the picture **Q** over the relative presentation $Q(r, n, k, s, q)$.

It is easy to see that any such dipole in **Q** arises from a pair of identical but oppositely oriented discs in **P** connected by an arc with label x_i for some i. Moreover, two bridge moves in **P** produce a cancelling pair of discs. This means that if **Q** has a pair of cancelling discs, then **P** has a pair of cancelling discs, too. Thus, the initial picture **P** must contain a dipole. Therefore, any nonempty spherical picture over $P(r, n, k, s, q)$ is equivalent to one having two fewer discs, hence this presentation is aspherical by induction. □

To study the asphericity of the relative presentation

$$Q(r, n, k, s, q) = \langle H, y : y^s\theta^A = \theta^B y^r\rangle$$

where $H = \langle \theta : \theta^n = 1\rangle$, we use the following algebraic criterion, due to Prishchepov, which is stated here in terms of relative presentations.

Theorem 6. *Let G be a group defined by the relative presentation*

$$\langle H, y : y^s\, d = \, a\, y^r \rangle$$

for some group H, where either $r > 2s > 0$ or $s > 2r > 0$. Then G is aspherical if one of conditions (i), (ii) and (iii) holds in H:

$$\textit{(i)} \begin{cases} a \;\; \textit{is of order at least } 3 \\ d \;\; \textit{is of order at least } 6 \\ ad^{\pm 1} \neq 1,\; ad^{\pm 2} \neq 1,\; ad^3 \neq 1,\; a^2 d \neq 1 \end{cases}$$

$$\textit{(ii)} \begin{cases} a \;\; \textit{is of order at least } 6 \\ d \;\; \textit{is of order at least } 3 \\ da^{\pm 1} \neq 1,\; da^{\pm 2} \neq 1,\; da^3 \neq 1,\; d^2 a \neq 1 \end{cases}$$

$$\textit{(iii)} \begin{cases} a \;\; \textit{is of order at least } 4 \\ d \;\; \textit{is of order at least } 4 \\ da^{\pm 1} \neq 1,\; da^2 \neq 1,\; d^2 a \neq 1. \end{cases}$$

In these cases, y is of infinite order in G and does not commute with any non-identity element of H.

We now apply Theorem 6 to our case where $a = \theta^B$, $d = \theta^A$, $A = k-1$ and $B = k - 1 - (r - s)q$. One can directly verify that cases (i), (ii) and (iii) of Theorem 6 produce the corresponding ones in the statement of Theorem 4. Finally, recall that the group presented by $Q(r, n, k, s, q)$ is infinite if and only if the group presented by $P(r, n, k, s, q)$ is infinite.

4. Topological results

Throughout the section let $G = G(r, n, k, s, q)$ denote the group defined by the symmetric presentation $P = P(r, n, k, s, q)$, and let $K = K(P)$ be the canonical 2-complex associated to P.

The following results were proved in [**8**].

Theorem 7. *Suppose that $r+s$ (≥ 3) is odd, and n (≥ 3) is odd and coprime with $2(k-1)+q(s-r)$. Then the Prishchepov group $G(r, n, k, s, q)$ cannot be the fundamental group of a hyperbolic 3-orbifold (in particular, a closed 3-manifold) of finite volume.*

Theorem 8. *The abelianization of the group $G(r, n, k, s, q)$ is infinite if and only if one of the following conditions holds:*

(i) $s = r$,
(ii) *there exists* $m \in \mathbb{Z}$, $m > 1$, $m \backslash n$, m *does not divide* qs, *with* $qs \equiv qr$ *(mod* m*), and* $k \equiv 1$ *(mod* m*),*
(iii) *there exists* $m \in \mathbb{Z}$, $m > 1$, $m \backslash n$, m *does not divide* qs, *with* $qs \equiv -qr$ *(mod* m*), and* $k + qs \equiv 1 + m/2$ *(mod* m*),* m *even.*

In the finite case, the natural HNN extension of $G(r, n, k, s, q)$ *is a* 3*-knot group.*

Recall that a *hyperbolic* 3*-orbifold* is the quotient space $\mathbb{H}^3/\Gamma$, where $\mathbb{H}^3$ is the hyperbolic 3-space and Γ is a discrete group of isometries of $\mathbb{H}^3$ (in particular, if Γ is torsion-free, then we get the notion of *hyperbolic* 3*-manifold*). A 3-*knot* is a locally flat topological embedding of $\mathbb{S}^3$ into $\mathbb{S}^5$.

Proposition 9. *Let* $P = P(r, n, k, s, q)$ *be orientable and satisfy one of the conditions in the statement of Theorem 4. Then the Prishchepov group* $G = G(r, n, k, s, q)$ *cannot be the fundamental group of a closed connected orientable* 3*-manifold.*

Proof: Suppose, on the contrary, that M^3 is a closed connected orientable 3-manifold such that $\pi_1(M) \cong G$. By Theorem 1 the canonical 2-complex $K = K(P)$ is aspherical, i.e., $K = K(G, 1)$. Since G is torsion-free, the prime factors of M are either aspherical or isomorphic to $\mathbb{S}^1 \times \mathbb{S}^2$ (or counterexamples to the Poincaré conjecture). So if G has k freely indecomposable free factors, then we have

$$1 = \chi(K) = \chi(G) = \chi(M) + 1 - k \leq 0$$

which is a contradiction. □

Theorem 10. *Let* $G = G(r, n, k, s, q)$ *be as in Proposition 9. Then there exists a smooth closed orientable spin* 4*-manifold* M^4 *such that:*

(1) $\chi(M) = 2$, $\pi_1(M) \cong G$, $\pi_2(M) \cong \overline{\mathrm{Ext}}^2_\Lambda(\mathbb{Z}, \Lambda) \cong \overline{H}^2(G; \Lambda)$, *where* $\Lambda = \mathbb{Z}[G]$ *is the integral group ring of* G *(for a right* Λ*-module* A*, the symbol* $\overline{A}$ *represents the associated left* Λ*-module induced by the canonical anti-automorphism* $-\colon \Lambda \to \Lambda$ *sending* g *to* g^{-1}*);*

(2) *M bounds a smooth compact 5-manifold $N^5 \subset \mathbb{R}^5$ such that $N \simeq K(G,1)$;*
(3) *The first k-invariant and the signature of M vanish;*
(4) *The integral homology of the universal cover $\widetilde{M}$ of M is $H_1(\widetilde{M}) \cong H_4(\widetilde{M}) \cong 0$, $H_2(\widetilde{M}) \cong \overline{H}^2(G;\Lambda)$, and $H_3(\widetilde{M}) \cong Z^{e(G)-1}$, where $e(G)$ is the number of ends of G.*

If $e(G) > 1$, then G is a nontrivial free product. If $e(G) = 1$ and $H^2(G;\Lambda)$ is finitely generated, then $\widetilde{M}$ is homotopy equivalent to $\mathbb{S}^2$ (hence $\pi_2(M) \cong \overline{H}^2(G;\Lambda) \cong \mathbb{Z}$ and $H^1(G;\Lambda) \cong 0$). If the abelianization G^{ab} of G is finite (see Theorem 8), then M^4 is a rational homology 4-sphere, and there is an epimorphism from $\pi_2(M)$ onto $H_2(M;\mathbb{Z}) \cong G^{\mathrm{ab}}$. If further $H^2(G;\Lambda)$ is finitely generated, then G^{ab} is finite cyclic (possibly null).

Proof: Embed the canonical 2-complex $K = K(P)$ into $\mathbb{R}^5$, and define M^4 to be the boundary of a regular neighborhood N^5 of K in $\mathbb{R}^5$. Since N collapses onto K, we have $N \simeq K(G,1)$ and $\chi(N) = 1$. One easily checks $\chi(M) = 2\chi(N) = 2$. By [**13**] and Corollary 5.2, p. 116, of [**15**] there are isomorphisms $\pi_2(M) \cong \overline{\mathrm{Ext}}^2_\Lambda(\mathbb{Z},\Lambda) \cong \overline{H}^2(G;\Lambda)$. Since G has cohomological dimension ≤ 2, we have $H^3(G;\pi_2(M)) \cong 0$, hence the first k-invariant of M vanishes. Furthermore, M is spin and its signature is zero as M embeds in $\mathbb{R}^5$. The integral homology of $\widetilde{M}$ is given by $H_1(\widetilde{M}) \cong 0$, $H_2(\widetilde{M}) \cong \pi_2(M)$, $H_3(\widetilde{M}) = H_3(M;\Lambda) \cong \overline{H}^1(M;\Lambda) \cong \overline{H}^1(G;\Lambda) \cong \mathbb{Z}^{e(G)-1}$, and $H_4(\widetilde{M}) \cong 0$ (recall that G is infinite). If the group G has more than one end, then it is isomorphic to a nontrivial generalized free product with amalgamation $U *_W V$ or an HNN extension $U *_W \phi$, where W is finite and $U \neq W \neq V$ (see for example [**12**, p. 11]). Since G is torsion free, we must have $W = 1$, hence G is isomorphic to either $U * V$ or $U * \mathbb{Z}$, where $U, V \neq 1$. Thus G is a nontrivial free product.

If $e(G) = 1$ and $H^2(G;\Lambda)$ is finitely generated, then $H_*(\widetilde{M};\mathbb{Z})$ is finitely generated. By Corollary C, p. 23, of [**12**], M is either aspherical or $\widetilde{M}$ is homotopy equivalent to $\mathbb{S}^2$ or $\mathbb{S}^3$ or $\pi_1(M)$ is finite. The first case cannot occur since otherwise $\chi(M) = \chi(G) = 1$ contradicts $\chi(M) = 2$. By Theorem 10 (i), p. 23, of [**12**], $\widetilde{M}$ is homotopy equivalent to $\mathbb{S}^3$ if and only if $e(G) = 2$ and $\chi(M) = 0$. Thus it remains only the case $\widetilde{M} \simeq \mathbb{S}^2$, hence $\pi_2(M) \cong \overline{H}^2(G;\Lambda) \cong \mathbb{Z}$ and $H^1(G;\Lambda) \cong 0$.

If the abelianization G^{ab} of G is finite, then $\chi(M) = 2 = 2-2\beta_1(M)+\beta_2(M) = 2 + \beta_2(M)$ implies that $\beta_2(M) = 0$. Thus M is a rational homology 4-sphere. Since G^{ab} is finite, we have also $\beta_1(K) = 0$. Then $\chi(K) = 1 = 1 + \beta_2(K)$ gives $\beta_2(K) = 0$, hence $H_2(K;\mathbb{Z}) \cong 0$. It follows that $H_2(G;\mathbb{Z}) \cong 0$ by the Hopf formula. In fact, this formula states that the number of generators of $H_2(G;\mathbb{Z})$ is $\alpha - \beta + \gamma$, where β is the number of generators and α the number of relations of G while γ is the rank of $H_1(G) = G^{\text{ab}}$ (see for example [**3**, p. 46]). In our case, we have $\alpha = \beta = n$ and $\gamma = 0$. Let us consider the terms of low degree of the spectral sequence of the universal cover of M, that is, the exact sequence

$$\cdots \longrightarrow H_2(\widetilde{M}) \cong \pi_2(M) \longrightarrow H_2(M) \longrightarrow H_2(G) \cong 0.$$

Since $H_2(M) \cong H^2(M) \cong FH_2(M) \oplus TH_1(M) \cong G^{\text{ab}}$, we have an epimorphism from $\pi_2(M) \cong \overline{H}^2(G;\Lambda)$ onto G^{ab}. Farrell [**9**] has shown that if G is finitely presentable and has an element of infinite order, then $H^2(G;\Lambda)$ is either 0, $\mathbb{Z}$, or is not finitely generated. So, if $H^2(G;\Lambda)$ is finitely generated, then G^{ab} is finite cyclic (possibly null). □

The following arises in a natural way:

Open problem. Compute $H^2(G;\Lambda)$ and determine the ends of the Prishchepov group $G = G(r,n,k,s,q)$ for arbitrary values of the parameters.

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References

[1] V. G. Bardakov and A. Yu. Vesnin, On a generalization of Fibonacci groups, (Russian), *Algebra Logika* **42(2)** (2003), 131–160; translation in: *Algebra Logic* **42(2)** (2003), 73–91.

[2] W. A. Bogley and S. J. Pride, Aspherical relative presentations, *Proc. Edinburgh Math. Soc. (2)* **35(1)** (1992), 1–39.

[3] K. S. Brown, *"Cohomology of Groups"*, Graduate Texts in Mathematics **87**, Springer-Verlag, New York-Berlin, 1982.

[4] C. M. Campbell and E. F. Robertson, On a class of finitely presented groups of Fibonacci type, *J. London Math. Soc. (2)* **11(2)** (1975), 249–255.

[5] A. Cavicchioli, Neuwirth manifolds and colourings of graphs, *Aequationes Math.* **44(2–3)** (1992), 168–187.

[6] A. Cavicchioli, F. Hegenbarth and A. C. Kim, A geometric study of Sieradski groups, *Algebra Colloq.* **5(2)** (1998), 203–217.

[7] A. Cavicchioli, F. Hegenbarth and D. Repovš, On manifold spines and cyclic presentations of groups, in: *"Knot Theory"* (Warsaw, 1995), Banach Center Publ. **42**, Polish Acad. Sci., Warsaw, 1998, pp. 49–56.

[8] A. Cavicchioli, D. Repovš and F. Spaggiari, Topological properties of cyclically presented groups, *J. Knot Theory Ramifications* **12(2)** (2003), 243–268.

[9] F. T. Farrell, The second cohomology group of G with $\mathbb{Z}_2[G]$ coefficients, *Topology* **13** (1974), 313–326.

[10] N. D. Gilbert and J. Howie, LOG groups and cyclically presented groups, *J. Algebra* **174(1)** (1995), 118–131.

[11] H. M. Hilden, M. T. Lozano and J. M. Montesinos-Amilibia, The arithmeticity of the figure eight knot orbifolds, in: *"Topology '90"* (Columbus, OH, 1990), Ohio State Univ. Math. Res. Inst. Publ. **1**, de Gruyter, Berlin, 1992, pp. 169–183.

[12] J. A. Hillman, *"The algebraic characterization of geometric 4-manifolds"*, London Mathematical Society Lecture Note Series **198**, Cambridge University Press, Cambridge, 1994.

[13] J. A. Hillman, Minimal 4-manifolds for groups of cohomological dimension 2, *Proc. Edinburgh Math. Soc. (2)* **37(3)** (1994), 455–461.

[14] D. L. Johnson, J. W. Wamsley and D. Wright, The Fibonacci groups, *Proc. London Math. Soc. (3)* **29** (1974), 577–592.

[15] S. Mac Lane, *"Homology"*, Die Grundlehren der mathematischen Wissenschaften **114**, Academic Press, Inc., Publishers, New York; Springer-Verlag, Berlin-Göttingen-Heidelberg, 1963.

[16] R. W. K. Odoni, Some Diophantine problems arising from the theory of cyclically-presented groups, *Glasg. Math. J.* **41(2)** (1999), 157–165.

[17] M. I. Prishchepov, Asphericity, atoricity and symmetrically presented groups, *Comm. Algebra* **23(13)** (1995), 5095–5117.

[18] A. Szczepański and A. Yu. Vesnin, On generalized Fibonacci groups with an odd number of generators, *Comm. Algebra* **28(2)** (2000), 959–965.

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