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FOURIER RESTRICTION TO CONVEX SURFACES OF REVOLUTION IN $\mathbb{R}^3$

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Abstract

If Γ is a C^3 hypersurface in $\mathbb{R}^n$ and $d\sigma$ is induced Lebesgue measure on Γ, then it is well known that a Tomas-Stein Fourier restriction estimate on Γ implies that Γ has a nowhere vanishing Gaussian curvature. In a recent paper, Carbery and Ziesler observed that if induced Lebesgue measure is replaced by affine surface area, then a Tomas-Stein restriction estimate on Γ implies that Γ satisfies the affine isoperimetric inequality. Since the only property needed for a hypersurface to satisfy the affine isoperimetric inequality is convexity, this raised the question of whether a Tomas-Stein restriction estimate can be obtained for flat but convex hypersurfaces in $\mathbb{R}^n$ such as $\Gamma(x) = (x, e^{-1/|x|^m})$, $m = 1, 2, \dots$. We prove that this is indeed the case in dimension $n = 3$.

1. Introduction

Let Γ be a C^3 hypersurface in $\mathbb{R}^n$ and $d\sigma$ a measure on Γ. A Tomas-Stein Fourier restriction estimate for the pair $(\Gamma, d\sigma)$ is an inequality of the form

$$\|\widehat{f}\|_{L^2(d\sigma)} \lesssim \|f\|_{L^{\frac{2n+2}{n+3}}(\mathbb{R}^n)} \qquad (1)$$

for $f \in C_0(\mathbb{R}^n)$.

The existence of restriction estimates such as (1), as well as their connection with the geometry of Γ, or with the decay of the Fourier transform of $d\sigma$, has been a subject of great interest. See [**9**, pp. 368–373] for some important applications of these estimates.

The choice of the measure $d\sigma$ is not completely arbitrary. It usually reflects some aspect of the geometry of Γ. Two important choices of $d\sigma$ are induced Lebesgue measure and affine surface area. In the former case, if Γ is assumed to have non-vanishing Gaussian curvature, (1) is a

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classical result of Tomas and Stein (see [**10**] and [**9**]). Conversely, if (1) holds with induced Lebesgue measure, then a result of Iosevich and Lu [**3**] (see also [**2**]), implies that Γ has non-vanishing Gaussian curvature. The proof of this converse uses, among other things, a Knapp-type scaling argument. To see how this argument goes, consider the special case where Γ is a surface of revolution given by $\Gamma(x) = (x, \phi(x))$, where $\phi(x) = \gamma(|x|)$, and $\gamma\colon [0,b) \to \mathbb{R}$ is increasing and satisfies $\gamma(0) = \gamma'(0) = 0$. For $0 < \delta < b$, let $S_\delta = \{(x, \gamma(|x|) : |x| \leq \delta\}$ and let f_δ be a smoothed-out characteristic function of S_δ. It is then easy to see that $\|f_\delta\|_{L^2(d\sigma)} \lesssim \delta^{(n-1)/2}$, and that $|\widehat{f_\delta\, d\sigma}| \gtrsim \delta^{n-1}$ on a $(C/\delta) \times \cdots \times (C/\delta) \times (C/\gamma(\delta))$ box in $\mathbb{R}^n$ (for a suitable constant C). Now if (1) holds then, by duality, the equivalent adjoint restriction estimate

$$\|\widehat{f\, d\sigma}\|_{L^{\frac{2n+2}{n-1}}(\mathbb{R}^n)} \lesssim \|f\|_{L^2(d\sigma)} \tag{2}$$

also holds. Applying (2) to f_δ we obtain

$$\delta^2 \lesssim \gamma(\delta) \tag{3}$$

and this implies that $\gamma''(0) \neq 0$. In particular γ cannot have vanishing Gaussian curvature at the origin. A more elaborate argument shows that the same conclusion holds in general.

In the latter case, say when $\Gamma(x) = (x, \phi(x))$, the affine surface area on Γ is given as the pushforward under Γ of the $(n-1)$-dimensional measure $|K_\phi(x)|^{1/(n+1)}\, dx$, where $K_\phi(x) = \det(\operatorname{Hess}\phi\,(x))$ is the affine curvature of Γ. To see what kind of geometry on Γ may be expected, take the case of a surface of revolution considered above. The radial assumption on ϕ, e.g. $\phi(x) = \gamma(|x|)$, simplifies matters and one computes that

$$K_\phi(x) = \gamma''(|x|) \left(\frac{\gamma'(|x|)}{|x|} \right)^{n-2}.$$

If we then take $d\sigma$ in the adjoint restriction estimate (2), which is equivalent to (1), to be affine surface area and use the function f_δ in it, we arrive [**1**] at the inequality

$$\int_0^\delta \left| \gamma''(r) \left(\frac{\gamma'(r)}{r} \right)^{n-2} \right|^{1/(n+1)} r^{n-2}\, dr \lesssim \left(\delta^{n-1} \gamma(\delta) \right)^{(n-1)/(n+1)}.$$

But now this inequality does not imply non-vanishing curvature. Rather, it is satisfied by any convex γ, regardless of how flat it is at the origin, e.g. it is satisfied by $\gamma(t) = e^{-1/t^m}$, m any positive integer. In fact, even if ϕ is not radial, there is a similar scaling argument that can be applied, and it leads to the conclusion that ϕ satisfies the affine isoperimetric inequality of affine differential geometry, which is certainly true whenever ϕ is convex. For more details we refer the reader to [**1**, pp. 409–410], [**5**, Chapter 5], and [**6**].

An earlier result of Sjölin [**8**] had already established that, if the dimension $n = 2$, and ϕ is convex, then the restriction inequality holds true for affine surface area. The strength of this result, along with the above considerations, suggested that, perhaps, the geometric condition of convexity of ϕ could imply a restriction result for affine surface area in higher dimensions. But if only convexity is to be used, functions such as $\phi(x) = e^{-1/|x|^m}$ have to be admitted. In attempting to prove this result, i.e. to show that convexity implies restriction, Carbery and Ziesler [**1**] considered the implications of a decay assumption on the Fourier transform of $d\sigma$.

Kenig, Ponce and Vega [**4**] proved that if the decay assumption

$$\left| \int_{B(0,b)} e^{-2\pi i \xi \cdot \Gamma(x)} |K_\phi(x)|^{\frac{1}{2}+i\alpha}\, dx \right| \lesssim \frac{(1+|\alpha|)^N}{|\xi_n|} \tag{4}$$

was true for all real α and some integer N, then (2) holds[1]. When testing (4) on $\phi(x) = e^{-1/|x|^m}$, Carbery and Ziesler [**1**] found that it did not hold true in dimension $n = 3$. This, of course, did not mean that there was no restriction result for $\phi(x) = e^{-1/|x|^m}$. More recently, the same restriction question was addressed in [**7**]. A consequence of the results there implies that if $\phi(\cdot) = \gamma(|\cdot|)$, where γ is convex, $\gamma(0) = \gamma'(0) = 0$, $\gamma^{(3)}(t)$ non-negative, and if

$$\sup_{0<t<b} \frac{t\gamma''(t)}{\gamma'(t)} \leq C < \infty,$$

then the restriction estimate (1) holds for affine surface area in dimension $n = 3$. Testing this last condition on $\gamma(t) = e^{-1/t^m}$, where $0 < t < b_m$, $b_m = m/(3m+3))$, one finds that

$$\sup_{0<t<b_m} \frac{t\gamma''(t)}{\gamma'(t)} = \sup_{0<t<b_m} \left(\frac{m}{t^m} - m - 1 \right) = \infty.$$

Once again, the function e^{-1/t^m} was precluded from the result.

It turns out that, at least for surfaces of revolution $\Gamma(x) = (x, \phi(x))$, $\phi(x) = \gamma(|x|)$, a Tomas-Stein restriction estimate for affine surface area

[1]This connection between decay and restriction is valid in dimensions $n = 2, 3$. In dimensions $n \geq 4$, one has to modify things slightly by inserting a smooth cut-off function into both (2) and (4), see [**1**] for further details.

does hold in the presence of convexity, if we add the condition that

$$\sup_{0<t<b} \frac{\gamma(t)\gamma''(t)}{\gamma'(t)^2} \le C < \infty. \tag{5}$$

Now testing this condition on $\gamma(t) = e^{-1/t^m}$ one finds that

$$\sup_{0<t<b_m} \frac{\gamma(t)\gamma''(t)}{\gamma'(t)^2} = \sup_{0<t<b_m} \left(1 - \frac{m+1}{m} t^m\right) \le 1. \tag{6}$$

We thus have a Tomas-Stein restriction result that includes the surfaces $\Gamma(x) = (x, e^{-1/|x|^m})$ in $\mathbb{R}^3$.

The purpose of this paper is to obtain restriction estimates for convex surfaces of revolution in $\mathbb{R}^3$. A major role is played by the function

$$\frac{\gamma(t)\gamma''(t)}{\gamma'(t)^2}$$

and our results only require the boundedness of certain L^{p_0} norms of this function. In particular, we obtain a Tomas-Stein restriction estimate for surfaces of revolution in $\mathbb{R}^3$ satisfying (5). We find it useful to prove our results in a little more general setting. In Section 2 we introduce a family of measures $d\sigma_\gamma$, state a general (L^p, L^q) restriction result for such measures, and obtain as a corollary the result on $\Gamma(x) = (x, e^{-1/|x|^m})$. In Section 3 we present the main component of our proof. In Section 4 we prove our results.

2. Statement of results

Let $0 < b \le \infty$, and denote by $B(0,b)$ the ball in $\mathbb{R}^2$ of center 0 and radius b. Let $\mathcal{C}([0,b))$ be the set of all real-valued functions $\gamma \in C^3([0,b))$ such that $\gamma(0) = \gamma'(0) = 0$, $\gamma''(t) > 0$ for $0 < t < b$, and $\gamma^{(3)}(t) \ge 0$ for $0 \le t < b$.

Suppose $0 \le \lambda \le 1$, $1 \le p$, $p_0 \le \infty$, $4 \le q \le \infty$, and $1/p + 2/q \le 1$. For $\gamma \in \mathcal{C}([0,b))$, let $d\sigma_\gamma$ be the pushforward under the map $x \to (x, \gamma(|x|)$ of the two-dimensional measure

$$\left(\frac{\gamma'(|x|)^{3-2\lambda}\gamma''(|x|)^\lambda}{|x|\gamma(|x|)^{1-\lambda}}\right)^{\frac{p'}{2q}} dx \tag{7}$$

with the understanding that when $p' = q = \infty$, $p'/(2q)$ is set to be equal to $1/4$; so that $p'/(2q) = 1/4$ on the sharp line $1/p + 2/q = 1$ including the point $(1/p, 1/q) = (1, 0)$.

Theorem 1. *If* $1/p+2/q=1-1/p_0$, *then*

$$\|\widehat{fd\sigma_\gamma}\|_{L^q(\mathbb{R}^3)} \le C_q \left\| \left(\frac{\gamma(|\cdot|)\gamma''(|\cdot|)}{\gamma'(|\cdot|)^2} \right)^{\frac{\lambda}{2q}} \right\|_{L^{p_0}(B(0,b))} \|f\|_{L^p(d\sigma_\gamma)} \tag{8}$$

for all $(f,\gamma) \in C_0(\mathbb{R}^3) \times \mathcal{C}([0,b))$, *where* $C_q = 4(2^{7/6}\pi)^{3/(2q)}$.

Notice that if $\lambda = 1$, then the density of the measure (7) is $|K_{\gamma(|\cdot|)}(x)|^{p'/(2q)}$, so if in addition $1/p+2/q=1$, then $d\sigma_\gamma$ is the same affine surface area measure we described in Section 1.

Corollary 1. *Suppose* $\gamma \in \mathcal{C}([0,b))$ *is such that*

$$\left\| \left(\frac{\gamma(|\cdot|)\gamma''(|\cdot|)}{\gamma'(|\cdot|)^2} \right)^{\frac{1}{2q}} \right\|_{L^{p_0}(B(0,b))} < \infty.$$

Let $\lambda = 1$ *and* $d\sigma = d\sigma_\gamma$. *If* $1/p+2/q=1-1/p_0$, *then*

$$\|\widehat{fd\sigma}\|_{L^q(\mathbb{R}^3)} \lesssim \|f\|_{L^p(d\sigma)}$$

for all $f \in L^p(d\sigma)$.

For example if $\gamma(t) = e^{-1/t^m}$, then by (6),

$$\left\| \left(\frac{\gamma(|\cdot|)\gamma''(|\cdot|)}{\gamma'(|\cdot|)^2} \right)^{\frac{1}{2q}} \right\|_{L^{p_0}(B(0,b_m))} \le (\pi b_m^2)^{1/p_0} < \infty$$

for $1 \le p_0 \le \infty$, and so the adjoint restriction estimate in Corollary 1 holds for $\gamma(t) = e^{-1/t^m}$ whenever $4 \le q \le \infty$ and $1/p+2/q \le 1$.

If, as another example, we take $\gamma(t) = -t\log(1-t)$, which is in $\mathcal{C}([0,1))$, then

$$\left\| \left(\frac{\gamma(|\cdot|)\gamma''(|\cdot|)}{\gamma'(|\cdot|)^2} \right)^{\frac{1}{2q}} \right\|_{L^{p_0}(B(0,1))} \approx \left\| \left(-\frac{\log(1-|\cdot|)}{|\cdot|} \right)^{\frac{1}{2q}} \right\|_{L^{p_0}(B(0,1))}$$

is finite for $1 \le p_0 < \infty$ but not for $p_0 = \infty$ (except if $q=\infty$), and so the adjoint restriction estimate in Corollary 1 holds for $\gamma(t) = -t\log(1-t)$ whenever $4 \le q \le \infty$ and $1/p+2/q < 1$.

3. Main estimate

Let $\tilde{B} = B(0,b) \cap \{x=(x_1,x_2) \in \mathbb{R}^2 \colon x_1, x_2 > 0\}$. The purpose of this section is to prove the following proposition.

Proposition 1. *Suppose* $0 < b \leq \infty$ *and* $\gamma \in \mathcal{C}([0,b))$. *Then*

$$\int_{\tilde{B}}\int_{\tilde{B}} h(u+v,\gamma(|u|)+\gamma(|v|))\left(\frac{\gamma'(|u|)^3}{|u|\gamma(|u|)}\frac{\gamma'(|v|)^3}{|v|\gamma(|v|)}\right)^{\frac{1}{4}}\,du\,dv$$
$$\leq (2^{7/6}\pi)^{3/2}\|h\|_{L^1(\mathbb{R}^3)}$$

for all Lebesgue measurable $h\colon \mathbb{R}^3 \to [0,\infty]$.

Proof: Denoting the integral on the left-hand side of the inequality by I, and changing into polar coordinates, we have

$$\mathrm{I}=\int_0^b\int_0^b\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} h(re^{i\theta}+se^{i\phi},\gamma(r)+\gamma(s))\,d\theta\,d\phi\left(\frac{r^3\gamma'(r)^3s^3\gamma'(s)^3}{\gamma(r)\gamma(s)}\right)^{\frac{1}{4}}\,dr\,ds.$$

The change of variable $x = re^{i\theta} + se^{i\phi}$ (cf [**7**]) shows that

$$\int_0^{\frac{\pi}{2}}\int_0^{\theta} h(re^{i\theta}+se^{i\phi},\gamma(r)+\gamma(s))\,d\phi\,d\theta$$
$$\leq \int_{\sqrt{r^2+s^2}<|x|<r+s}\frac{2\,h(x,\gamma(r)+\gamma(s))}{\sqrt{(|x|^2-(r-s)^2)((r+s)^2-|x|^2)}}\,dx.$$

So

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} h(re^{i\theta}+se^{i\phi},\gamma(r)+\gamma(s))\,d\phi\,d\theta$$
$$\leq \int_{\sqrt{r^2+s^2}<|x|<r+s}\frac{4\,h(x,\gamma(r)+\gamma(s))}{\sqrt{(|x|^2-(r-s)^2)((r+s)^2-|x|^2)}}\,dx$$
$$\leq \int_{\sqrt{r^2+s^2}<|x|<r+s}\frac{4\,h(x,\gamma(r)+\gamma(s))}{\sqrt{(2rs)((r+s)^2-|x|^2)}}\,dx$$
$$\leq \int_{|x|<r+s}\frac{2\,h(x,\gamma(r)+\gamma(s))}{(rs)^{\frac{3}{4}}\sqrt{r+s-|x|}}\,dx,$$

where we have used the inequality $r+s \geq 2\sqrt{rs}$. It follows that

$$\begin{aligned}
\mathrm{I} &\leq 2\int_0^b \int_0^b \int_{|x|<r+s} \frac{h(x,\gamma(r)+\gamma(s))}{\sqrt{r+s-|x|}}\,dx \left(\frac{\gamma'(r)^3\gamma'(s)^3}{\gamma(r)\gamma(s)}\right)^{\frac{1}{4}} dr\,ds \\
&= 2\int_{B(0,2b)}\int_0^b\int_0^b h(x,\gamma(r)+\gamma(s))\frac{\chi_E(r,s)}{\sqrt{r+s-x}}\left(\frac{\gamma'(r)^3\gamma'(s)^3}{\gamma(r)\gamma(s)}\right)^{\frac{1}{4}} dr\,ds\,dx \\
&= 4\int_{B(0,2b)}\int_0^b\int_0^b h(x,\gamma(r)+\gamma(s))\frac{\chi_F(r,s)}{\sqrt{r+s-x}}\left(\frac{\gamma'(r)^3\gamma'(s)^3}{\gamma(r)\gamma(s)}\right)^{\frac{1}{4}} dr\,ds\,dx \\
&= 4\int_{B(0,2b)} \mathrm{II}\,dx,
\end{aligned}$$

where $E=\{(r,s)\in(0,b)\times(0,b) : r+s>|x|\}$, $F=\{(r,s)\in E : s<r\}$, and

$$\mathrm{II} = \int_0^b\int_0^b h(x,\gamma(r)+\gamma(s))\frac{\chi_F(r,s)}{\sqrt{r+s-x}}\left(\frac{\gamma'(r)^3\gamma'(s)^3}{\gamma(r)\gamma(s)}\right)^{\frac{1}{4}} dr\,ds.$$

To estimate II, we shall first apply the change of variable

$$\begin{aligned}
r &= r(t,y) = \gamma^{-1}(y\sin^2 t)\\
s &= s(t,y) = \gamma^{-1}(y\cos^2 t),
\end{aligned}$$

which is defined on the open set

$$\Omega = \left\{(t,y)\in\mathbb{R}^2 : \frac{\pi}{4}<t<\frac{\pi}{2},\ y>0\right\};$$

so, with a slight abuse of notation, (r,s) is now a mapping from Ω to $\mathbb{R}^2$. The Jacobian of this mapping is

$$J_{(r,s)}(t,y) = \frac{2y\sin t\cos^3 t + 2y\sin^3 t\cos t}{\gamma'(\gamma^{-1}(y\sin^2 t))\gamma'(\gamma^{-1}(y\cos^2 t))} = \frac{y\sin 2t}{\gamma'(r)\gamma'(s)}.$$

But[2]

$$\gamma(r) = y\sin^2 t \quad\text{and}\quad \gamma(s) = y\cos^2 t, \tag{9}$$

so

$$\gamma'(r)\frac{\partial r}{\partial t} = y\sin 2t \quad\text{and}\quad \gamma'(s)\frac{\partial s}{\partial t} = -y\sin 2t, \tag{10}$$

[2]To simplify the notation, we are writing r, s, $\partial r/\partial t$, and $\partial s/\partial t$ for $r(t,y)$, $s(t,y)$, $\partial r/\partial t(t,y)$, and $\partial s/\partial t(t,y)$ respectively.

and so

$$y\sin 2t = \sqrt{\gamma'(r)\gamma'(s)}\sqrt{\frac{\partial r}{\partial t}\left|\frac{\partial s}{\partial t}\right|}.$$

Thus

$$J_{(r,s)}(t,y) = \frac{1}{\sqrt{\gamma'(r)\gamma'(s)}}\sqrt{\frac{\partial r}{\partial t}\left|\frac{\partial s}{\partial t}\right|}.$$

But also

$$\frac{\gamma'(r)\gamma'(s)}{\gamma(r)\gamma(s)}\frac{\partial r}{\partial t}\left|\frac{\partial s}{\partial t}\right| = \frac{y^2\sin^2 2t}{(y\sin^2 t)(y\cos^2 t)} = 4,$$

so

$$\left(\frac{\gamma'(r)^3\gamma'(s)^3}{\gamma(r)\gamma(s)}\right)^{\frac{1}{4}} J_{(r,s)}(t,y) = \left(4\frac{\partial r}{\partial t}\left|\frac{\partial s}{\partial t}\right|\right)^{\frac{1}{4}}.$$

Next, to determine the domain of integration in the ty-plane, we make the following observations. By the convexity of γ, $\gamma(r)+\gamma(|x|-r)$, as a function of r, increases on the interval $(|x|/2,|x|)$. So

$$2\,\gamma(\frac{|x|}{2}) \le \gamma(r)+\gamma(|x|-r) < \gamma(r)+\gamma(s)$$

whenever $|x|/2 < r < |x|$ and $|x|-r < s$, which are in turn satisfied whenever $s < r < |x| < r+s$. Also by the convexity of γ,

$$2\,\gamma(\frac{|x|}{2}) \le \gamma(|x|) \le \gamma(r) < \gamma(r)+\gamma(s)$$

whenever $r \ge |x|$ and $s > 0$. Thus

$$2\,\gamma(\frac{|x|}{2}) < \gamma(r)+\gamma(s) < 2\,\gamma(b)$$

whenever $0 < s < r < b$ and $|x| < r+s$. But, by the definition of the mapping (r,s),

$$y = \gamma(r)+\gamma(s)$$

for all $(t,y) \in \Omega$, so

$$2\,\gamma(\frac{|x|}{2}) < y < 2\,\gamma(b)$$

whenever $0 < s < r < b$ and $|x| < r+s$. For any such (fixed) y, the range of (r,s) is a curve in $\mathbb{R}^2$ that "enters" the closure of the domain of integration of II when $t = \pi/4$ (i.e. when $s = r$) and "leaves" when $t = \tau(y)$ for some $\tau(y) \in (\pi/4, \pi/2]$. Thus

$$\begin{aligned} \mathrm{II} &= \int_{2\,\gamma(\frac{|x|}{2})}^{2\,\gamma(b)} \int_{\frac{\pi}{4}}^{\tau(y)} h(x,y) \frac{1}{\sqrt{r+s-|x|}} \left(4 \frac{\partial r}{\partial t} \left|\frac{\partial s}{\partial t}\right|\right)^{\frac{1}{4}} dt\, dy \\ &= \int_{2\,\gamma(\frac{|x|}{2})}^{2\,\gamma(b)} h(x,y) \int_{\frac{\pi}{4}}^{\tau(y)} \frac{\sqrt{2}}{\sqrt{r+s-|x|}} \left(\frac{\partial r}{\partial t} \left|\frac{\partial s}{\partial t}\right|\right)^{\frac{1}{4}} dt\, dy. \end{aligned}$$

Now, by the definition of $\tau(y)$,

$$r + s = r(t,y) + s(t,y) \geq |x| \quad \text{for} \quad \frac{\pi}{4} \leq t \leq \tau(y),$$

so, in particular,

$$r(\tau(y),y) + s(\tau(y),y) \geq |x|,$$

and hence

$$r + s - |x| \geq r + s - (r(\tau(y),y) + s(\tau(y),y)) \quad \text{for} \quad \frac{\pi}{4} < t < \tau(y).$$

Thus

$$\mathrm{II} \leq \int_{2\,\gamma(\frac{|x|}{2})}^{2\,\gamma(b)} h(x,y) \int_{\frac{\pi}{4}}^{\tau(y)} \frac{\sqrt{2}}{\sqrt{r+s-r(\tau(y),y) - s(\tau(y),y)}} \left(\frac{\partial r}{\partial t} \left|\frac{\partial s}{\partial t}\right|\right)^{\frac{1}{4}} dt\, dy.$$

The rest of the proof will be devoted to estimating

$$\frac{1}{\sqrt{r+s-r(\tau(y),y) - s(\tau(y),y)}} \left(\frac{\partial r}{\partial t} \left|\frac{\partial s}{\partial t}\right|\right)^{\frac{1}{4}}$$

for $2\gamma(|x|/2) < y < 2\gamma(b)$ and $\pi/4 < t < \tau(y)$.

We start by examining the function $\partial r/\partial t + \partial s/\partial t$. By (10),

$$\frac{\partial r}{\partial t} + \frac{\partial s}{\partial t} = \frac{y \sin 2t}{\gamma'(r)} - \frac{y \sin 2t}{\gamma'(s)}$$

is negative for $\pi/4 < t < \pi/2$ (since $\gamma'(s) < \gamma'(r)$), so

$$\begin{aligned} \left|\frac{\partial r}{\partial t} + \frac{\partial s}{\partial t}\right| &= \frac{y \sin 2t}{\gamma'(s)} - \frac{y \sin 2t}{\gamma'(r)} \\ &= 2y \left(\frac{\cos t}{\gamma'(s)} \sin t - \frac{\sin t}{\gamma'(r)} \cos t\right) \\ &= 2y \sqrt{\frac{\cos^2 t}{\gamma'(s)^2} + \frac{\sin^2 t}{\gamma'(r)^2}} \sin(t - \phi), \end{aligned}$$

where $\phi = \phi(t)$ is defined by

$$\sin\phi = \frac{(\sin t)/\gamma'(r)}{\sqrt{\frac{\cos^2 t}{\gamma'(s)^2} + \frac{\sin^2 t}{\gamma'(r)^2}}}, \quad \cos\phi = \frac{(\cos t)/\gamma'(s)}{\sqrt{\frac{\cos^2 t}{\gamma'(s)^2} + \frac{\sin^2 t}{\gamma'(r)^2}}}.$$

We shall need precise information about ϕ and $\partial^2 r/\partial t^2 + \partial^2 s/\partial t^2$. For this we need the following easy, but important, observation. By integration by parts,

$$\int_0^\rho 2\gamma'(\alpha)\gamma''(\alpha)\,d\alpha = 2\gamma(\rho)\gamma''(\rho) - 2\int_0^\rho \gamma(\alpha)\gamma^{(3)}(\alpha)\,d\alpha$$

for $0 < \rho < b$, and since $\gamma^{(3)}$ is nonnegative, we get

$$\gamma'(\rho)^2 \le 2\gamma(\rho)\gamma''(\rho) \quad \text{for} \quad 0 < \rho < b. \tag{11}$$

(This is the only place where we use the assumptions that γ is C^3 and $\gamma^{(3)}$ is nonnegative; everywhere else we need only require of γ to be C^2 and convex.)

Differentiating both sides of (10) with respect to t, we have

$$\gamma''(r)\left(\frac{\partial r}{\partial t}\right)^2 + \gamma'(r)\frac{\partial^2 r}{\partial t^2} = 2y\cos 2t$$

and

$$\gamma''(s)\left(\frac{\partial s}{\partial t}\right)^2 + \gamma'(s)\frac{\partial^2 s}{\partial t^2} = -2y\cos 2t.$$

This combined with (11) gives

$$\frac{\gamma'(r)^2}{2\gamma(r)}\left(\frac{\partial r}{\partial t}\right)^2 + \gamma'(r)\frac{\partial^2 r}{\partial t^2} \le 2y\cos 2t$$

and

$$\frac{\gamma'(s)^2}{2\gamma(s)}\left(\frac{\partial s}{\partial t}\right)^2 + \gamma'(s)\frac{\partial^2 s}{\partial t^2} \le -2y\cos 2t.$$

But by (9) and (10),

$$\frac{\gamma'(r)^2}{2\gamma(r)}\left(\frac{\partial r}{\partial t}\right)^2 = 2y\cos^2 t \quad \text{and} \quad \frac{\gamma'(s)^2}{2\gamma(s)}\left(\frac{\partial s}{\partial t}\right)^2 = 2y\sin^2 t,$$

so

$$\gamma'(r)\frac{\partial^2 r}{\partial t^2} \le -2y\sin^2 t \quad \text{and} \quad \gamma'(s)\frac{\partial^2 s}{\partial t^2} \le -2y\cos^2 t,$$

and it follows that

$$\gamma'(r)\left(\frac{\partial^2 r}{\partial t^2}+\frac{\partial^2 s}{\partial t^2}\right)\le -2y<0 \tag{12}$$

for $\pi/4 < t < \pi/2$.

Going back to ϕ, we have

$$\tan\phi = \frac{\gamma'(s)}{\gamma'(r)}\tan t.$$

Now if we let $m(\rho)=\gamma'(\rho)^2/\gamma(\rho)$, $0<\rho<b$, then by (11), $m'(\rho)\ge 0$ and it follows that $m(s)\le m(r)$. Hence

$$\tan\phi = \frac{\sqrt{m(s)\gamma(s)}}{\sqrt{m(r)\gamma(r)}}\tan t \le \frac{\sqrt{\gamma(s)}}{\sqrt{\gamma(r)}}\tan t = \cot t\tan t = 1,$$

and hence $0<\phi\le\pi/4$. Thus

$$\begin{aligned}
\left|\frac{\partial r}{\partial t}+\frac{\partial s}{\partial t}\right| &\ge 2y\sqrt{\frac{\cos^2 t}{\gamma'(s)^2}+\frac{\sin^2 t}{\gamma'(r)^2}}\,\sin(t-\frac{\pi}{4})\\
&= 2y\sqrt{\frac{\cos^2 t}{(y\sin 2t)^2}\left(\frac{\partial s}{\partial t}\right)^2+\frac{\sin^2 t}{(y\sin 2t)^2}\left(\frac{\partial r}{\partial t}\right)^2}\,\sin\left(t-\frac{\pi}{4}\right)\\
&= \sqrt{\frac{1}{\sin^2 t}\left(\frac{\partial s}{\partial t}\right)^2+\frac{1}{\cos^2 t}\left(\frac{\partial r}{\partial t}\right)^2}\,\sin\left(t-\frac{\pi}{4}\right)\\
&> \sqrt{\left(\frac{\partial r}{\partial t}\right)^2+\left(\frac{\partial s}{\partial t}\right)^2}\,\sin\left(t-\frac{\pi}{4}\right)\\
&\ge \sqrt{2\,\frac{\partial r}{\partial t}\left|\frac{\partial s}{\partial t}\right|}\left(\frac{2}{\pi}\right)\left(t-\frac{\pi}{4}\right)
\end{aligned}$$

for $\pi/4 < t < \pi/2$. Thus

$$\frac{\sqrt{\frac{\partial r}{\partial t}\left|\frac{\partial s}{\partial t}\right|}}{\left|\frac{\partial r}{\partial t}+\frac{\partial s}{\partial t}\right|} < \frac{\pi}{2\sqrt{2}}\,\frac{1}{t-\frac{\pi}{4}}$$

for $\pi/4 < t < \pi/2$.

As we saw above, $\partial r/\partial t+\partial s/\partial t$ is negative on the interval $(\pi/4,\pi/2)$. Also by (12),

$$\frac{\partial^2 r}{\partial t^2}+\frac{\partial^2 s}{\partial t^2}<0,$$

so $\partial r/\partial t + \partial s/\partial t$, as a function of t, is decreasing on $(\pi/4, \pi/2)$, and so $|\partial r/\partial t + \partial s/\partial t|$ is increasing there. Now applying the mean value theorem, we obtain

$$r + s - r(\tau(y), y) - s(\tau(y), y) \geq (\tau(y) - t)\left|\frac{\partial r}{\partial t} + \frac{\partial s}{\partial t}\right|$$

for $\pi/4 < t < \tau(y)$. Thus

$$\begin{aligned}\frac{\left(\frac{\partial r}{\partial t}\left|\frac{\partial s}{\partial t}\right|\right)^{\frac14}}{\sqrt{r + s - r(\tau(y), y) - s(\tau(y), y)}} &\leq \frac{1}{\sqrt{\tau(y) - t}}\frac{\left(\frac{\partial r}{\partial t}\left|\frac{\partial s}{\partial t}\right|\right)^{\frac14}}{\left|\frac{\partial r}{\partial t} + \frac{\partial s}{\partial t}\right|^{\frac12}} \\ &< \sqrt{\frac{\pi}{2\sqrt{2}}}\frac{1}{\sqrt{\tau(y) - t}}\frac{1}{\sqrt{t - \frac{\pi}{4}}}\end{aligned}$$

for $\pi/4 < t < \tau(y)$. Thus

$$\begin{aligned}&\int_{\frac{\pi}{4}}^{\tau(y)} \frac{\sqrt{2}}{\sqrt{r + s - r(\tau(y), y) - s(\tau(y), y)}}\left(\frac{\partial r}{\partial t}\left|\frac{\partial s}{\partial t}\right|\right)^{\frac14} dt \\ &\qquad \leq \sqrt{\frac{\pi}{\sqrt{2}}}\int_{\frac{\pi}{4}}^{\tau(y)} \frac{dt}{\sqrt{(\tau(y) - t)(t - \frac{\pi}{4})}} \\ &\qquad = \frac{\pi^{3/2}}{2^{1/4}}.\end{aligned}$$

Thus

$$\mathrm{II} \leq \frac{\pi^{3/2}}{2^{1/4}}\int_{2\,\gamma(\frac{|x|}{2})}^{2\,\gamma(b)} h(x, y)\,dy$$

and consequently

$$\mathrm{I} \leq 2^{7/4}\pi^{3/2}\int_{B(0,2b)}\int_{2\,\gamma(\frac{|x|}{2})}^{2\,\gamma(b)} h(x, y)\,dy\,dx \leq (2^{7/6}\pi)^{3/2}\,\|h\|_{L^1(\mathbb{R}^3)}. \qquad \square$$

4. Proof of Theorem 1

Let f be a continuous function on $\mathbb{R}^3$ which is compactly supported in the third variable, and let $\gamma \in \mathcal{C}([0, b))$. It is enough to show that

$$\|\widehat{f d\sigma}\|_{L^q(\mathbb{R}^3)} \leq (2^{7/6}\pi)^{3/(2q)}\left\|\left(\frac{\gamma(|\cdot|)\gamma''(|\cdot|)}{\gamma'(|\cdot|)^2}\right)^{\frac{\lambda}{2q}}\right\|_{L^{p_0}(B(0,b))}\|f\|_{L^p(d\sigma)},$$

where $d\sigma = \chi_E\,d\sigma_\gamma$ and $E = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1, x_2 \geq 0\}$. If $q = \infty$, then this follows easily from Hölder's inequality. So we may assume

$q < \infty$. Then the relation $1/p + 2/q = 1 - 1/p_0$ tells us that $p, p_0 > 1$. Also, since

$$\|\widehat{fd\sigma}\|_{L^q(\mathbb{R}^3)} = \|\widehat{fd\sigma}\,\widehat{fd\sigma}\|^{1/2}_{L^{q/2}(\mathbb{R}^3)} = \|\widehat{fd\sigma * fd\sigma}\|^{1/2}_{L^{q/2}(\mathbb{R}^3)},$$

and since $q/2 \geq 2$, it is enough by the Hausdorff-Young inequality to establish that

$$\int h|f|d\sigma * |f|d\sigma$$
$$\leq (2^{7/6}\pi)^{3/q} \left\| \left(\frac{\gamma(|\cdot|)\gamma''(|\cdot|)}{\gamma'(|\cdot|)^2} \right)^{\frac{\lambda}{2q}} \right\|^2_{L^{p_0}(B(0,b))} \|f\|^2_{L^p(d\sigma)} \|h\|_{L^{q/2}(\mathbb{R}^3)}$$

for any nonnegative Lebesgue measurable function h on $\mathbb{R}^3$. But by Hölder's inequality,

$$\int h|f|d\sigma * |f|\, d\sigma \leq \|f\|^2_{L^p(d\sigma)} \|h\|_{L^{p'}(d\sigma * d\sigma)},$$

so we need to have

$$\|h\|_{L^{p'}(d\sigma*d\sigma)} \leq (2^{7/6}\pi)^{3/q} \left\| \left(\frac{\gamma(|\cdot|)\gamma''(|\cdot|)}{\gamma'(|\cdot|)^2} \right)^{\frac{\lambda}{2q}} \right\|^2_{L^{p_0}(B(0,b))} \|h\|_{L^{q/2}(\mathbb{R}^3)}.$$

Now this follows from Proposition 1 by writing

$$\|h\|^{p'}_{L^{p'}(d\sigma*d\sigma)}$$
$$= \int_{\tilde{B}} \int_{\tilde{B}} h^{p'}(x+y, \gamma(|x|)+\gamma(|y|)) M(x)^{\frac{p'}{2q}} M(y)^{\frac{p'}{2q}} \frac{N(x)^{\frac{\lambda p'}{2q}} N(y)^{\frac{\lambda p'}{2q}}}{N(x)^{\frac{\lambda p'}{2q}} N(y)^{\frac{\lambda p'}{2q}}} \, dx \, dy,$$

where

$$M(\cdot) = \frac{\gamma'(|\cdot|)^{3-2\lambda}\gamma''(|\cdot|)^{\lambda}}{|\cdot|\gamma(|\cdot|)^{1-\lambda}} \quad \text{and} \quad N(\cdot) = \frac{\gamma(|\cdot|)\gamma''(|\cdot|)}{\gamma'(|\cdot|)^2},$$

and applying Hölder's inequality to get

$$\|h\|^{p'}_{L^{p'}(d\sigma * d\sigma)}$$

$$\le \left(\int_{\tilde B}\int_{\tilde B} N(x)^{\frac{\lambda r p'}{2q}} N(y)^{\frac{\lambda r p'}{2q}}\,dx\,dy\right)^{\frac{1}{r}}$$

$$\times \left(\int_{\tilde B}\int_{\tilde B} h^{q/2}(x+y,\gamma(|x|)+\gamma(|y|))\frac{M(x)^{\frac14}M(y)^{\frac14}}{N(x)^{\frac{\lambda}{4}}N(y)^{\frac{\lambda}{4}}}\,dx\,dy\right)^{\frac{2p'}{q}}$$

$$= \left(\int_{\tilde B} N(x)^{\frac{\lambda p_0}{2q}}\,dx\right)^{\frac{2}{r}}$$

$$\times \left(\int_{\tilde B}\int_{\tilde B} h^{q/2}(x+y,\gamma(|x|)+\gamma(|y|))\left(\frac{\gamma'(|x|)^3}{|x|\gamma(|x|)}\frac{\gamma'(|y|)^3}{|y|\gamma(|y|)}\right)^{\frac14}dx\,dy\right)^{\frac{2p'}{q}}$$

$$\le \left(\int_{\tilde B} N(x)^{\frac{\lambda p_0}{2q}}\,dx\right)^{\frac{2p'}{p_0}} \left((2^{7/6}\pi)^{3/2}\|h^{q/2}\|_{L^1(\mathbb{R}^3)}\right)^{\frac{2p'}{q}}$$

$$\le (2^{7/6}\pi)^{3p'/q}\left\|\left(\frac{\gamma(|\cdot|)\gamma''(|\cdot|)}{\gamma'(|\cdot|)^2}\right)^{\frac{\lambda}{2q}}\right\|^{2p'}_{L^{p_0}(B(0,b))}\|h\|^{p'}_{L^{q/2}(\mathbb{R}^3)},$$

where r is the dual exponent to $q/(2p')$ (so that $rp' = p_0$).

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