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LINEAR GROUPS WITH THE MAXIMAL CONDITION ON SUBGROUPS OF INFINITE CENTRAL DIMENSION

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Abstract

Let A a vector space over a field F and let H be a subgroup of $\mathrm{GL}(F,A)$. We define $\mathrm{centdim}_F H$ to be $\dim_F(A/C_A(H))$. We say that H has *finite central dimension* if $\mathrm{centdim}_F H$ is finite and we say that H has *infinite central dimension* otherwise. We consider soluble linear groups, in which the (ordered by inclusion) set of all subgroups having infinite central dimension satisfies the maximal condition.

Introduction

Let F be a field, A a vector space over F. The group $\mathrm{GL}(F,A)$ of all automorphisms of A and its distinct subgroups (the linear groups) are the oldest subjects of investigation in Group Theory. The investigation of the case when A has finite dimension over F was the initial natural step. In this case, every element of $\mathrm{GL}(F,A)$ (a non-singular linear transformation) defines a non-singular $n \times n$-matrix over F where $n = \dim_F A$. Thus, for the finite-dimensional case the theory of linear groups is exactly the theory of matrix groups. That is why the theory of finite dimensional linear groups is one of the best developed in Algebra. However, in the case when $\dim_F A$ is infinite, the situation is completely different. The study of this case requires some essential additional restrictions. The circumstances here are similar to those that appeared in the early period of development of the Infinite Group Theory. One of the most fruitful approaches here is the application of finiteness conditions to the study of infinite groups. The celebrated problem of O. Yu. Šmidt regarding an infinite group with all proper subgroups finite has determined in many respects the further development of the theory of groups with finiteness conditions (see, for example, [**CS**]). The following two valuable generalizations follow from this: the problem of S. N. Chernikov

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on groups with the minimal condition on subgroups, and the problem of R. Baer on groups with the maximal condition on subgroups. These problems have been solved under some natural restrictions related to generalized solubility. However, in the general case, these problems have no solution yet. Moreover, A. Yu. Ol'shanskiĭ [**OA**, Chapter 9] has constructed a series of his brilliant examples showing that, in general, the description of Schmidt's groups is an extremely complicated problem. We shall consider a similar problem for linear groups.

As you would expect, we are trying to employ the finiteness conditions to the study of infinite dimensional linear groups. More precisely, we want to investigate such classes of linear groups that can be extracted by applying finiteness conditions on some systems of infinite dimensional subgroups. At the beginning we need to clarify the concepts of infinite dimensional and finite dimensional linear groups. If $\dim_F A$ is finite, then everything is clear. If $\dim_F A$ is infinite, then there are several different approaches. For instance, we can consider the following one. If H is a subgroup of $\mathrm{GL}(F, A)$, then H really acts on the factor-space $A/C_A(H)$. *We say that H has finite central dimension, if* $\dim_F(A/C_A(H))$ *is finite. In this case* $\dim_F(A/C_A(H))$ *will be called the central dimension of the subgroup H and will be denoted* $\mathrm{centdim}_F(H)$.

At the risk of being overly pedantic, we remark that it is crucial that H is a subgroup of a *particular* general linear group. It is easy to construct embeddings of a group H in two general linear groups such that H has finite central dimension when it is viewed as a subgroup of the first group, and has infinite central dimension as a subgroup of the second group. Consequently, we may not speak of "the class of groups of finite central dimension". In order to avoid such kind of misunderstandings connecting with different linear representations we fix from now the concrete field F, the concrete vector space A over F, and will consider only subgroups of $\mathrm{GL}(F, A)$.

Let H has finite central dimension, then $A/C_A(H)$ is finite dimensional. Put $C = C_H(A/C_A(H))$; then, clearly, C is a normal subgroup of H and H/C is isomorphic to some subgroup of $\mathrm{GL}_n(F)$ where $n = \dim_F(A/C_A(H))$. Each element of C acts trivially in every factor of series $\langle 0\rangle \leq C_A(H) \leq A$, so that, C is an abelian subgroup. Moreover, if $\mathrm{char}\, F = 0$, then C is torsion-free; if $\mathrm{char}\, F = p > 0$, then C is an elementary abelian p-subgroup. Hence, in general, the structure of H is defined by the structure of an ordinary finite dimensional linear group H/C.

A group $G \leq \mathrm{GL}(F, A)$ is called a finitary linear group if for each element $g \in G$ the factor-space $A/C_A(g)$ has finite dimension. Finitary groups are a linear analogy of the FC-groups (the groups with finite conjugacy classes). The theory of finitary linear groups is developed rather intensively and became rich with many interesting results (see, for example, the survey [**PR**]). This is a good example of effectiveness of finiteness conditions in the study of infinite dimensional linear groups. Some other approaches that also based on the use of finiteness conditions in infinite dimensional (near to irreducible) linear groups have been realized in [**KS1**], [**KS2**].

Let $G \leq \mathrm{GL}(F, A)$ and let $\mathrm{L}_{\mathrm{icd}}(G)$ be the set of all subgroups of G having infinite central dimension. As the first expected step we will consider such linear groups G close to finite dimensional, in which the set $\mathrm{L}_{\mathrm{icd}}(G)$ is "very small" in some particular sense. According to the above analogy with the groups with finiteness conditions, the following problems logically arise:

- the study of linear groups in which every proper subgroup has finite central dimension (a linear analogy of the Šmidt's problem);
- the study of linear groups, in which the set of all subgroups having infinite central dimension satisfies the minimal condition (a linear analogy of the Chernikov's problem);
- the study of linear groups, in which the set of all subgroups having infinite central dimension satisfies the maximal condition (a linear analogy of the Baer's problem).

The first and the second problems have been solved for the locally soluble linear groups in the paper [**DEK**].

We say that *a group* $G \leq \mathrm{GL}(F, A)$ *satisfies the condition* Max-id*, if the family* $\mathrm{L}_{\mathrm{icd}}(G)$ *ordered by inclusion satisfies the maximal condition.*

In the current paper we investigate the soluble linear groups G satisfying Max-id. The general case immediately splits into the following two cases:

- the case of groups which have no finite sets of generators; and
- the case of finitely generated groups.

The first case is described in the following two theorems.

Theorem A (Theorem 2.6)**.** *Let* $G \leq \mathrm{GL}(F, A)$ *and suppose that* G *is a soluble group satisfying* Max-id*. If* G *has infinite central dimension and* $G/[G, G]$ *is not finitely generated, then* G *satisfies the following conditions:*

(1) *A has a finite series of FG-submodules*

$$\langle 0 \rangle = C_0 \leq C_1 = C \leq C_2 \cdots \leq C_n = A$$

such that $\dim_F(A/C)$ *is finite and* C_{j+1}/C_j *is a simple* FG*-module for every* j, $0 \le j \le n-1$, $Q = G/C_G(C)$ *is a Prüfer* q*-group.*

(2) $H = C_G(C_1) \cap C_G(C_2/C_1) \cap \cdots \cap C_G(C_n/C_{n-1})$ *is a nilpotent normal subgroup of* G*; moreover, if* $\operatorname{char} F = 0$*, then* H *is torsion-free; if* $\operatorname{char} F = p > 0$*, then* H *is a bounded* p*-subgroup.*

(3) $G/H \le Q \times S_1 \times \cdots \times S_{n-1}$ *where* Q *is a Prüfer* q*-group,* $q \ne \operatorname{char} F$*,* $S_1, \dots, S_{n-1}$ *are finite-dimensional irreducible linear groups. In particular,* G *has the normal subgroups* $H \le R \le V$ *such that* G/V *is finite,* V/R *is a Prüfer* q*-group,* $q \ne \operatorname{char} F$*,* $\operatorname{centdim}_F(R)$ *is finite,* R/H *is finitely generated and* V/H *is abelian.*

(4) C *has an* FG*-submodules* $B = \bigoplus_{n \in N} B_n$ *where each* B_n *is a simple* FG*-submodules having finite* F*-dimension such that* $C(\omega FG) \le B$*.*

Here ωFG denote the augmentation ideal of the group ring FG.

Theorem B (Theorem 2.9)**.** *Let* $G \le \mathrm{GL}(F, A)$ *and suppose that* G *is a soluble group satisfying* Max-id*. If* G *has infinite central dimension and* G *is not finitely generated, then* G *has a normal subgroup* S *such that* G/S *is abelian-by-finite and finitely generated but* $S/[S, S]$ *is not finitely generated.*

The case of finitely generated soluble linear groups satisfying Max-id is described in the following two theorems. In their study we will heavily use the following subgroup.

Let $G \le \mathrm{GL}(F, A)$*. Put*

$$FD(G) = \{x \in G / \langle x \rangle \text{ has finite central dimension}\}.$$

Further we will see that $FD(G)$ is a normal subgroup of G.

Let G be a group, H be a subgroup of G, and U be an H-invariant subgroup of G. We say that *a subgroup* U *satisfies the condition* Max-H, *if the family of all* H*-invariant subgroups of* U*, ordered by inclusion, satisfies the maximal condition.*

Theorem C (Theorem 2.10)**.** *Let* $G \le \mathrm{GL}(F, A)$*, and suppose that* G *is a finitely generated soluble group satisfying* Max-id*. If* G *has infinite central dimension but* $\operatorname{centdim}_F(FD(G))$ *is finite, then the following conditions hold:*

(1) *G has a normal subgroup U such that G/U is polycyclic.*

(2) *There is a number* $m \in N$ *such that* $A(x-1)^m = \langle 0 \rangle$ *for each* $x \in U$*; in particular,* U *is nilpotent.*

(3) *If* $\operatorname{char} F = 0$*, then* U *is torsion-free, if* $\operatorname{char} F = p > 0$*, then* U *is a bounded* p*-subgroup.*

(4) *If* $\langle 1\rangle = Z_0 \leq Z_1 \leq \cdots \leq Z_m = U$ *is the upper central series of* U*, then* $Z_j + 1/Z_j$ *is a noetherian* $Z\langle g\rangle$*-module for each element* $g \in G \setminus FD(G)$, $0 \leq j \leq m-1$. *In particular,* U *satisfies* Max-$\langle g\rangle$ *for each element* $g \in G \setminus FD(G)$.

Theorem D (Theorem 2.12)**.** *Let* $G \leq \mathrm{GL}(F, A)$*, and suppose that* G *is a finitely generated soluble group satisfying* Max-id. *If both* $\mathrm{centdim}_F(G)$ *and* $\mathrm{centdim}_F(FD(G))$ *are infinite, then* G *has a normal subgroup* L *satisfying the following conditions:*

(1) G/L *is abelian-by-finite.*

(2) $L \leq FD(G)$*, and* L *has an infinite central dimension.*

(3) $L/[L, L]$ *is not finitely generated.*

(4) L *satisfies* Max-$\langle g\rangle$ *for each element* $g \in G \setminus FD(G)$.

Finally, in the last part of the article, we consider the structure of soluble linear groups satisfying Max-id for some specific kinds of fields which make the structure of such groups more transparent. The following two results illustrate this situation.

Theorem E (Theorem 3.8)**.** *Let* $G \leq \mathrm{GL}(F, A)$ *and suppose that* G *is a soluble group satisfying* Max-id. *Suppose that* G *is not finitely generated. Then in each of the following cases* G *has finite central dimension:*

(1) F *is a field satisfying the following conditions: the Sylow* q*-subgroup of* $U(F)$ *is finite and non-identity for each prime* q*, and the Sylow* 2*-subgroup has order at least* 4.

(2) F *is a field of characteristic* $p > 0$ *such that the periodic part of* $U(F)$ *is finite.*

(3) F *is a finitely generated field.*

(4) F *is a finite field extension of the* p*-adic field* Q_p.

(5) F *is a finite field extension of a field* L *where* L *is the field of rational functions over a finite extension of* Q.

Here $U(F)$ denotes the multiplicative group of a field F.

Theorem F (Theorem 3.9)**.** *Let* $G \leq \mathrm{GL}(F, A)$ *and suppose that* G *is a soluble group satisfying* Max-id. *Suppose that* G *is finitely generated. Then in each of the following cases the finitary radical of* G *has finite central dimension:*

(1) F *is a field satisfying the following conditions: the Sylow* q*-subgroup of* $U(F)$ *is finite and non-identity for each prime* q*, and the Sylow* 2*-subgroup has order at least* 4.

(2) *F is a field of characteristic $p > 0$ such that the periodic part of $U(F)$ is finite.*

(3) *F is a finitely generated field.*

(4) *F is a finite field extension of the p-adic field Q_p.*

(5) *F is a finite field extension of a field L where L is the rational functions field over a finite extension of Q.*

1. Preliminary results

Lemma 1.1. *Let $G \leq \mathrm{GL}(F, A)$.*

(i) *If $L \leq H \leq G$ and $\mathrm{centdim}_F(H)$ is finite, then $\mathrm{centdim}_F(L)$ is also finite.*

(ii) *If H and L have finite central dimension, then $\mathrm{centdim}_F(\langle H, L\rangle)$ is likewise finite.*

In fact, if $\mathrm{centdim}_F(H)$ is finite, then $C_A(H)$ has a finite codimension. Thus if L is a subgroup of H, then $C_A(L) \geq C_A(H)$; so that, $C_A(L)$ has finite codimension too. If H and L have finite central dimension, then the both subspaces $C_A(H)$ and $C_A(L)$ have finite codimension. It follows that $\dim_F(A/(C_A(H)C_A(L))$ is finite too.

Corollary 1.2. *Let $G \leq \mathrm{GL}(F, A)$. Then the set*

$$FD(G) = \{x \in G/\langle x\rangle \textit{ has finite central dimension}\}$$

is a normal subgroup of G.

In fact, by Lemma 1.1 $FD(G)$ is a subgroup. Let $x \in FD(G)$, $g \in G$. Since $C_A(x^g) = C_A(x)g$, $C_A(x^g)$ also has finite codimension.

Note that $G = FD(G)$ if and only if G is a finitary linear group. *Therefore the subgroup $FD(G)$ is called the finitary radical of a linear group G.*

Lemma 1.3. *Let $G \leq \mathrm{GL}(F, A)$ and suppose that G satisfies* Max-id.

(i) *If H is a subgroup of G, then H satisfies* Max-id.

(ii) *If $H_1 < H_2 < \cdots < Hn < \cdots$ is a chain of subgroup, then each subgroup H_n has finite central dimension.*

(iii) *If H has infinite central dimension, then an ordered by inclusion set $L[H, G]$ of all subgroups containing H, satisfies the maximal condition.*

(iv) *Either every finitely generated subgroup of G has finite central dimension or G is a finitely generated group. In particular, if G is not finitely generated, it is a finitary linear group.*

Proof: (i) is obvious. To prove (ii) we note that there is a number d such that every subgroup H_n has finite central dimension for $n \geq d$. By Lemma 1.1 the subgroups $H_1, \ldots, H_{d-1}$ have also finite central dimensional; and (iii) is also obvious.

Finally, we will prove (iv). Suppose that G has no finite sets of generators. Let L be a finitely generated subgroup of G. Then $G \setminus L \neq \emptyset$. Let $a_1 \in G \setminus L$, $L_1 = \langle L, a_1 \rangle$. Since L_1 is finitely generated, $G \neq L_1$; that is, $G \setminus L_1 \neq \emptyset$. Using the similar arguments, we can construct a strictly ascending series $L < L_1 < \cdots < L_n < \cdots$ of finitely generated subgroups. By Lemma 1.1 every subgroup L_n has finite central dimensional, in particular, $\mathrm{centdim}_F(L)$ is also finite. □

Corollary 1.4. *Let G be a group satisfying* Max-id*, H a subgroup of G, and K a normal subgroup of H. If H/K does not satisfies the maximal condition, then K has finite central dimension.*

Corollary 1.5. *Let G be a group satisfying* Max-id*, H a subgroup of G and K a normal subgroup of H. If $H/K = Dr_{\lambda \in \Lambda}(H_\lambda/K)$ where $H_\lambda \neq K$ for every $\lambda \in \Lambda$, and the set Λ is infinite, then K has finite central dimension.*

Corollary 1.6. *Let G be a group satisfying* Max-id*, H a subgroup of G and K an H-invariant subgroup of G. If* $\mathrm{centdim}_F(H)$ *is infinite and $H \cap K$ satisfies the maximal condition for subgroups, then K satisfies* Max-H.

Proof: If K satisfies Max (the maximal condition for all subgroups), then all is proved. Therefore, suppose that K has a strictly ascending series

$$L_1 < \cdots < L_n < \cdots$$

of H-invariant subgroups of K. Consider the ascending chain

$$HL_1 \leq \cdots \leq HL_n \leq \cdots .$$

Suppose that there exists a positive integer m such that $HL_m = HL_n$ for all $n \geq m$. Since $H \cap K$ satisfies Max, there exists a positive integer t such that $H \cap L_t = H \cap L_n$ for all $n \geq t$. Let $d = \max\{m, t\}$ and $n \geq t$. The inclusion $L_t \leq L_n$ implies

$$L_n = L_t(H \cap L_n) = L_t(H \cap L_t) = L_t.$$

This fact contradicts to our assumption concerning the chain $\{L_n \mid n \in N\}$. This contradiction shows that the chain $\{HL_n \mid n \in N\}$ is strictly ascending. It follows that there exists a positive integer k such that HL_t has finite central dimension. By Lemma 1.1 $\mathrm{centdim}_F(H)$ is finite. □

Lemma 1.7. *Let G be a group satisfying* Max-id*, H a subgroup of G and K a normal subgroup of H. If* $\mathrm{centdim}_F(K)$ *is infinite, then H/K satisfies the maximal condition. In particular, if H is locally (soluble-by-finite), then H/K is polycyclic-by-finite.*

Lemma 1.8. *Let $G \leq \mathrm{GL}(F, A)$ and suppose that G satisfies* Max-id*. Suppose also that L and H are subgroups of G satisfying the following properties:*

- *$L = Dr_{\lambda\in\Lambda} L_\lambda$ where L_λ is a non-identity H-invariant subgroup of L for every $\lambda \in \Lambda$; and*
- *$H \cap L \leq Dr_{\lambda\in M} L_\lambda$.*

If the set $\Gamma = \Lambda\ M$ is infinite, then the subgroup H has finite central dimension.

Proof: Since Γ is infinite, it has an infinite ascending chain

$$\Gamma_1 \subseteq \Gamma_2 \subseteq \cdots \subseteq \Gamma_n \subseteq \cdots$$

of infinite subsets. Since $H \cap Dr_{\lambda\in\Lambda} L_\lambda = \langle 1\rangle$, we come to an ascending chain of subgroups

$$\langle H, L_\lambda \mid \lambda \in \Gamma_1\rangle < \langle H, L_\lambda \mid \lambda \in \Gamma_2\rangle < \cdots < \langle H, L_\lambda \mid \lambda \in \Gamma_n\rangle < \cdots .$$

There is a number d such that the subgroup $\langle H, L_\lambda \mid \lambda \in \Gamma_d\rangle$ has finite central dimension. By Lemma 1.1 H also has finite central dimension. □

Lemma 1.9. *Let $G \leq \mathrm{GL}(F, A)$ and suppose that G satisfies* Max-id*. Let H and Q be subgroups of G satisfying the following conditions:*

- *Q is a normal subgroup of H; and*
- *$H/Q = B/Q \times C/Q$.*

If the subgroups B/Q and C/Q do not satisfy the maximal condition, then the subgroup H has finite central dimension.

Proof: Since $H/B \cong C/Q$, it does not satisfies Max. Corollary 1.4 yields that $\mathrm{centdim}_F(B)$ is finite. By the same reasons $\mathrm{centdim}_F(C)$ is finite. Since $H = BC$, Lemma 1.1 implies that H has finite central dimension as well. □

Corollary 1.10. *Let $G \leq \mathrm{GL}(F, A)$ and suppose that G satisfies* Max-id*. Let H and Q are the subgroups of G satisfying the following conditions:*

- *Q is a normal subgroup of H; and*
- *$H/Q = Dr_{\lambda\in\Lambda}(L_\lambda/Q)$ where $L_\lambda \neq Q$ for every $\lambda \in \Lambda$.*

If the set Λ is infinite, then $\mathrm{centdim}_F(H)$ *is finite.*

Proof: There are two subsets Γ and Δ of Λ with the following properties: $\Gamma \cup \Delta = \Lambda$, $\Gamma \cap \Delta = \emptyset$, Γ, Δ are infinite. It follows that Γ (respectively Δ) has an infinite ascending chain

$$\Gamma_1 \subseteq \Gamma_2 \subseteq \cdots \subseteq \Gamma_n \cdots \text{ (respectively } \Delta_1 \subseteq \Delta_2 \subseteq \cdots \subseteq \Delta_n \subseteq \cdots)$$

of infinite subsets. Put $U/Q = Dr_{\lambda\in\Lambda}(L_\lambda/Q)$, $V/Q = Dr_{\lambda\in\Lambda}(L_\lambda/Q)$. Then we obtain two ascending chains of subgroups

$$\langle L_\lambda \mid \lambda \in \Delta_1\rangle < \langle L_\lambda \mid \lambda \in \Delta_2\rangle < \cdots < \langle L_\lambda \mid \lambda \in \Delta_n\rangle < \cdots$$

and

$$\langle L_\lambda \mid \lambda \in \Gamma_1\rangle < \langle L_\lambda \mid \lambda \in \Gamma_2\rangle < \cdots < \langle L_\lambda \mid \lambda \in \Gamma_n\rangle < \cdots .$$

It follows that the groups U/Q and V/Q do not satisfy Max. Since $H/Q = U/Q \times V/Q$, Lemma 1.9 gives that H has finite central dimension. □

Lemma 1.11. *Let G be a soluble subgroup of* $\mathrm{GL}(F, A)$ *and suppose that G satisfies* Max-id*. Then $G/FD(G)$ is polycyclic.*

Proof: Let $\langle 1\rangle = D_0 \le D_1 \le \cdots \le D_n = G$ be the derived series of G. If G is not finitely generated, then Lemma 1.3 yields that $G = FD(G)$. Suppose that G is finitely generated. Then G/D_{n-1} is finitely generated. If D_{j+1}/D_j is finitely generated for each j, $0 \le j \le n-1$, then G is polycyclic. Therefore, we may assume that there is a number $m \in N$ such that G/D_m is polycyclic and D_m/D_{m-1} is not finitely generated. In particular, D_m is not finitely generated and $D_m \le FD(G)$ by Lemma 1.3. □

2. The general structure of the soluble linear groups satisfying Max-id

Recall that a group G is said to be *quasicyclic or a Prüfer p-group*, where p is a prime, if $G = \langle a_n \mid a_1^p = 1, a_n + 1^p = a_n, n \in N\rangle$. A group G is said to be a *Chernikov group*, if G has a normal subgroup of finite index, which is decomposed into a direct product of finitely many Prüfer groups.

Lemma 2.1. *Let $G \le \mathrm{GL}(F, A)$ and suppose that G satisfies* Max-id*. If G has infinite central dimension and $G \ne [G, G] = D$, then either $G_{ab} = G/D$ is finitely generated or it has a finitely generated subgroup S/D such that G/S is a Prüfer p-group for some prime p.*

Proof: Suppose that G/D is not finitely generated. Let T/D be the periodic part of G/D. Choose in $Q = G/T$ a maximal $\mathbb{Z}$-independent set of elements $\{u_\lambda \mid \lambda \in \Lambda\}$. Then the subgroup $U = \langle u_\lambda \mid \lambda \in \Lambda\rangle$

is free abelian, $U = Dr_{\lambda\in\Lambda}\langle u_\lambda\rangle$, and Q/U is periodic. Assume that the set Λ is infinite. Since a free abelian group is projective (see, for example, [**FL**, Theorem 14.6]), U has a subgroup Y such that U/Y is a direct product of countable many copies of Prüfer p-groups. Then $Q/Y = U/Y \times W/Y$ for some subgroup W (see, for example, [**FL**, Theorem 21.2]), thus Q/W is a direct product of countable many copies of Prüfer p-groups. Corollary 1.10 yields that, in this case, $\mathrm{centdim}_F(G)$ is finite. This contradiction shows that Λ is finite, i.e. $r_0(Q)$ is finite.

Let V/D be a preimage of a subgroup U in G/D. Then by the selection of U the factor-group $(G/D)/(V/D) \cong G/V$ is periodic. If the set $\Pi(G/V)$ of primes which occur as orders of elements of G/V is infinite, then by Corollary 1.10 $\mathrm{centdim}_F(G)$ is finite. Hence, (G/V) is finite. We denote by S_p/V a Sylow p-subgroup of G/V for each prime p and put

$$\vartheta = \{p \mid p \in \Pi(G/V) \text{ and } S_p/V \text{ is infinite}\}.$$

Lemma 1.9 implies that $\vartheta = \{p\}$. Put $P/V = Dr_{q\neq p}S_q/V$. Then $G/P \cong S_p/V$ is infinite and P/V is finite. If we suppose that $(G/P)/(G/P)^p$ is infinite Corollary 1.10 implies that $\mathrm{centdim}_F(G)$ is finite. It shows that $(G/P)/(G/P)^p$ is finite. It follows that $S_p/V = C_p/V \times K_p/V$ where K_p/V is finite, C_p/V is a divisible p-subgroup [**KL2**, Lemma 3]. Lemma 1.9 implies that C_p/V is a Prüfer p-group. Hence, the subgroup $S/V = P/V \times K_p/V$ is finite, that is S/D is finitely generated, and G/S is a Prüfer p-group. □

Corollary 2.2. *Let $G \leq \mathrm{GL}(F, A)$, and suppose that G satisfies* Max-id. *If G has infinite central dimension, $G \neq [G, G]$, and $G/[G, G]$ is not finitely generated, then*

$$G = \cup_{n\in N} G_n, \text{ where } G_1 \leq G_2 \leq \cdots \leq G_n \leq \cdots$$

is an ascending series of normal subgroups having finite central dimension.

Proof: Let S be a normal subgroup of G such that G/S is a Prüfer p-group and $S/[G, G]$ is finitely generated. The existence of such a subgroup follows from Lemma 2.1. Then there is an ascending series

$$S = G_0 < G_1 < \cdots < G_n < \cdots$$

such that G_n/S is a cyclic subgroup of order p^n, $n \in N$, $G = \cup_{n\in N} G_n$. By Lemma 1.3 every subgroup G_n has finite central dimension. □

Lemma 2.3. *Let $G \leq \mathrm{GL}(F, A)$, and suppose that G is a finitary linear group. If G has infinite central dimension, then every subgroup of finite index of G has infinite central dimension.*

Proof: Suppose the contrary: let assume that G has a subgroup H of finite index such that $\mathrm{centdim}_F(H)$ is finite. Put $K = \mathrm{Core}_G(H)$; then K is a normal subgroup of finite index. By Lemma 1.1 K has finite central dimension. There is a finitely generated subgroup L such that $G = KL$. Since G is a finitary linear group, L has finite central dimension. Using Lemma 1.1 we obtain that G has finite central dimension; a contradiction. □

We could not found the needed references for the following form of the well known result below. Therefore, we decided to place a proof of it here.

Proposition 2.4. *Let G be a soluble group, F be a field and A a simple FG-module. If $\dim_F(A) = n$ is finite, then $G/C_G(A)$ has a normal subgroup of finite index, which is isomorphic to some subgroup of $U_1 \times \cdots \times U_n$ where U_j is isomorphic to the multiplicative group of a field K for certain finite field extension K of the field F, $1 \le j \le n$.*

Proof: We can assume that $C_G(A) = \langle 1 \rangle$; that is, we can identify G as a subgroup of $\mathrm{GL}_n(F)$ where $n = \dim_F A$. By a Maltsev's Theorem (see, for example, [**WB**, Lemma 3.5]) G has a normal abelian subgroup U of finite index. Suppose now that A is a non simple FE-module. By Clifford's Theorem (see, for example, [**WB**, Theorem 1.7]) A has a simple FU-submodule L and $A = \bigoplus_{x \in S} L_x$ for some finite subset S. Then clearly $\cap_{x \in S} C_G(L_x) = C_G(A) = \langle 1 \rangle$, and Remak's Theorem gives the imbedding $E \le Dr_{x \in S} G/C_G(L_x)$. Since L_x is a simple FU-submodule and U is abelian, there exists a finite field K extension of F such that $G/C_G(L_x)$ is isomorphic to some subgroup of the multiplicative group of K (see, for example, [**WB**, Corollary 1.3]) for each $x \in S$. □

Lemma 2.5. *Let Q be a Prüfer q-group, F a field, $\mathrm{char}\, F \ne q$, A a simple FQ-module of finite F-dimension. If $U(F)$ has a Prüfer q-subgroup, then $\dim_F(A) = 1$.*

Proof: Let $Q = \langle z_n \mid z_1^q = 1, z_{n+1}^q = z_n, n \in N \rangle$, R be a Prüfer q-subgroup of $U(F)$, $f\colon Q \to R$ an isomorphism. Denote by $\chi_n(X)$ the characteristic polynomial of a linear transformation induced on A by the element z_n, $n \in N$. Since z_n is a root of $\chi_n(X)$, $\lambda_n = f(z_n)$ is also a root of $\chi_n(X)$. It follows that there is an element $0 \ne a \in A$ such that $az_n = \lambda_n a$. In turn, it follows that an F-subspace $A(n) = \{b \in A \mid bz_n = \lambda_n b\}$ is non-zero. Since G is abelian, $A(\lambda_n)$ is an FG-submodule. Thus $A = A(\lambda_n)$. In other words, for each element $a \in A$ we have $aF\langle z_n \rangle = aF$. Since it is valid for every $n \in N$, the equation $Q = \cup_{n \in N} \langle z_n \rangle$ implies that $aFQ = aF$ for each element $a \in A$.

Since A is a simple FG-module $A = aFQ = aF$ for every $0 \neq a \in A$. It follows that $\dim_F(A) = 1$. □

Theorem 2.6. *Let $G \leq \mathrm{GL}(F, A)$, and suppose that G is a soluble group satisfying* Max-id. *If G has infinite central dimension and $G/[G, G]$ is not finitely generated, then G satisfies the following conditions:*

(1) *A has a finite series of FG-submodules*

$$\langle 0 \rangle = C_0 \leq C_1 = C \leq C_2 \leq \cdots \leq C_n = A$$

such that $\dim_F(A/C)$ is finite and C_{j+1}/C_j is a simple FG-module for every j, $0 \leq j \leq n-1$, $Q = G/C_G(C)$ is a Prüfer q-group.

(2) *$H = C_G(C_1) \cap C_G(C_2/C_1) \cap \cdots \cap C_G(C_n/C_{n-1})$ is a nilpotent normal subgroup of G. Moreover, if $\mathrm{char}\, F = 0$, then H is torsion-free; if $\mathrm{char}\, F = p > 0$, then H is a bounded p-subgroup.*

(3) *$G/H \leq Q \times S_1 \times \cdots \times S_{n-1}$ where Q is a Prüfer q-group, $q \neq \mathrm{char}\, F$, $S_1, \ldots, S_{n-1}$ are finite-dimensional irreducible linear groups. In particular, G has the normal subgroups $H \leq R \leq V$ such that G/V is finite, V/R is a Prüfer q-group, $q \neq \mathrm{char}\, F$, $\mathrm{centdim}_F(R)$ is finite, R/H is finitely generated and V/H is abelian.*

(4) *C has an FG-submodules $B = \bigoplus_{n \in N} B_n$ where B_n is a simple FG-submodules having finite F-dimension for each $n \in N$, such that $C(\omega FG) \leq B$.*

Proof: By Lemma 2.1 G has a normal subgroup S such that G/S is a Prüfer q-group for some prime q. By Lemma 1.3 G is finitary linear group. Corollary 1.4 yields that $\mathrm{centdim}_F(S)$ is finite; so that, $C = C_A(S)$ has finite codimension. Furthermore, $C_G(C) \geq S$, in particular, $G/C_G(C)$ is a Prüfer q-group. Lemma 5.1 of [**DEK**] proves that $q \neq \mathrm{char}\, F$. Since S is a normal subgroup of G, C is an FG-submodule of A and $\dim_F(A/C)$ is finite. Hence A has a finite series of FG-submodules

$$\langle 0 \rangle = C_0 \leq C_1 = C \leq C_2 \leq \cdots \leq C_n = A$$

such that $C_2/C_1, \ldots, C_n/C_{n-1}$ are simple FG-modules having finite dimension over F. Put

$$H = C_G(C_1) \cap C_G(C_2/C_1) \cap \cdots \cap C_G(C_n/C_{n-1}).$$

Then, by Remak's Theorem

$$G/H \leq G/C_G(C_1) \times G/C_G(C_2/C_1) \times \cdots \times C_G(C_n/C_{n-1}),$$

where $Q = G/C_G(C_1)$ is a Prüfer q-group, $S_j = C_G(C_{j+1}/C_j)$ is an irreducible finite dimensional linear group, $1 \leq j \leq n-1$.

The inclusion $H \leq C_G(C_1) \leq S$ implies that $\mathrm{centdim}_F(H)$ is finite. Each element of H acts trivially on every factor C_{j+1}/C_j, $0 \leq j \leq n-1$. It follows that H is a nilpotent subgroup, moreover, if $\mathrm{char}\, F = 0$, then H is torsion-free; if $\mathrm{char}\, F = p > 0$, then H is a bounded p-subgroup (see, for example, [**KW**, Proposition 1.C.3] and [**FL**, Section 43]).

Since G is soluble, by a Maltsev's Theorem (see, for example, [**WB**, Lemma 3.5]) all factor-groups $G/C_G(C_2/C_1), \ldots, G/C_G(C_n/C_{n-1})$ are abelian-by-finite. Then by Remak's Theorem G/H has a normal abelian subgroup V/H of finite index. By Lemma 1.3 and Lemma 2.3 $\mathrm{centdim}_F(V)$ is infinite. By Lemma 2.1 V/H has a finitely generated subgroup W/H such that V/W is a Prüfer q-group for some prime q. Put $R/H = (W/H)^{G/H}$. Since G/V is finite, R/H is likewise finitely generated. Lemma 1.1 yields that R has finite central dimension; that is, $E = C_A(R)$ has finite codimension. Furthermore, $C_G(E) \geq R$, in particular, $G/C_G(E)$ is a Prüfer q-group. By Lemma 1.3 G is a finitary linear group. Lemma 5.1 of [**DEK**] proves that $q \neq \mathrm{char}\, F$.

As we have already noted G has a subgroup $S \geq [G,G]$ such that $S/[G,G]$ is finitely generated and G/S is a Prüfer q-group. We have already mentioned that G is a finitary linear group. By Corollary 1.4 $\mathrm{centdim}_F(S)$ is finite. Put again $C = C_A(S)$; then $\dim_F(A/C)$ is finite. Put now $U = C_G(C)$; then $U \geq H$; so that, either $G/U = Q$ is a Prüfer q-group, or $U = G$. But in the last case, $\mathrm{centdim}_F(G)$ is finite. By Lemma 5.1 of [**DEK**] $q \neq \mathrm{char}\, F$. Let $Q = \langle z_n \mid z_1^p = 1,\ z_{n+1}^p = z_n,\ n \in N\rangle$. If z is an arbitrary element of Q, then by Maschke's Theorem (see, for example, [**WB**, Theorem 1.5]) $C = \bigoplus_{\lambda\in\Lambda} D_\lambda$, where D_λ is a simple $F\langle z\rangle$-submodule, $\lambda \in \Lambda$. In particular, either $D_\lambda(z-1) = D_\lambda$ or $D_\lambda(z-1) = \langle 0\rangle$. It implies the decomposition $C = C_C(z) \oplus C(z-1)$. Since Q is abelian, $C_C(z)$ and $C(z-1)$ are FG-submodules of C. Since G is a finitary linear group, $\dim_F(C/C_C(z))$ is finite, in particular, $C(z-1)$ is finite-dimensional for each element $z \in Q$. Put $E_1 = C(z_1 - 1)$, $L_1 = C_C(z_1)$, then E_1 is an FG-submodule of finite F-dimension. By Maschke's Theorem (see, for example, [**WB**, Theorem 1.5]) $E_1 = \bigoplus_{\gamma\in\Gamma} B_\gamma$, where $\Gamma = \{1, \ldots, n_1\}$ and B_γ is a simple FG-submodule, $\gamma \in \Gamma$. As above, $L_1 = E_2 \oplus L_2$ where $E_2 = E_1(z_2 - 1)$, $L_2 = L_1 \cap C_C(z_2)$. Again $\dim_F(E_2)$ is finite; so that, $E_2 = \bigoplus_{\sigma\in\Sigma} B_\sigma$, where $\sum = \{n_1 + 1, \ldots, n_2\}$ and B_σ is a simple FG-submodule, $\sigma \in \Sigma$. In particular, $C = (\bigoplus_{\sigma\in\Sigma\cup\Sigma} B_\sigma) \oplus C_C(z_3)$. Continue in the similar way we construct an FG-submodule $B = \bigoplus_{n\in N} B_n$ where B_n is a simple FG-submodules having finite F-dimension for every $n \in N$, and $C(\omega FG) \leq B$. $\square$

By Proposition 2.4 Q is isomorphic to a subgroup of $U(K)$, where K is some finite field extension of F. In particular, $U(K)$ has a Prüfer q-subgroup and using Lemma 2.5 we obtain

Corollary 2.7. *Let $G \leq \mathrm{GL}(F, A)$ and suppose that G is a soluble group satisfying* Max-id. *If G has an infinite central dimension and $G/[G, G]$ is not finitely generated, then there is a finite field extension K of F such that $D = A \bigotimes_F K$ has the KG-submodules $U \geq V = \bigoplus_{n \in N} E_n$ where $\dim_K(D/U)$ is finite, $\dim_K(E_n) = 1$, $n \in N$, and $U(\omega FG) \leq V$.*

Recall some needed concepts. Let A be a divisible Chernikov normal subgroup of a group G. Following B. Hartley [**HB2**] we say that A is divisible irreducible in G, if G has no any proper non-identity G-invariant divisible subgroup. It follows that every proper G-invariant subgroup of G is finite.

Let R be an integral domain, F the field of fractions of R, G a group, A an RG-module, which is a torsion-free R-module of finite R-rank. *We say that A is rationally irreducible, if $A \bigotimes_R F$ is a simple FG-module.*

Lemma 2.8. *Let G be a group having a finite series of normal subgroups*

$$\langle 1 \rangle = H_0 \leq H_1 \leq H_2 \leq \cdots \leq H_n = G,$$

every infinite factor of which is either a divisible Chernikov group or a torsion-free abelian group of finite 0-rank. Suppose also that every divisible factor of this series is divisible irreducible in G and every torsion-free factor is rationally irreducible in G. Then G has a normal nilpotent subgroup L such that G/L is abelian-by-finite.

Proof: Let $L = \cap_{0 \leq j \leq n-1} C_G(H_{j+1}/H_j)$. If $A_j = H_{j+1}/H_j$ is torsion-free, then we can consider $A_j \bigotimes_Z Q$ as a simple QG-module. Using theorems of Clifford (see, for example, [**WB**, Theorem 1.7]) and Maltsev (see, for example, [**WB**, Lemma 3.5]), we obtain that $G/C_G(H_{j+1}/H_j)$ is abelian-by-finite. Consider now the case, when $Aj = H_{j+1}/H_j$ is a divisible Chernikov group. Let R denote the ring of all integer p-adic numbers, K the field of fraction of R, R_0 the R-module F/R, $B_j = \mathrm{Hom}_R(A_j, R_0)$. Then B_j is a rationally irreducible RG-module [**HB2**, Lemma 2.1]. Using the theorems of Clifford and Maltsev, we obtain that $G/C_G(H_{j+1}/H_j)$ is abelian-by-finite. Then by Remak's Theorem

$$G/L \leq G/C_G(H_1) \times G/C_G(H_2/H_1) \times \cdots \times C_G(H_n/H_{n-1}),$$

so that G/L is abelian-by-finite. Each element of L acts trivially on every factor H_{j+1}/H_j, $0 \leq j \leq n-1$. It follows that L is a nilpotent subgroup (see, for example, [**KW**, Proposition 1.C.3]). ☐

Theorem 2.9. *Let $G \leq \mathrm{GL}(F, A)$ and suppose that G is a soluble group satisfying* Max-id. *If G has infinite central dimension and G is not finitely generated, then G has a normal subgroup S such that G/S is abelian-by-finite and finitely generated, and $S/[S, S]$ is not finitely generated.*

Proof: Let

$$\langle 1 \rangle = D_0 \leq D_1 \leq D_2 \leq \cdots \leq D_n = G$$

be the derived series of the group G. Since G is not finitely generated, there is a number t such that G/D_t is polycyclic, while D_t/D_{t-1} is not finitely generated. Put $H = D_t$. Since G/H is finitely generated, Lemma 1.3 and Lemma 1.1 show that $\dim_F(A/C_A(H))$ is infinite. The factor-group $H/[H, H]$ is not finitely generated. By Lemma 2.1 H has a finitely generated subgroup $L \geq [H, H]$ such that H/L is a Prüfer group. In other words, there is a series of normal subgroups between $[H, H]$ and G, every infinite factor of which is either a divisible Chernikov group or a torsion-free abelian group of finite 0-rank. We can assume that every divisible factor of this series is divisible irreducible in G, and every torsion-free factor is rationally irreducible in G. If it is not so, we can take the respectively refinement. Using Lemma 2.8 we obtain that G has a normal subgroup $R \geq W = [H, H]$ such that R/W is nilpotent and G/R is abelian-by-finite. Let A be a normal subgroup of G such that $R \leq A$, G/A is finite and A/R is abelian. If A/R is not finitely generated, then put $S = A$. Suppose now that A/R is finitely generated. Since G/W is not finitely generated, this means that R/W is not finitely generated. Then $(R/W)/[R/W, R/W]$ is not finitely generated (see, for example, [**RD1**, Corollary of Theorem 2.26]) as well, in particular, $R/[R, R]$ is not finitely generated. For this case we put $S = R$. □

In [**DEK**, Section 5] it has been constructed a group $G \leq \mathrm{GL}(F, A)$ satisfying the following conditions:

- $G = M \rtimes Q$ is a Charin group; that is, M is a minimal normal abelian p-subgroup, Q is a Prüfer q-group, p, q are primes, and $p \neq q$;
- G is a finitary subgroup of $\mathrm{GL}(F, A)$;
- M has a finite central dimension and Q has an infinite central dimension.

It is not hard to see, that this group is not finitely generated and satisfies Max-id. Using the same construction, we can build a finitary subgroup $G = M \rtimes Q$ of $\mathrm{GL}(F, A)$ satisfying Max-id where M is a minimal normal abelian p-subgroup, Q is the group of q-adic numbers, p, q are primes, and $p \neq q$.

In the paper [**MPP**] U. Meierfrankenfeld, R. E. Phillips and O. Puglisi considered the locally soluble finitary linear groups. In particular, they proved that locally soluble finitary linear groups is unipotent-by-abelian-by-(locally finite). Since we consider more concrete cases, we are able to get more complete description of these factors.

The next natural step is consideration of finitely generated case.

Theorem 2.10. *Let $G \leq \mathrm{GL}(F, A)$, and suppose that G is a finitely generated soluble group satisfying* Max-id. *If G has infinite central dimension but* $\mathrm{centdim}_F(FD(G))$ *is finite, then the following conditions hold:*

(1) *G has a normal subgroup U such that G/U is polycyclic.*

(2) *There is a number $m \in N$ such that $A(x-1)^m = \langle 0 \rangle$ for each $x \in U$, in particular, U is nilpotent.*

(3) *If* $\mathrm{char}\, F = 0$*, then U is torsion-free, if* $\mathrm{char}\, F = p > 0$*, then U is a bounded p-subgroup.*

(4) *If $\langle 1 \rangle = Z_0 \leq Z_1 \leq \cdots \leq Z_m = U$ is an upper central series of U, then Z_{j+1}/Z_j is a noetherian $Z\langle g \rangle$-module for each element $g \in G \setminus FD(G)$, $0 \leq j \leq m-1$.*

In particular, U satisfies Max-$\langle g \rangle$ *for each element $g \in G \setminus FD(G)$.*

Proof: Put $C = C_A(FD(G))$. Since $\dim_F(A/C)$ is finite, A has a finite series of FG-submodules $\langle 0 \rangle = C_0 \leq C_1 = C \leq C_2 \leq \cdots \leq C_m = A$ such that $C_2/C_1, \ldots, C_m/C_{m-1}$ are simple FG-modules having finite dimension over F. By a Maltsev's Theorem (see, for example, [**WB**, Lemma 3.5]) $G/C_G(C_2/C_1), \ldots, G/C_G(C_n/C_{n-1})$ are abelian-by-finite groups and hence polycyclic, because G is finitely generated. Put

$$U = C_G(C_1) \cap C_G(C_2/C_1) \cap \cdots \cap C_G(C_m/C_{m-1}).$$

Clearly, $C_G(C_1) \geq FD(G)$, so $G/C_G(C_1)$ is polycyclic by Lemma 1.1. Then the embedding

$$G/U \leq G/C_G(C_1) \times G/C_G(C_2/C_1) \times \cdots \times C_G(C_n/C_{n-1})$$

proves that G/U is polycyclic. Each element of U acts trivially in every factor C_{j+1}/C_j, $0 \leq j \leq m-1$. It follows that U is a nilpotent subgroup, moreover, if $\mathrm{char}\, F = 0$, then H is torsion-free; if $\mathrm{char}\, F = p > 0$, then H is a bounded p-subgroup (see, for example, [**KW**, Proposition 1.C.3] and [**FL**, Section 43]). The assertion (4) follows from Corollary 1.6. □

The example below helps us to illustrate these results.

Example 2.11. Let F be a field, which is not locally finite, and put $A = \bigoplus_{n\in N} A_n$ where $A_n \cong F$ for each $n \in N$. Choose in $U(F)$ an element g of infinite order and consider the following infinite matrix $\gamma = \|u_{jm}\|_{j,m\in N}$, where $u_{jj} = g^j$ for all $j \in N$ and that $u_{jm} = 0$ for $j \neq m$. Consider now the set Σ of all matrices $\alpha = \|u_{jm}\|_{j,m\in N}$ such that

$$u_{jm} = 0 \text{ if } (j,m) \notin \{(1,2),(j,j) \mid (j \in N\}.$$

If $\beta = \|v_{jm}\|_{j,m\in N}$ is the other matrix of Σ, then $\alpha\beta = \|w_{jm}\|_{j,m\in N}$ where

$$w_{jj} = u_{jj}v_{jj} \in N, \quad w_{12} = u_{11}v_{12} + u_{12}v_{22},$$
$$w_{jm} = 0 \text{ if } (j,m) \notin \{(1,2),(j,j) \mid j \in N\}.$$

It follows that $\alpha\beta \in \Sigma$. Furthermore, if $\alpha^{-1} = \|y_{jm}\|_{j,m\in N}$, then $y_{jj} = u_{jj}^{-1}$, $j \in N$, $y_{12} = u_{11}^{-1}v_{22}^{-1}u_{12}$, in particular, $\alpha^{-1} \in \Sigma$. This means, that Σ is a subgroup of $\mathrm{GL}(F,A)$. Consider now the set Φ of all matrices $\alpha = \|u_{jm}\|_{j,m\in N}$ such that $u_{jj} = 1$ for all $j \in N$, $u_{jm} = 0$ if $(j,m) \notin \{(1,2)\}$. If $\beta = \|v_{jm}\|_{j,m\in N}$ is another matrix of Φ, then $\alpha\beta = \|w_{jm}\|_{j,m\in N}$ where $w_{jj} = 1$ for all $j \in N$, $w_{12} = v_{12} + u_{12}$. It follows that $\alpha, \beta \in \Phi$. Moreover, Φ is isomorphic to the additive group of the field F. Let

$$\alpha = \|u_{jm}\|_{j,m\in N} \in \Sigma, \quad \beta = \|v_{jm}\|_{j,m\in N} \in \Phi$$

then $\alpha^{-1}\beta\alpha = \|w_{jm}\|_{j,m\in N}$ where $w_{jj} = 1$ for all $j \in N$, $w_{12} = u_{11}^{-1}u_{22}v_{12}$. It follows that Φ is a normal subgroup of Σ. Let $\tau = \|u_{jm}\|_{j,m\in N}$ such that $u_{jj} = 1$ for all $j \in N$, $u_{12} = 1$, and $u_{jm} = 0$ if $(j,m) \notin \{(1,2)\}$. Put $\Gamma = \langle \gamma, \tau \rangle$. Then $\Gamma = T \rtimes \langle \gamma \rangle$ where T is isomorphic to the subgroup of the additive group of F generated by the elements $\{g^n \mid n \in Z\}$. We can consider T as a $Z\langle\gamma\rangle$-module. Since $Z\langle\gamma\rangle$ is a noetherian ring, a cyclic $Z\langle\gamma\rangle$-module is noetherian. It is easy to see that Γ satisfies Max-id.

We have considered the easiest case. However, the construction above allows the normal unipotent subgroup T has as much as desired length of nilpotency.

Theorem 2.12. *Let $G \leq \mathrm{GL}(F,A)$, and suppose that G is a finitely generated soluble group satisfying* Max-id*. If* $\mathrm{centdim}_F(G)$ *and* $\mathrm{centdim}_F(FD(G))$ *are infinite, then G has a normal subgroup L satisfying the following conditions:*

(1) *G/L is abelian-by-finite.*
(2) *$L \leq FD(G)$, and L has an infinite central dimension.*
(3) *$L/[L,L]$ is not finitely generated.*
(4) *L satisfies* Max-$\langle g \rangle$ *for each element $g \in GFD(G)$.*

Proof: Let

$$\langle 1 \rangle = D_0 \leq D_1 \leq \cdots \leq D_n = G$$

be the derived series of G. If G is polycyclic, then $FD(G)$ is likewise polycyclic, and Lemma 1.1 proves that $\dim_F(A/C_A(FD(G))$ is finite. This contradiction shows that there is a number $m \in N$ such that G/D_m is polycyclic and D_m/D_{m-1} is not finitely generated. Repeating the arguments of the proof of Theorem 2.9, we can construct a normal subgroup L such that G/L is abelian-by-finite and $L/[L,L]$ is not finitely generated. In particular, L is not finitely generated, so that $L \leq FD(G)$ by Lemma 1.3. If we suppose that $\dim_F(A/C_A(L))$ is finite, then Lemma 1.1 yields that $\dim_F(A/C_A(FD(G)))$ is finite, because $FD(G)/L$ is finitely generated. Finally the assertion (4) follows from Corollary 1.6. □

3. The structure of the soluble linear groups with Max-id over some specific fields

The structure of finite dimensional soluble linear groups is often defined by the structure of the multiplicative group of the field, over which such groups are considered. More precisely, it is defined by not only the structure of this multiplicative group, but also by the structure of multiplicative groups of finite extensions of the base fields. So it is reasonable to expect, that the same dependence takes place also for infinite dimensional linear groups satisfying Max-id. Proposition 2.4 and Theorem 2.6 logically lead us to the consideration of the following types of fields.

If G is an abelian subgroup, then as usually, we will denote by $t(G)$ the maximal periodic subgroup of G, the periodic part of G.

Lemma 3.1. *Let G be a soluble subgroup of* $\mathrm{GL}(F,A)$ *satisfying* Max-id. *Suppose that the field F satisfies the following condition:*

(RE) *for each finite field extension E of F (in particular, for $E = F$) the subgroup $t(U(E))$ has only finite Sylow q-subgroup for each prime q.*

Then either G has finite central dimension or $G/[G,G]$ is finitely generated.

Proof: Put $D = [G,G]$, and suppose that $\dim_F(A/C_A(G))$ is infinite and G/D is not finitely generated. Lemma 2.1 yields that G has a normal subgroup $L \geq D$ such that L/D is finitely generated and G/L is a Prüfer q-group for some prime q. By Lemma 1.3 G is a finitary linear group. Corollary 1.4 yields that L has finite central dimension; so that,

$C = C_A(L)$ has finite codimension. Furthermore, $C_G(C) \geq L$; in particular, either $G = C_G(C)$ or $G/C_G(C)$ is a Prüfer q-group. In a first case, $\mathrm{centdim}_F(G)$ is finite. Hence, $G/C_G(C)$ is a Prüfer q-group. Lemma 5.1 of [**DEK**] proves that $q \neq \mathrm{char}\, F$. Since L is a normal subgroup of G, C is an FG-submodule of A. Put

$$G/C_G(C) = \langle g_n C_G(C) \mid g_1^p \in C_G(C),\, g_{n+1}^p \in g_n C_G(C),\, n \in N\rangle.$$

Since $G/C_G(C)$ is abelian, $Y_n = C_A(g_n) = C_A(\langle g_n\rangle)$ is an FG-submodule of C for each $n \in N$. By Lemma 1.3 G is a finitary linear group; so that, $\dim_F(C/Y_n)$ is finite for each $n \in N$. We have already noted that $q \neq \mathrm{char}\, F$, therefore by Maschke's Theorem (see, for example, [**WB**, Corollary 1.6]) $C/Y_n = M_1/Y_n \oplus \cdots \oplus M_k/Y_n$, where M_j/Y_n is a simple FG-submodule of finite F-dimension. By Proposition 2.4 $G/C_G(M_j/Y_n)$ is isomorphic to a subgroup of $U_1 \times \cdots \times U_n$ where U_j is isomorphic to a multiplicative group of a field E for certain finite field extension E of the field F, $1 \leq j \leq n$. Since $C_G(M_j/Y_n) \geq C_G(C)$, then either $G = C_G(M_j/Y_n)$ or $G/C_G(M_j/Y_n)$ is a Prüfer q-group. By our condition (RE) the group $U(E)$ does not have a Prüfer q-subgroup. This implies that $G = C_G(M_j/Y_n)$ for each $n \in N$. In turn, it follows that $G = C_G(C/Y_n)$. Since it is valid for each $n \in N$, $G = \cap_{n\in N} C_G(C/Y_n) = C_G(C)$. Thus we again obtain that G has finite central dimension. This contradiction proves our lemma. □

Proposition 3.2. *Let G be a soluble subgroup of* $\mathrm{GL}(F, A)$ *satisfying* Max-id *and suppose that a field F satisfies the condition (RE).*

(1) *If G is not finitely generated, then G has finite central dimension.*

(2) *If G is finitely generated, then $FD(G)$ has finite central dimension.*

Proof: Let

$$\langle 1\rangle = D_0 \leq D_1 \leq \cdots \leq D_k = G$$

be the derived series of G. If every factor of this series is finitely generated, then G satisfies Max. Therefore assume that there is a number t such that G/D_t is polycyclic, but D_t/D_{t-1} is not finitely generated. In particular, $G = \langle D_t, S\rangle$ for some finite subset S. By Lemma 3.1 $\dim_F(A/C_A(D_t))$ is finite. If G is not finitely generated, then Lemma 1.3 shows that G is a finitary group and the equation $G = \langle D_t, S\rangle$ coupled with Lemma 1.1 imply that G has finite central dimension. If G is finitely generated, then $D_t \leq FD(G)$. Since $FD(G)/D_t$ is finitely generated, Lemma 1.1 proves that $FD(G)$ has finite central dimension. □

R. M. Guralnick kindly provided us with the following results about the structure of the multiplicative groups of some types of fields.

Proposition 3.3. *Let F be a field and suppose that the Sylow q-subgroup of $U(F)$ is finite and non-identity for each prime q, and the order of the Sylow 2-subgroup is at least 4. If E is a finite field extension of F, then the Sylow q-subgroup of $U(E)$ is finite for each prime q.*

Proof: Let S_q (respectively R_q) be a Sylow q-subgroup of $U(F)$ (respectively $U(E)$). Then by Lemma 4.2 of [**GW**] R_q/S_q is finite. It follows that R_q is finite for each prime q. □

The multiplicative groups of rings considered in Proposition 3.3 have a very large (although reduced) periodic parts. In this connection, it will be interesting to consider the dual situation; that is, the case when the periodic part of the multiplicative group of a field is finite. But the example of the complex field C, which is a finite extension of the field R, shows that not every field F for which $t(U(F))$ is finite satisfies the condition (RE) (note that $t(U(R))$ has order 2, but $t(U(C))$ is divisible). Note that in this example we are dealing with the fields of characteristic zero. The following theorem shows that for fields of positive characteristic we have much better situation.

Theorem 3.4. *Let F be a field of characteristic $p > 0$ and suppose that the periodic part of $U(F)$ is finite. If E is a finite field extension of F, then the periodic part of $U(E)$ is likewise finite.*

Proof: Let P be the prime subfield of F. We may assume that E is a normal extension of F (this just enlarges E). We may also assume that extension E is separable (if not, we can replace E by $F(T)$ where $T = t(U(E))$; this is a separable Galois extension). Let G be the Galois group of this extension.

Consider $E^* = P(T)$. Then $F^* = F \cap E^* = P(F \cap T)$ and by assumption, this is a finite field. We also see that F^* is the fixed field of G acting on E^* and so E^* is a finite Galois extension of F^* and so E^* is finite, whence the periodic part of $U(E)$ is finite. □

Further, if F is a field, the multiplicative group of which is residually finite, then every Sylow q-subgroup of $U(F)$ is finite for each prime p. In connection with Proposition 3.2 the following question is naturally raised: is a Sylow q-subgroup of the multiplicative group of an arbitrary finite field extension of F finite for each prime q? In particular, is the multiplicative group of an arbitrary finite field extension of F residually finite?

R. M. Guralnick provided us with the following counterexamples for this question.

Proposition 3.5. *There exist fields F and E satisfying the following conditions:*

(1) $\operatorname{char} F = 0$.

(2) *$U(F)$ is residually finite and $t(U(F))$ is a finite subgroup of order 2.*

(3) *E is a finite field extension of F.*

(4) *$U(E)$ has a Prüfer q-subgroup for some prime q, in particular, $U(E)$ is not residually finite.*

Proof: Let q be a prime. Let S be a Prüfer q-subgroup of $U(C)$ and put $E = Q(S)$. Clearly $U(E)$ has S, in particular, $U(E)$ is not residually finite. Let F be the fixed field of E under complex conjugation. So $[E : F] = 2$.

We show that $U(F)$ is residually finite, the periodic part of $U(F)$ is finite (has order 2), but, as we have recently noted, $U(E)$ has a Prüfer q-subgroup. Since R contains F, the periodic part of $U(F)$ has order 2. So $U(F) = T \times V$ where $|T| = 2$, V is a torsion-free subgroup (see, for example, [**FL**, Theorem 27.5]). Hence it suffices to prove that V is residually finite. Since V/V^n is bounded, it is decomposed into a direct product of finite cyclic groups by Prüfer's Theorem (see, for example, [**FL**, Theorem 17.2]). If we prove that $\cap_{n\in N} V^n = \langle 1\rangle$, it will implies that V is residually finite. Let $1 \neq v \in V$. Since $v \in E$, v is algebraic over Q, so $Q(v)$ is a finite extension of Q. Then $U(Q(v))$ is a direct product of a finite cyclic subgroup and a free abelian subgroup (see, for example, [**KG**, Chapter 4, Corollary 5.7]). Thus we can choose a sufficiently large prime r such that $v \notin (Q(v))^r$. Suppose that $v \in V^r$, then there must exists an element $w \in F$ such that $v = w^r$. Then $Q(w)$ is a nontrivial abelian extension of $Q(v)$ (since E is generated over Q by roots of unity). The field $Q(w)$ must contain at least two roots of a polynomial $X^r - v$ and their ratio is an r-th root of 1. This implies that $Q(w)$ is not fixed by complex conjugation and so w is not in F. This contradiction shows that $v \notin V^r$, as required. □

In this example the periodic part of the multiplicative group of a basic field is very small —the least possible. In the following proposition the multiplicative group is periodic.

Proposition 3.6. *There exists fields F and E satisfying the following conditions:*

(1) *F and E are locally finite fields.*

(2) *For each prime r the Sylow r-subgroup of $U(F)$ is finite.*

(3) *E is a finite field extension of F.*

(4) *There exists a prime q such that the Sylow q-subgroup of $U(F)$ is identity, but the Sylow q-subgroup of $U(E)$ is a Prüfer q-subgroup, in particular, $U(E)$ is not residually finite.*

(5) *For each prime $r \neq q$ the Sylow r-subgroup of $U(E)$ is finite.*

Proof: Let p be a prime and q a distinct prime with $\mathrm{GCD}(q, p-1) = 1$. Let F be the union of the finite fields of size $p^m e$ where $m_e = q^e$, $e \in N$. By the choice of q the Sylow q-subgroup of F is identity.

Let $E = F[a]$ where a is a q-th root of 1 and let E_0 be the subfield of E generated over the prime subfield. Then E_0 is finite, say $|E_0| = p^d$. So $p^d \equiv 1 \pmod{q}$. Then E contains a subfield of degree $s_e = dq^e$ over the prime field for each $e \in N$. However, q^e divides $p^s e - 1$ for all $e \in N$ and so the Sylow q-subgroup E_q of $U(E)$ is infinite. This means that E_q is a Prüfer q-subgroup.

We note now that the Sylow r-subgroup F_r of $U(F)$ is finite for each prime r. If r divides $p^m e - 1$ with e minimal, then we pick no more powers of r for any larger e, and so $U(F)$ is residually finite. However, $U(E)$ is not residually finite, because it has a Prüfer q-subgroup.

We prove now that that the Sylow r-subgroup E_r of $U(E)$ is finite for each prime $r \neq q$. Choose the smallest s_e so that the field of size $p^s e$ contains r-th roots of 1 (or 4-th roots of 1 if $r = 2$; if no such field exists, we are done); we are then passing to a series of extensions of degree q and we pick up no further r-torsion (see Lemma 4.2 in [**GW**]). □

The following theorem is concerned with the class of fields for which residual finiteness of the multiplicative group is inherited by every finite field extension.

Theorem 3.7. *Let F be a field with a discrete valuation v and K be the residue field. Suppose that the following conditions hold:*

(1) *K has positive characteristic.*

(2) *$U(P)$ is residually finite for every finite extension P of K (in particular, for $P = K$).*

If E is a finite field extension of F, then $U(E)$ is residually finite.

Proof: Let $\mathrm{char}\, K = p$. There is no harm in taking E is a normal extension of F. It is straightforward to reduce to the case that E is a separable extension of F (and so Galois) (if v is an n-th power for all n, the same is true for v^p; and so, we may take v to be separable over F —then the v is a p^a power for every a, and if some such element were not separable over E, they would generate larger and larger extensions).

Now, since E is separable extension of F, we see that the discrete valuation has only finitely many extensions to E, which are all discrete. So if $v \in U(E)$ is an n-th power for all n, the valuation of v must be zero.

The condition on the residue fields implies that $v = 1$ modulo each maximal ideal over the maximal ideal of F. It is straightforward to see that such elements (other than 1) are not p^a powers for a sufficiently large. □

Now we will consider fields with some other properties which imply additional conditions on linear groups with Max-id over such fields.

Let F be a field. We say that F *has a property (FAE) if for each finite field extension E of F (in particular, for $E = F$) the factor-group $U(E)/t(U(E))$ is free abelian.*

Lemma 3.8. *Let G be a subgroup of* $\mathrm{GL}(F, A)$ *satisfying* Max-id. *Suppose that G has subgroups V and H with the following properties:*

(1) *H is a normal subgroup of V.*

(2) *H is a nilpotent bounded p-subgroup for some prime p.*

(3) *V/H is a Prüfer q-group, $q \neq p$.*

(4) $\dim_F(A/C_A(V))$ *is infinite.*

Then H has a finite V-composition series.

Proof: By Lemma 1.D.4 of the book [**KW**] $V = H \rtimes Q$ where Q is a Prüfer q-subgroup, so that, $Q = \langle z_n \mid z_1^p = 1, z_{n+1}^p = z_n, n \in N\rangle$. If H is finite, then all is proved. Suppose that H is infinite. It follows that $H/[H,H]$ is infinite (see, for example, [**RD1**, Corollary of Theorem 2.26]). Corollary 1.5 yields that $\dim_F(A/C_A(H))$ is finite. The equation $V = HQ$ combined with Lemma 1.1 imply that $\dim_F(A/C_A(Q))$ is infinite. Let C, D be the V-invariant subgroup of H such that $D \leq C$, and $B = C/D$ is an elementary abelian p-group. If z is an arbitrary element of Q, then by Maschke's Theorem (see, for example, [**WB**, Theorem 1.5]) $B = \bigoplus_{\lambda\in\Lambda} B_\lambda$ where B_λ is the minimal $\langle z\rangle$-invariant subgroup of B for each $\lambda \in \Lambda$. In particular, either $[B_\lambda, z] = B_\lambda$ or $[B_\lambda, z] = \langle 1\rangle$. It follows that $B = C_B(z) \times [B, z]$. Since Q is abelian, $C_B(z)$ and $[B, z]$ are the Q-invariant subgroups of B. If y is an element of Q such that $\langle z\rangle \leq \langle y\rangle$, then $C_B(y)C_B(z)$, and $[B, z] \leq [B, y]$. Thus we have an ascending series

$$[B, z_1] \leq [B, z_2] \leq \cdots \leq [B, z_n] \leq \cdots$$

of Q-invariant subgroups. By Corollary 1.6 there is a number $m \in N$ such that $[B, z_n] = [B, z_m]$ for all $n \geq m$. The equation $B = C_B(z_j) \times [B, z_j]$ implies that $C_B(z_n) = C_B(z_m)$ for all $n \geq m$. In other words, B satisfies the minimal condition for centralizers. By Theorem A of paper [**HB1**] B is a semisimple F_pQ-module. By Corollary 1.6 B satisfies Max-Q, hence B is a direct sum of minimal Q-invariant subgroups. In turn, it follows that H has finite Q-composition series, because H is nilpotent and bounded. □

Proposition 3.9. *Let F be a field of positive characteristic p and G be a soluble subgroup of* $\mathrm{GL}(F, A)$ *satisfying* Max-id. *Suppose that G has infinite central dimension and is not finitely generated. If F satisfies (FAE), then*

(1) *G has the normal subgroups $H \leq R$ such that G/R is polycyclic, R/H is a Prüfer q-group, $q \neq \operatorname{char} F$.*

(2) *H is a nilpotent bounded p-subgroup.*

(3) *H has a finite G-composition series.*

Proof: Suppose first that $G/[G, G]$ is not finitely generated. Using Theorem 2.6, we obtain that G has a normal nilpotent bounded p-subgroup H, such that $G/H \leq Q \times S_1 \times \cdots \times S_{n-1}$ where Q is a Prüfer q-group, $q \neq \operatorname{char} F$, $S_1, \ldots, S_{n-1}$ are ordinary finite-dimensional irreducible linear groups. By Proposition 2.4 S_j has a normal subgroup of finite index, which is isomorphic to some subgroup of $U_1 \times \cdots \times U_m$ where U_t is isomorphic to the multiplicative group of a field E for certain finite field extension E of the field F, $1 \leq t \leq m$. Since F satisfies (FAE), $U_t/t(U_t)$ is free abelian for each t, $1 \leq t \leq m$. It follows that G has a normal subgroup $R \geq H$ such that G/R is abelian-by-finite and finitely generated, R/H is a Prüfer q-group, $q \neq \operatorname{char} F$. Since G is not finitely generated, Lemma 1.3 proves that G is a finitary linear group. Lemma 1.1 yields that R has infinite central dimension. By Lemma 3.10 H has finite R-composition length, therefore H has finite G-composition length. For the general case, it is sufficient to apply Theorem 2.9. □

Now we will consider some particular fields satisfying (FAE).

Proposition 3.10. *Let F be a finitely generated extension of locally finite field L. Then $U(F)/t(U(F))$ is free abelian.*

Proof: Note that the property (FAE) is valid for any subfield. Thus, there is no harm in replacing L by K where K is the algebraic closure of L (and is still locally finite) and F by $FK = P$ (note that the hypothesis still hold). Now P is a finite extension of $K(x_1, \ldots, x_r)$ where the x_j are algebraically independent, $1 \leq j \leq r$. We can apply Noether's Lemma and see that we may choose the x_j so that P is the quotient field of the integral closure D of $K[x_1, \ldots, x_r]$.

Let V be the set of discrete valuations corresponding to the places of D. Then we have the map $1 \to U(D) \to U(P) \to G$ where $G \cong \bigoplus_{v \in V} Y_v$ and $Y_v \cong Z$, $v \in V$, and V is the set of valuations on P (given by $u \to \sum v(u)$). Now it is well known that $U(D)/U(K)$ is a finitely generated abelian [**GW**], and the image of the map is contained in the free abelian group G and so is also free abelian.

Thus, the map splits and we see that $U(P)/U(K) \cong A \oplus B$ where A is finitely generated and B is free abelian. Since $U(K)$ is the periodic part of $U(P)$, the result is proved. □

The following lemma gives us an example of a field of characteristic zero, satisfying (FAE).

Lemma 3.11. *Let F be a field and suppose that $E = F(X \mid \lambda \in \Lambda)$ is a field extension of F such that $\{X \mid \lambda \in \Lambda\}$ are algebraically independent. Then $U(E) \cong U(F) \times B$ where B is free abelian.*

Proof: We have the obvious F-valuations on E (corresponding to irreducible polynomials). This gives a map $1 \to U(D) \to U(E) \to A$ where A is free abelian. Thus, $U(E) \cong U(F) \times B$ where B is the image of the map and so free. □

Proposition 3.12. *Let L be a rational functions field over a finite extension of Q, and let F be a finite extension of L. Then $U(L)/T$ is free abelian, where T is the periodic part of $U(L)$. Moreover, T is finite.*

Proof: We have $F = L(a)$ for some element a. The minimal polynomial of a involves only finitely many elements of the transcendence base for L. So we see that F is a rational functions field over a field finitely generated over Q and so it suffices to prove the result for L finitely generated over Q. The result is well known in this case (i.e. there are enough discrete valuations). □

The results above allow us to obtain concrete information about the groups satisfying Max-id for the following types of fields.

Theorem 3.13. *Let $G \leq \mathrm{GL}(F, A)$ and G be a soluble group satisfying* Max-id. *Suppose that G is not finitely generated. Then G has finite central dimension in each of the following cases:*

(1) *F is a field satisfying the following conditions: the Sylow q-subgroup of $U(F)$ is finite and non-identity for each prime q, and the Sylow 2-subgroup has order at least 4.*

(2) *F is a field of characteristic $p > 0$ such that the periodic part of $U(F)$ is finite.*

(3) *F is a finitely generated field.*

(4) *F is a finite field extension of the p-adic field Q_p.*

(5) *F is a finite field extension of a field L where L is the rational functions field over a finite extension of Q.*

Theorem 3.14. *Let $G \leq \mathrm{GL}(F, A)$ and suppose that G is a soluble group satisfying* Max-id . *Suppose that G is finitely generated. Then the finitary radical of G has finite central dimension in the following cases:*

(1) *F is a field satisfying the following conditions: the Sylow q-subgroup of $U(F)$ is finite and non-dentity for each prime q, and the Sylow 2-subgroup has order at least 4.*

(2) *F is a field of characteristic $p > 0$ such that the periodic part of $U(F)$ is finite.*

(3) *F is a finitely generated field.*

(4) *F is a finite field extension of the p-adic field Q_p.*

(5) *F is a finite field extension of a field L where L is a rational functions field over a finite extension of Q.*

In these two theorems Assertion (1) follows from Proposition 3.2 and Proposition 3.3, and Assertion (2) follows from Proposition 3.2 and Theorem 3.4. Let F be a finitely generated field. Then every finite field extension E of F is finitely generated. Then $U(K)$ is a direct product of a finite cyclic subgroup and a free abelian subgroup [**MW**, Proposition], and we can apply Proposition 3.2. Assertion (4) is a straightforward consequence of Proposition 3.2 and Theorem 3.7. Assertion (5) follows from Proposition 3.2 and Proposition 3.12.

Propositions 3.10 and 3.9 imply

Theorem 3.15. *Let G be a soluble subgroup of* GL(F, A) *satisfying* Max-id*. Suppose that a field F is a finitely generated extension of locally finite field L. If G has infinite central dimension and is not finitely generated, then*

(1) *G has the normal subgroups $H \leq R$ such that G/R is polycyclic, R/H is a Prüfer q-group, $q \neq \operatorname{char} F$.*

(2) *H is a nilpotent bounded p-subgroup.*

(3) *H has a finite G-composition series.*

Corollary 3.16. *Let G be a soluble subgroup of* GL(F, A) *satisfying* Max-id *and suppose that F is a locally finite field. If G has infinite central dimension and is not finitely generated, then*

(1) *G has the normal subgroups $H \leq R$ such that G/R is finite, R/H is a Prüfer q-group, $q \neq \operatorname{char} F$.*

(2) *H is a nilpotent bounded p-subgroup.*

(3) *H has a finite G-composition series.*

In particular, G satisfies Min-id.

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