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TRAFFIC PLANS

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Abstract

In recent research in the optimization of transportation networks, the problem was formalized as finding the optimal paths to transport a measure μ^+ onto a measure μ^- with the same mass. This approach is realistic for simple good distribution networks (water, electric power,...) but it is no more realistic when we want to specify "who goes where", like in the mailing problem or the optimal urban traffic network problem. In this paper, we present a new framework generalizing the former approaches and permitting to solve the optimal transport problem under the "who goes where" constraint. This constraint is formalized as a transference plan from μ^+ to μ^- which we handle as a boundary condition for the "optimal traffic problem".

1. Introduction

Many systems designed by humans can be viewed as supply-demand distribution networks designed to transport goods from one place (the supply) to another (the demand). This is obviously the case with distribution networks such as communication networks [**11**], electric power supply, etc. The same can be said of many natural flow networks which connect a finite size volume to a source. This happens for example with drainage networks [**14**], gas pipeline [**3**], actual plants and trees, bronchial systems or cardiovascular systems.

The Monge-Kantorovitch problem.

A first mathematical transportation problem was formalized by Monge, then given a relaxed formulation by Kantorovitch [**16**], [**13**]. The problem he considered was the one of moving a pile of sand from a place to another with the less possible work. In the Monge-Kantorovitch framework, μ^+ and μ^- are measures on $\mathbb{R}^N$ which model, respectively,

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the supply and demand mass distributions. Here, to transport μ^+ onto μ^- means to tell where the supplied mass is sent, i.e. to give a measure π on $\mathbb{R}^N \times \mathbb{R}^N$ where $\pi(A \times B)$ represents the amount of mass going from A to B. This measure π is called a transference plan. To evaluate the efficiency of a transference plan, we consider the cost function $c\colon \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ where $c(x,y)$ is the cost of transporting a unit mass from x to y. The cost associated with a transference plan is $\int_{\mathbb{R}^N \times \mathbb{R}^N} c(x,y)\, d\pi(x,y)$. The minimization of this functional is the Monge-Kantorovitch problem.

As an example, consider the cost function $c(x,y) = |x-y|^2$, and the supply and demand measures $\mu^+ = \delta_x$ and $\mu^- = \frac{1}{2}(\delta_{y_1} + \delta_{y_2})$. The minimizer π is the measure on $\mathbb{R}^N \times \mathbb{R}^N$ such that $\pi(\{x\} \times \{y_1\}) = \frac{1}{2}$ and $\pi(\{x\} \times \{y_2\}) = \frac{1}{2}$. The actual transportation, for the real problem of transporting sand, is achieved along geodesics between x, y_1 and y_2 as represented in Figure 1.

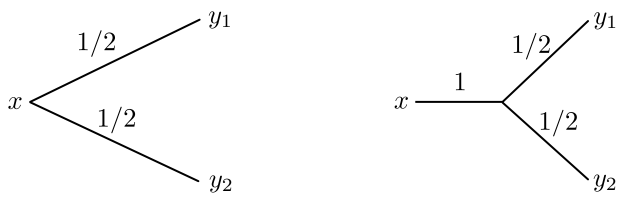

Figure 1. The transport from δ_x to $\frac{1}{2}(\delta_{y_1} + \delta_{y_2})$. Monge-Kantorovich versus Q. Xia's solution.

The Gilbert-Steiner problem and the irrigation problem.

In the Monge-Kantorovitch framework, the cost of the structure (trucks along roads, buckets, tubes...) achieving the transport is not taken into account. Indeed, with this formulation, the cost behaves as if every single particle of sand goes straight from its starting to its ending point. In the case of real supply-demand distribution problems, achieving this kind of single particle transport would be very costly. This is why the structure of the transporting network has to be included in the formulation of the cost.

The Steiner problem which consists in minimizing the total length of a network connecting a given set of points could be such a model. However, this cost is not realistic since it does not discriminate the cost of high or low capacity edges (a road has not the same cost as a highway). The first model taking into account capacities of edges was proposed by

Gilbert [**11**] in the case of communication networks. This author models the network as a graph such that each edge e is associated with a capacity c_e. Let $f(c)$ denote the cost per unit length of an edge with capacity c. It is assumed that $f(c)$ is subadditive and increasing, i.e., $f(a) + f(b) \geq f(a+b) \geq \max(f(a), f(b))$. Gilbert then considers the problem of minimizing the cost of networks supporting a given set of flows between terminals. The subadditivity of the cost f translates the fact that it is more advantageous to transport flows together. Thus, it leads to delay bifurcations. In the fluid mechanics context, this subadditivity follows from Poiseuille's law, according to which the resistance of a tube increases when it gets thinner (we refer to [**2**], [**7**] for a study of irrigation trees in this context). The Gilbert model was adapted to the study of optimal pipeline or drainage networks [**3**], [**14**]. From a numerical point of view, a backtrack algorithm exploring relevant Steiner topologies is proposed in [**21**] to solve a problem of water treatment network. A different algorithmic approach can be found in [**22**].

Recently, the discrete Gilbert-Steiner model was set in a more general continuous framework [**19**], [**15**] where the wells and sources are arbitrary measures, instead of a finite sum of Dirac masses. Qinglan Xia [**19**] models the transportation network as an embedded graph with a countable number of vertices and satisfying Kirchhoff's law. The network can then represented as a one-dimensional flat current G with possibly non integer multiplicity satisfying

$$\partial G = \mu^+ - \mu^-.$$

The multiplicity of each edge of the graph represents the fluid flow, or equivalently the mass conveyed along the vertex. The condition $\partial G = \mu^+ - \mu^-$ implies that Kirchhoff's law is satisfied at each vertex of the graph. To penalize the fact of not transporting mass together, Qinglan Xia considers the energy

$$E^\alpha(G) = \sum_{e \text{ edge of } G} w(e)^\alpha \operatorname{length}(e), \tag{1}$$

where $0 < \alpha < 1$. This cost corresponds to a cost per unit length of $w(e)^\alpha$ for each edge e. It is subadditive because of the concavity of $f(x) = x^\alpha$.

As a simple example, the minimizer of (1) with $\mu^+ = \delta_x$ and $\mu^- = \frac{1}{2}(\delta_{y_1} + \delta_{y_2})$ is represented in Figure 1. Let us consider another example which will show the difference with the traffic plan approach: take $\mu^+ = \frac{1}{2}(\delta_{x_1} + \delta_{x_2})$ and $\mu^- = \frac{1}{2}(\delta_{y_1} + \delta_{y_2})$. The locations of x_1, x_2, y_1 and y_2 and the minimizer are represented in Figure 2.

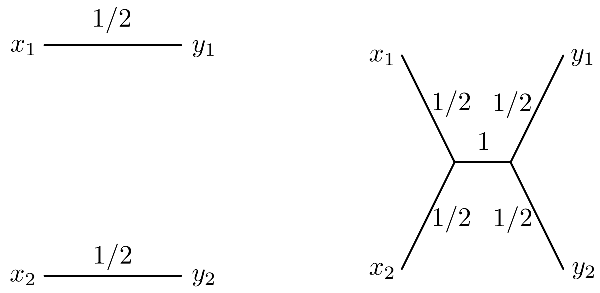

FIGURE 2. Irrigation problem minimizer versus traffic problem minimizer.

The very same problem with the same energy was given a different (Lagrangian) formulation in [**15**], with a single source supply $\mu^+ = \delta_S$. The authors model the transportation network as a set of "fibers" $\chi(p, \cdot)$, where $\chi(p,t)$ represents the location of a particle $p \in \Omega$ at time t (Ω is an abstract probability space). The irrigated measure can then be defined as the density measure of the fibers stopping in any given volume. Let us mention that the cost functional defined in [**15**] is slightly different from the energy (1). Indeed, both functionals coincide on trees, and [**15**] only handles such tree like objects, since $\chi(p,t)$ has a filtration structure. To see why the two costs are different, let us consider $\mu^- = \frac{2}{5}\delta_{y_1} + \frac{2}{5}\delta_{y_2} + \frac{1}{5}\delta_{y_3}$ and $\mu^+ = \delta_x$. The left-hand side of Figure 3 shows that once two fibers get separated, they are considered to be separated until the end, even if they coincide geometrically afterwards. Thus, the cost of the segment part of the graph irrigating y_3 is $2l(1/10)^\alpha$ on the left-hand side of Figure 3 and $l(1/5)^\alpha$ on the right-hand side. Now, this difference does not matter, as it is easily shown [**11**], [**19**] that optimal networks are loop free. Considering the simplest example of transportation with two Dirac masses as a demand (see Figure 1), Maddalena-Morel-Solimini's solution coincides with the Qinglan Xia's one displayed in Figure 1. In this case the solution is given by the set of fibers $\chi\colon [0,1] \times [0,\infty) \to \mathbb{R}^2$, where $\chi(p,t)$ is either the path from x to y_1 (if $p \in [0,1/2]$), or the path from x to y_2 (if $p \in (1/2, 1]$). In any case, we may parameterize these paths by arc length. Another difference which makes the model in [**15**] slightly more restrictive than the Gilbert and the Qinglan Xia models is the fact that the source is a single Dirac mass.

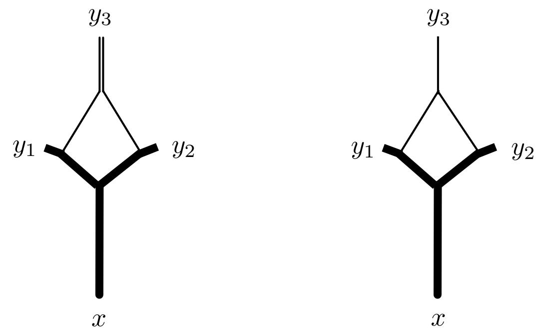

Figure 3. Maddalena-Morel-Solimini's versus Qinglan Xia's model of the irrigation problem with $\mu^+ = \delta_x$ and $\mu^- = \frac{2}{5}\delta_{y_1} + \frac{2}{5}\delta_{y_2} + \frac{1}{5}\delta_{y_3}$. The two geometric objects are the same but on the left-hand side, once fibers separate, they are considered to be separated until they stop. This difference, however, is irrelevant for optimal networks, which are loop free.

Optimal urban transportation problem.

In [**5**], [**6**] and [**4**], a different extension of the Monge-Kantorovitch problem has been proposed to model urban transportation network. In [**5**], a transportation network is modelled as a connected closed set Σ. The users can either walk or join and use Σ. Thus, the cost for going from x to y is $d_\Sigma(x, y) := d(x, y) \wedge (\text{dist}(x, \Sigma) + \text{dist}(y, \Sigma))$, i.e. the minimum between the euclidian (walking) distance $d(x, y)$ and the sum of distances from x and y to the network. Notice that the distance d_Σ describes how the euclidian distance is twisted by the network. Given a population density μ^+ and a density of workplaces μ^-, the cost of this transportation network is given by the Monge-Kantorovitch distance between μ^+ and μ^-, for the cost $d_\Sigma(x, y)$. The authors of [**5**] then consider optimal transportation networks, i.e. transportation networks with a minimal cost among all transportation networks with length less than a prescribed length L, and study their qualitative topological and geometrical properties.

The traffic problem.

Neither the irrigation nor the optimal urban transportation models incorporate a transference plan constraint, that is to say, a "who is going where" set of constraints. In case that we incorporate them, we call this generalization the traffic problem and its solution a traffic plan. This problem was briefly addressed in [**19**], but its solution is not satisfactory, to the best our knowledge. We explain in the next paragraph why. In order to understand the discussion, it is good to consider the very basic problem where $\mu^+ = \delta_{x_1} + \delta_{x_2}$ and $\mu^- = \delta_{y_1} + \delta_{y_2}$ as in Figure 2, i.e. $d(x_1, y_1) = d(x_2, y_2)$ is small compared to $d(y_1, y_2) = d(x_1, x_2)$. From the irrigation problem viewpoint, the solution is the same as the Monge-Kantorovitch one since it is not efficient to group the mass of μ^+ together. It is not if instead we want to find the best transportation network with the "who goes where" constraint that all the mass in x_1 is sent onto y_2, and all the mass in x_2 onto y_1.

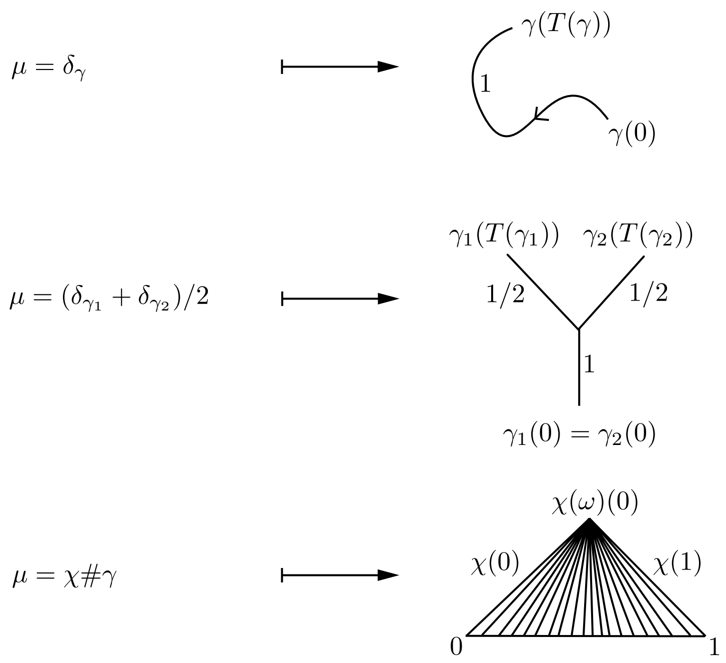

FIGURE 4. Three traffic plans and their associated embedding: a Dirac measure on γ, a tree with one bifurcation, a spread tree irrigating Lebesgue's measure on the segment $[0, 1] \times \{0\}$ of the plane. Let us detail this last example. In that case, to $\omega \in [0, 1]$ corresponds $\chi(\omega) \in K$, the path parameterized by its length from the Dirac mass located at $(1/2, 1)$, to the point $(\omega, 0)$.

The Lagrangian formalism of "fibers" [**15**] is well adapted to that problem since it keeps track of every mass particle. So, in this paper, we propose to generalize this model. We define a traffic plan as a measure on the set of all possible paths. Figure 4 shows three examples of traffic plans: a Dirac mass on a finite length path γ (which means that a unit mass is transported from $\gamma(0)$ to $\gamma(L)$), a traffic plan with "Y" shape, and a traffic plan transporting a Dirac mass to the Lebesgue measure on a segment of the plane. In the same way as for the "Y" shape, a weighted graph can easily be modelled by an atomic measure on the space of paths in the graph. The solution of the traffic problem versus the solution of the irrigation problem is displayed in Figure 2.

We call "traffic plan" any feasible solution for the general traffic problem, that is, a traffic plan is a probability measure on the set of paths. This very handy object is more general than the trees of paths considered in [**15**] since it allows individual spread fibers to exist, like in the third example of Figure 4. It permits also to recover the existence results for the irrigation model obtained using 1-dimensional rectifiable currents in [**19**]. It is also adapted to the urban transportation model proposed by [**5**]. Our main result is that for those models and more general ones, "there exists an optimal traffic plan associated with each transference plan".

A traffic plan as a compatible pair of a transport path and a transference plan.

As mentioned in the previous paragraph, a graph approach modelling the traffic (or mailing) problem was presented in Section 7 of [**19**]. To express the transference plan constraint, Qinglan Xia considers what he calls "compatible pairs" of a transport path and a transference plan. A piecewise rectilinear curve γ can be viewed as a graph with starting and ending points denoted by γ_i^- and γ_i^+. Given an atomic transference plan π, a transport path (a weighed finite graph in that case) is said to be compatible with π if it can be decomposed as a sum of curves γ_i with weight w_i so that $\pi(\gamma_i^-, \gamma_i^+) = w_i$. Notice that the notion of traffic plan is a convenient way to handle such compatible pairs. Indeed, the traffic plan $\sum_i w_i \delta_{\gamma_i}$ contains both the transference plan and the transport path information and is such that they are automatically "compatible". Qinglan Xia then extends this compatibility definition to more general, non atomic, irrigating and irrigated measures. A transport path T and a transference π from μ^+ to μ^- are said to be compatible if

- There exist atomic measures a_i and b_i such that $a_i \rightharpoonup \mu^+$ and $b_i \rightharpoonup \mu^-$.
- There exists a compatible pair (G_i, π_i) of transport path and transference plan from a_i to b_i such that $G_i \rightharpoonup T$ and $\pi_i \rightharpoonup \pi$.

We were not able to find a way to make this definition consistent with the discrete case. Indeed, a pair of a transport path with a transference plan can be both at a time compatible with respect to this last general definition but not compatible with respect to the atomic case definition.

To prove that, let us consider $\mu^+ = \mu^- = \frac{1}{2}\delta_x + \frac{1}{2}\delta_y$. Let T be the null transport path i.e. the one associated with an empty graph. It is such that $\operatorname{div}(T) = \mu^+ - \mu^- = 0$ so that T is a transport path from μ^+ to μ^-. Let π be the transference plan such that $\pi(x, y) = \frac{1}{2}$ and $\pi(y, x) = \frac{1}{2}$. This means that the mass in x and the mass in y are swapped by π. Thus defined, T and π form a compatible pair with respect to the general definition. Indeed, take G_i the graph made of and edge (x, y) with weight $\frac{1}{2}$ and of an edge (y_i, x_i) with weight $\frac{1}{2}$, parallel to (x, y) where y_i and x_i are getting closer and closer of x and y (see Figure 5). Then G_i is weakly converging to $T = 0$. Let us define $a_i = \frac{1}{2}\delta_x + \frac{1}{2}\delta_{y_i}$ and $b_i = \frac{1}{2}\delta_y + \frac{1}{2}\delta_{x_i}$ so that a_i and b_i are weakly converging to μ^+ and μ^-. Finally, let π_i be the transference plan such that $\pi_i(x, y) = \frac{1}{2}$ and $\pi_i(y_i, x_i) = \frac{1}{2}$ so that π_i is weakly converging to π (see Figure 6). Since, G_i and π_i form a compatible pair, it follows that T and π are compatible. However, considered as a pair of a transport path and transference plan irrigating atomic measures, they are no more compatible with respect to the atomic case definition. This proves that the general definition of a compatible pair does not fit with what Qinglan Xia wants a compatible pair in the atomic case to be.

FIGURE 5. On the left hand side: the transport path $G_i = \frac{1}{2}[[e]] + \frac{1}{2}[[e_i]]$ where $[[e]]$ is the vector measure $\mathcal{H}^1\lfloor_e \mathbf{e}$ with $\mathbf{e}$ the unit directional vector of the edge e. On the right hand side: the weak limit of G_i is the null transport path.

FIGURE 6. On the left hand side: the transference plan π_i is such that $\pi_i(x,y) = \pi_i(y_i,x_i) = \frac{1}{2}$. On the right hand side: the limit of π_i is the transference plan π such that $\pi(x,y) = \pi(y,x) = \frac{1}{2}$.

Thus, it seems to us that the traffic plan object is a more convenient way to handle the transference plan constraint since it conveys both at a time the transport path and the transference plan information. Let us give the plan of the present paper. In Section 2, we define traffic plans and transference plans. In Section 3, we model probability measures in a Lagrangian way as sets of particles indexed by $[0,1]$. In Section 4, we prove semicontinuity results, and sequential compactness properties of traffic plans. Section 5 is devoted to the proof of existence of minimizers of the Monge-Kantorovitch problem within our framework. In Section 6, we prove the existence of a minimizer for both the irrigation and the traffic problems. This result in particular retrieves the existence results of [**15**] and [**19**] in a more general setting.

2. Traffic plans with prescribed transference plans

Let $X \subset \mathbb{R}^N$ be a compact set.

Definition 2.1. Let us denote by K the set of 1-Lipschitz maps $\gamma\colon \mathbb{R}^+ \to X$ endowed with the distance

$$d(\gamma,\gamma') := \sup_{k\in\mathbb{N}^*} \frac{1}{k}||\gamma-\gamma'||_{L^\infty([0,k])}.$$

From now on, we consider $\mathcal{B}$, the Borel σ-algebra on K.

Definition 2.2. Let $\gamma \in K$. We define its stopping time as

$$T(\gamma) := \inf\{t : \gamma \text{ constant on } [t,\infty[\}.$$

Remark 2.1. Observe that the stopping time $T\colon K \to \bar{\mathbb{R}}$ is measurable. Indeed, using Lemma 4.2 below, T is lower semicontinuous. This means that $T^{-1}(]A,+\infty])$ is open, then measurable. Thus, T is measurable.

Lemma 2.1. *The metric space (K,d) is compact.*

Proof: The space K is complete and the precompactness is a straightforward consequence of Ascoli-Arzela's Theorem.

Definition 2.3. We define a traffic plan μ as a probability measure on $(K, \mathcal{B})$ such that

$$\int_K T(\gamma)\, d\mu(\gamma) < \infty. \tag{2}$$

We denote by $TP(X)$ the set of all traffic plans in X. We denote by $TP_C(X)$ the set of traffic plans μ such that $\int_K T(\gamma)\, d\mu(\gamma) \leq C$. We shall omit the mention of X in the following.

Remark 2.2. This definition is realistic for a traffic plan, as $T(\gamma)$ represents a transportation time and we don't want the average transportation time to be infinite! Observe that (2) implies that $T(\gamma) < \infty$, μ-almost everywhere.

Definition 2.4. With any traffic plan μ is associated a transference plan, that is to say a probability measure on $X \times X$ that we denote by π_μ and define by

$$\langle \pi_\mu, \phi \rangle := \int_K \phi(\gamma(0), \gamma(T(\gamma)))\, d\mu(\gamma),$$

where $\phi \in C(X \times X, \mathbb{R})$. In an informal way, $\pi_\mu(A \times B)$ is the mass carried from A to B by means of the traffic plan μ. We denote by $TP(\pi)$ the set of traffic plans μ such that $\pi_\mu = \pi$. This is the set of traffic plans with prescribed transference plan π.

Definition 2.5. If μ is a traffic plan, we define its irrigating and irrigated measure by

$$\langle \mu^+, \phi_1 \rangle := \langle \pi_\mu, \phi_1 \otimes 1\!\!1_X \rangle \text{ and } \langle \mu^-, \phi_2 \rangle := \langle \pi_\mu, 1\!\!1_X \otimes \phi_2 \rangle \quad \phi_1, \phi_2 \in C(X).$$

We denote by $TP(\nu^+, \nu^-)$ the set of traffic plans μ such that $\mu^+ = \nu^+$ and $\mu^- = \nu^-$.

3. Parameterization of a probability measure on a precompact metric space

The aim of this section is to show that we can associate with any probability measure a system of "elementary particles" such that $\mu_n \rightharpoonup \mu$ becomes "almost every elementary particle of μ_n tends to an elementary particle of μ". In an abstract setting, we assume in this section that (K, d) is a precompact metric space equipped with the σ-algebra of its Borel sets. We presume that the results in this section must be known.

Unfortunately, we were not able to find any reference. The results of this section will be applied to traffic plans but it is convenient to develop them in a more general setting.

Definition 3.1. Let μ be a probability measure on K. We call parameterization of μ a measurable application $\chi\colon \omega \in [0,1] \to K$ such that $\mu = \chi\#\lambda$ where λ is the Lebesgue measure on $[0,1]$. That is to say $\mu(A) = \lambda(\chi^{-1}(A))$. Observe that if $\phi\colon K \to \mathbb{R}^+$ is a μ-measurable function, then $\int_K \phi(\gamma)\, d\mu(\gamma) = \int_\Omega \phi(\chi(\omega))\, d\omega$ [**1**, Definition 1.70, p. 32].

Remark 3.1. As an illustrative example, if $K = [-1,1]$, the Dirac mass at 0 is parameterized by the null constant application on $[0,1]$. In the same way, an atomic measure $\sum_1^n a_i\delta_{x_i}$ can be parameterized by the piecewise constant function $\chi(\omega) = x_1$ on $[0,a_1]$, $\chi(\omega) = x_2$ on $]a_1,a_2]$ and so on.

Remark 3.2. Recall that the function $\chi\colon [0,1]\times\mathbb{R}^+ \to \mathbb{R}^N$ is called a Carathéodory function if $\chi(\omega,t)$ is a continuous function of t for almost every $\omega \in [0,1]$ and is measurable in ω for every $t \in \mathbb{R}^+$. As it is well-known, Carathéodory functions are measurable as functions of (ω,t) [**8**]. As a function of (ω,t), when K is given by Definition 2.1, the parameterization χ defined in Definition 3.1 is a Carathéodory function. Observe that, as a consequence of Proposition 4.1, both concepts coincide for functions $\chi\colon [0,1] \to K$.

In Lemma 3.2, we shall construct a filtration on K of a special kind which gives us a parameterization of μ (see Lemma 3.3). For that, we first prove that we can construct a filtration on K whose sets have a specified diameter. Then, in Lemma 3.2, we prove that we can adapt the filtration so that μ does not charge the boundaries of its elements.

Lemma 3.1. *There exists a filtration of K made of finite partitions $\mathcal{F}_l = \{F_j^l : 1 \le j \le J_l\}$, where $J_l \in \mathbb{N}^*$, such that the diameters of the sets F_j^l are less than 2^{-l}.*

Proof: We construct this filtration recursively. In order to construct $\mathcal{F}_1$, we cover K with a finite number of balls of radii $1/4$. Let us denote by B_i, where $1 \le i \le n$, the intersection of these balls with K. Let us find a partition of $K = \cup_i B_i$ with at most n elements. To do this, we denote $\tilde{F}_1^1 := B_1$ and, in a recursive way, we define $\tilde{F}_{i+1}^1 := B_{i+1} \setminus \cup_{j\le i} B_j$. If any of the $\tilde{F}_i^1$ is empty, we do not take it into account, so that we obtain a family of non empty elements F_i^1 where $i \le J_1$. Since the F_i^1 are precompact, we can iterate the above process by covering them with balls of radius 1/8. Proceeding iteratively we construct the desired filtration. □

Lemma 3.2. *Let μ be a probability measure on K. There exists a filtration made of finite partitions $\mathcal{F}_l = \{F_j^l : 1 \leq j \leq J_l\}$, $J_l \in \mathbb{N}^*$, such that the diameters of F_j^l are less than 2^{-l+1} and $\mu(\partial F_j^l) = 0$ for all l and $j \leq J_l$.*

Proof: To obtain this filtration, we slightly modify the construction of Lemma 3.1. We only need to request in addition that $\mu(\partial F_j^l) = 0$ for all l and $j \in J_l$. For that, it is enough to perturb the radii $r_l = 2^{-l}(1 + \epsilon_l)$, with $\epsilon_l \leq 1$ so that μ does not charge the boundaries of the balls with radius r_l used to construct $\mathcal{F}_l$. □

The filtration obtained in Lemma 3.2 allows us to define a canonical parameterization of μ. The idea is to group together the ω's whose images are close.

Lemma 3.3. *Let μ be a probability measure on K and $\mathcal{F}$ be the filtration constructed in Lemma 3.2. There exists a parameterization χ of μ such that for all l, the sets*

$$\Omega_{j,l} = \{\omega : \chi(\omega) \in F_j^l\}$$

are intervals ordered in an increasing way with j.

Proof: We construct χ by successive approximations χ_n using the filtration of Lemma 3.2.

Step 1: Definition of χ_n. Let $t_0^n := 0$ and $t_j^n := \sum_{i \leq j} \mu(F_i^n)$ where $1 \leq j \leq J_n$. The application χ_n is defined as a piecewise constant function sending each interval $[t_{j-1}^n, t_j^n[$ onto an arbitrary element of F_j^n. By construction, $\Omega_{j,l} := \{\omega : \chi_n(\omega) \in F_j^l\} = [t_{j-1}^l, t_j^l[$ for all $j \leq J_l$. We notice that the intervals $[t_{j-1}^l, t_j^l[$ where $1 \leq j \leq J_l$, are intervals ordered in an increasing way when j goes from 1 to J_l, so that their union is $[0, 1[$. Notice also that $\mu(F_j^l) = \lambda(\Omega_{j,l})$.

Step 2: The sequence $\chi_n(\omega)$ converges for all ω. Let us prove that χ_n is a Cauchy sequence. Let us first observe that $\chi_n(\Omega_j^m) \subset F_j^m$ for any $n \geq m$. Indeed, let us fix m and $n \geq m$. By the definition of filtration, Ω_j^m is the union of Ω_k^n where k describes the set of indices such that $F_k^n \subset F_j^m$. Thus, χ_n sends every element of Ω_k^n to an element of $F_k^n \subset F_j^m$. A fortiori, the image of Ω_j^m under χ_n is in F_j^m. Now, since the sets F_j^m have diameter less than 2^{-m}, we deduce that $d(\chi_n(\omega), \chi_m(\omega)) < 2^{-m}$ for all $m \leq n$. Thus, $\chi_n(\omega)$ is a Cauchy sequence.

Let χ be the pointwise limit of χ_n. Observe that χ is measurable as a pointwise limit of measurable functions.

Step 3: The measure $\chi\#\lambda$ is exactly μ. We have to show that $\chi\#\lambda(F_j^l) = \mu(F_j^l)$ for all (j,l). The measures μ and $\chi\#\lambda$ will then be equal on the sets F_j^l which form a Π-system. Then the extension theorem of Π-systems [**18**, Lemma 1.6, p. 19] shows that $\mu = \chi\#\lambda$ on the σ-algebra generated by this Π-system, that is, on the σ-algebra of Borel sets of K.

Let us fix l, $j \leq J_l$, and let us define

$$G_p := \{\gamma \in F_j^l : d(\gamma, \partial F_j^l) \geq 1/p\}.$$

This is a non decreasing sequence of sets such that $\cup_p G_p = F_j^l \setminus \partial F_j^l$. Fix $\epsilon > 0$. For a sufficiently large p, we have

$$\mu(G_p) \geq \mu(F_j^l) - \epsilon. \tag{3}$$

Now, consider an l' such that $2^{-l'} < \frac{1}{2p}$. For any $y \in G_p$, there exists k so that $y \in F_k^{l'}$. Since the diameter of $F_k^{l'}$ is less than $\frac{1}{2p}$, $F_k^{l'} \subset G_{2p}$ so that $\bar{F}_k^{l'} \subset F_j^l$. For $n \geq l'$, the construction of χ_n ensures that $\chi_n(\Omega_k^{l'}) \subset F_k^{l'}$. Since χ is the pointwise limit of χ_n,

$$\chi(\Omega_k^{l'}) \subset \bar{F}_k^{l'} \subset F_j^l. \tag{4}$$

We obtain a covering of G_p with sets of the form $F_k^{l'}$ satisfying (4), and, using (3), we have $\chi\#\lambda(F_j^l) \geq \mu(F_j^l) - \epsilon$. This being true for all $\epsilon > 0$, we deduce that $\chi\#\lambda(F_j^l) \geq \mu(F_j^l)$. Since these sets form a partition for $1 \leq j \leq J_l$, and $\chi\#\lambda$ is a probability measure, the inequality is indeed an equality, that is: $\chi\#\lambda(F_j^l) = \mu(F_j^l)$. As a consequence, we have $\chi^{-1}(F_j^l) = \Omega_{j,l}$ modulo a null set. □

Definition 3.2. Let $(\mu_n)_n$ and μ be probability measures on (K, d). We say that μ_n tends to μ "pointwise" whenever there exist parameterizations χ_n and χ of μ_n and of μ, respectively, such that $d(\chi_n(\omega), \chi(\omega)) \to 0$ almost everywhere in $[0,1]$.

Theorem 3.1. *Let $(\mu_n)_n$ be a sequence of probability measures on (K, d). Then μ_n weakly-$*$ converges to μ if and only if μ_n to μ tends to μ "pointwise".*

Proof: Assume that μ_n converges to μ "pointwise", and let χ_n, χ denote the parameterizations of μ_n and μ, respectively. Since $\chi_n(\omega)$ converges to $\chi(\omega)$ for almost every ω, using Lebesgue's theorem, for all $\phi \in C(K)$, we have

$$\langle \mu_n, \phi \rangle = \int_K \phi(\gamma)\, d\mu_n(\gamma) = \int_{[0,1]} \phi(\chi_n(\omega))\, d\omega$$
$$\to \int_{[0,1]} \phi(\chi(\omega))\, d\omega = \int_K \phi(\gamma)\, d\mu(\gamma) = \langle \mu, \phi \rangle.$$

Conversely, let μ_n be weakly-$*$ converging to μ. Let us consider the filtration associated with μ constructed in Lemma 3.2. Since $\mu(\partial F_j^l) = 0$, we deduce that $\mu_n(F_j^l)$ converges to $\mu(F_j^l)$. Next, applying Lemma 3.3 to measures μ_n and μ, we get applications χ_n and χ such that $\chi_n\#\lambda = \mu_n$ and $\chi\#\lambda = \mu$. The fact that $\mu_n(F_j^l)$ converges to $\mu(F_j^l)$ implies that $\lambda(\Omega_{j,l}^n)$ converges to $\lambda(\Omega_{j,l})$, where $\Omega_{j,l}^n := \{\omega : \chi_n(\omega) \in F_j^l\}$ and $\Omega_{j,l} := \{\omega : \chi(\omega) \in F_j^l\}$. This convergence of measures implies the convergence of intervals $\Omega_{j,l}^n$ to some intervals $\Omega_{j,l}$, ordered in an increasing way with j.

We are now in a position to prove that for almost all ω the sequence $\chi_n(\omega)$ converges to $\chi(\omega)$. Notice that for almost all ω and for any $l \in \mathbb{N}$, there exists a $j \leq J_l$ such that ω is in the interior of $\Omega_{j,l}$. Indeed, there is a finite number of such intervals at each rank of the filtration, and, thus, the set of its endpoints is countable, hence of measure zero. Thus, for n large enough, we have that $\omega \in \Omega_{j,l}^n$, i.e., $\chi_n(\omega) \in F_j^l$. This yields $d(\chi_n(\omega), \chi(\omega)) < 2^{-l}$. □

4. Stability properties of traffic plans

From now on, we will denote $|A| := \lambda(A)$ the Lebesgue measure of a measurable set $A \subset [0,1]$. Throughout this section, (K,d) is the compact metric space of Definition 2.1. According to Lemma 3.3, we can associate with a traffic plan μ a parameterization $\chi\colon \Omega \to K$. We set $\chi(\omega,t) := \chi(\omega)(t)$. It is easy to check that χ is a measurable function from $\Omega \times \mathbb{R}^+ \to X$. Indeed, this is true, since χ is a Carathéodory function (see Remark 3.2). Moreover, if a function $\chi\colon [0,1] \to K$ is measurable as a function of (ω,t), then it is measurable as a function from $[0,1]$ to (K,d). Being a simple argument, we include it here for the sake of completeness.

Proposition 4.1. *The application $\chi\colon \Omega \times \mathbb{R}^+ \to X$ is measurable if and only if the application $\omega \in [0,1] \mapsto \chi(\omega,\cdot) \in K$ is measurable.*

Proof: Let $\chi\colon \Omega \times \mathbb{R}^+ \to X$ be a measurable function. Observe that

$$\begin{aligned}\chi^{-1}(B(\gamma, r)) &= \{\omega : d(\chi(\omega), \gamma) \leq r\} \\ &= \left\{\omega : \forall\, k,\, \frac{||\chi(\omega) - \gamma||_{L^\infty([0,k])}}{k} \leq r\right\} \\ &= \cap_k \{\omega : ||\chi(\omega) - \gamma||_{L^\infty([0,k])} \leq kr\} \\ &= \cap_k \cap_{t \in \mathbb{Q} \cap [0,k]} \{\omega : |\chi(\omega)(t) - \gamma(t)| \leq kr\}.\end{aligned}$$

This last expression is a countable intersection of measurable sets since the maps $\tilde{\chi}\colon \omega \mapsto \tilde{\chi}(\omega, t)$ are measurable for any $t \in [0, 1]$. □

This shows that if $\chi\colon \Omega \times \mathbb{R}^+ \to \mathbb{R}^N$ is measurable, we can define its associated traffic plan $\mu := \chi \# \lambda$. Of course, as we can deduce from the preceding section, a traffic plan can have many different parameterizations.

Definition 4.1. Let μ_n be a sequence of traffic plans. We shall say that μ_n converges to a traffic plan μ if one of the equivalent relationships is satisfied:

$$\mu_n \rightharpoonup \mu,$$

$$\chi_n(\omega) \to \chi(\omega) \text{ in } K \text{ for almost all } \omega \in \Omega,$$

where μ_n and μ are parameterized using a common filtration constructed as in Lemma 3.2, such that $\mu_n(\partial F_j^l) = \mu(\partial F_j^l) = 0$ for any j, l.

Remark 4.1. An immediate adaptation of Lemma 3.2 permits to use the same filtration to construct the parameterizations of all measures μ_n and μ.

4.1. Lower semicontinuity of length, stopping time, averaged length and averaged stopping time.

Lemma 4.1. *Let μ_n be a sequence of probability measures on a compact metric space K and such that μ_n weakly converges to μ. Let $\gamma \mapsto f(\gamma)$ be a lower semicontinuous function on K. Then,*

$$\int_K f(\gamma)\, d\mu(\gamma) \leq \liminf \int_K f(\gamma)\, d\mu_n(\gamma).$$

Proof: This is a straightforward application of the fact that any lower semicontinuous function f on a metric compact space is the increasing limit of a sequence of continuous functions [**1**, Lemma 1.61, p. 27], and the monotone convergence theorem. □

Lemma 4.2. *Let $L(\gamma)$ denote the length of $\gamma \in K$. If the sequence $\gamma_n \in K$ converges to γ for the metric d, then*

$$T(\gamma) \leq \liminf T(\gamma_n),$$

and

$$L(\gamma) \leq \liminf L(\gamma_n).$$

Proof: For all $t \geq s > \liminf T(\gamma_n)$, there exists an increasing sequence of indices n_k going to infinity such that $T(\gamma_{n_k}) < s \leq t$. This ensures that $\gamma_{n_k}(t) = \gamma_{n_k}(s)$. Considering the limit of this equality, we obtain $\gamma(t) = \gamma(s)$. Then γ is constant on $]\liminf T(\gamma_n), +\infty[$, so that $T(\gamma) \leq \liminf T(\gamma_n)$. The lower semicontinuity of the length functional is well-known and we shall omit the details. □

Lemma 4.3. *If a sequence of traffic plans μ_n converges to μ, then*

$$\int_K T(\gamma)\, d\mu(\gamma) \leq \liminf \int_K T(\gamma)\, d\mu_n(\gamma)$$

and

$$\int_K L(\gamma)\, d\mu(\gamma) \leq \liminf \int_K L(\gamma)\, d\mu_n(\gamma).$$

Proof: Because of Lemma 4.2, the applications $\gamma \mapsto T(\gamma)$ and $\gamma \mapsto L(\gamma)$ are lower semicontinuous. The desired inequalities then directly come from Lemma 4.1. □

4.2. Multiplicity of a traffic plan and its upper semicontinuity.

Definition 4.2. Let μ be a traffic plan. We call multiplicity of μ at a point $x \in \mathbb{R}^N$ the number

$$|x|_\mu := \mu(\{\gamma : \exists\, t,\, \gamma(t) = x\}).$$

If χ is a parameterization of μ, then we define the path class of $x \in \mathbb{R}^N$ as the set

$$[x]_\chi := \{\omega : \exists\, t,\, \chi(\omega, t) = x\}.$$

Since $\chi \# \lambda = \mu$, we have that $|[x]_\chi| = |x|_\mu$.

Remark 4.2. The multiplicity is well defined since the set $\{\gamma : \exists\, t,\, \gamma(t) = x\}$ is a Borel set of K. Indeed, $\{\gamma : \exists\, t,\, \gamma(t) = x\} = \cup_n \{\gamma : \exists\, t \leq n,\, \gamma(t) = x\}$ is a union of closed sets in K.

Proposition 4.2 ([**15**, Lemma 6.2])**.** *Let χ_n be a sequence of parameterizations of traffic plans converging to χ. Suppose further that there is $C > 0$ such that $\int_\Omega T(\chi_n(\omega))\, d\omega \le C$. Then, for almost all ω,*

$$\limsup |[\chi_n(\omega,t)]_{\chi_n}| \le |[\chi(\omega,t)]_\chi|.$$

Proof: Set $\epsilon = C/M$. By Markov's inequality,

$$|\{\omega : T(\chi_n(\omega)) > M\}| \le \frac{C}{M} = \epsilon.$$

Let us define an approximate multiplicity by

$$[\chi(\omega,t)]^\epsilon_\chi := \{\omega' \in [\chi(\omega,t)]_\chi : T(\chi(\omega')) \le M\}.$$

Next, let us take an element ω' in $\cap_k \cup_{n>k} [\chi_n(\omega,t)]^\epsilon_{\chi_n}$. This means that there exists a sequence of indices n_i which goes to infinity, and times $s_i \le T(\chi_{n_i}(\omega)) \le M$ such that $\chi_{n_i}(\omega', s_i) = \chi_{n_i}(\omega,t)$. Since s_i is bounded, it is possible to extract $s_i \to s$ and because of uniform convergence of $\chi_{n_i}(\omega', \cdot)$ on $[0,M]$, we obtain $\chi(\omega', s) = \chi(\omega,t)$, hence $\omega' \in [\chi(\omega,t)]_\chi$. This shows that $\cap_k \cup_{n>k} [\chi_n(\omega,t)]^\epsilon_{\chi_n} \subset [\chi(\omega,t)]_\chi$, so that

$$\limsup |[\chi_n(\omega,t)]^\epsilon_{\chi_n}| \le |[\chi(\omega,t)]_\chi|.$$

Thus,

$$\limsup |[\chi_n(\omega,t)]_{\chi_n}| - \epsilon \le |[\chi(\omega,t)]_\chi|. \qquad \square$$

We prove another kind of upper semicontinuity which will be useful to prove Corollary 4.1.

Lemma 4.4. *Let χ be a parametrization of a traffic plan μ. Then, the function $\phi\colon x \mapsto |[x]_\chi|$ is upper semicontinuous.*

Proof: Let us show that for each x such that $|[x]_\chi| < r$, there is a ball $B(x,\epsilon)$ such that for all y in $B(x,\epsilon)$, $|[y]_\chi| < r$. This will prove that $\phi^{-1}([0,r[)$ is an open set, and therefore that ϕ is upper semicontinuous. Suppose that it is not the case. Then, for each ball $B_n := B(x, 1/n)$, there is a $y_n \in B_n$ so that $|[y_n]_\chi| \ge r$. Notice that y_n tends to x when n goes to infinity. Let us consider

$$\tilde\Omega := \cap_n \cup_{m\ge n} [y_m]_\chi.$$

Then, modulo a null set, $\tilde\Omega \subset [x]_\chi$. Indeed, for almost every ω, $T(\chi(\omega)) < \infty$. For such an ω in $\tilde\Omega$, this means that for all n, there is an $m \ge n$ such that $\omega \in [y_m]_\chi$, that is, there is a t_m such that $\chi(\omega, t_m) = y_m$. Since $T(\chi(\omega)) < \infty$, the sequence $(t_m)_m$ can be supposed to be bounded. Thus, it is possible to extract a convergent subsequence $t_m \to t$ such that $\chi(\omega,t) = x$, i.e., $\omega \in [x]_\chi$. Thus $|\tilde\Omega| \le |[x]_\chi| < r$ and $|\tilde\Omega| = \lim_n |\cup_{m\ge n} [y_n]_\chi| \ge r$. This contradicts our initial assumption. $\square$

Corollary 4.1. *Let χ be a parametrization of a traffic plan μ. The function $(\omega,t)\mapsto |[\chi(\omega,t)]_\chi|$ is measurable.*

Proof: This a consequence of the measurability of $x\mapsto |[x]_\chi|$ (Lemma 4.4). Indeed, we have

$$\begin{aligned}\{(\omega,t):|[\chi(\omega,t)]_\chi|<r\} &= \{(\omega,t):\chi(\omega,t)=x \text{ and } |[x]_\chi|<r\}\\ &= \chi^{-1}(\{x:|[x]_\chi|<r\}).\end{aligned}$$

□

4.3. Sequential compactness of traffic plans.

Theorem 4.1. *If $(\mu_n)_n$ is a sequence of TP_C such that $\mu_n \rightharpoonup \mu$, then $\pi_{\mu_n}\rightharpoonup \pi_\mu$. Hence, given a sequence $(\mu_n)_n$ of TP_C, it is possible to extract a convergent subsequence such that π_{μ_n} converges.*

Proof: Set $\epsilon = C/M$. By Markov's inequality, we have $\mu_n(K\setminus K_\epsilon)\le \frac{C}{M}=\epsilon$ where $K_\epsilon := \{\gamma : T(\gamma)\le M\}$. Because of Lemma 4.3, we also have that $\int_K T(\gamma)\,d\mu(\gamma)\le C$, and, thus, $\mu(K\setminus K_\epsilon)<\epsilon$. Let $\phi\in C(X\times X,\mathbb{R})$. Since, by definition of the distance on K, the map $\gamma\mapsto \phi(\gamma(0),\gamma(M))$ is continuous from K to $\mathbb{R}$, then, by definition of the transference plan associated with a traffic plan, we have

$$\begin{aligned}\limsup_n\langle \pi_{\mu_n},\phi\rangle &\le \limsup_n\left(\int_{K_\epsilon}\phi(\gamma(0),\gamma(T(\gamma)))\,d\mu_n(\gamma)+\epsilon||\phi||_\infty\right)\\
&= \limsup_n\int_{K_\epsilon}\phi(\gamma(0),\gamma(M))\,d\mu_n(\gamma)+\epsilon||\phi||_\infty\\
&\le \limsup_n\int_{K}\phi(\gamma(0),\gamma(M))\,d\mu_n(\gamma)+2\epsilon||\phi||_\infty\\
&= \int_{K}\phi(\gamma(0),\gamma(M))\,d\mu(\gamma)+2\epsilon||\phi||_\infty\\
&\le \int_{K}\phi(\gamma(0),\gamma(T(\gamma)))\,d\mu(\gamma)+4\epsilon||\phi||_\infty\\
&= \langle\pi_\mu,\phi\rangle+4\epsilon||\phi||_\infty.\end{aligned}$$

In the same way,

$$\liminf_n\langle\pi_{\mu_n},\phi\rangle \ge \langle\pi_\mu,\phi\rangle-4\epsilon||\phi||_\infty.$$

□

Corollary 4.2. *Let π be a probability measure on $X\times X$. There exists a traffic plan μ such that $\pi_\mu=\pi$.*

Proof: Let us first prove this property in the case of finite atomic measures π. Let $(a_i)_{i=1}^k$ and $(b_j)_{j=1}^l$ the elements of the support of the two marginals of π. Let us denote by $\pi_{i,j}$ the values $\pi(\{a_i\} \times \{b_j\})$. We now define $\gamma_{i,j} \in K$, the segment joining a_i to b_j, i.e. $\gamma_{i,j}(0) = a_i$, for $t \in]0, |a_i - b_j|]$,

$$\gamma_{i,j}(t) := \frac{t}{|a_i - b_j|} b_j + \frac{1-t}{|a_i - b_j|} a_i$$

and $\gamma_{i,j}$ is constant on $[|a_i - b_j|, \infty[$. The traffic plan $\mu := \sum_{i,j} \pi_{i,j} \delta_{\gamma_{i,j}}$ is such that $\pi_\mu = \pi$ by construction.

Let us now consider a general transference plan π and a sequence of atomic measures π_n such that $\pi_n \rightharpoonup \pi$. The first part of the proof tells that there are traffic plans μ_n such that $\pi_{\mu_n} = \pi_n$. By Theorem 4.1, we can extract a converging subsequence from $(\mu_n)_n$ such that μ_n converges to μ with $\pi_{\mu_n} \rightharpoonup \pi_\mu$. Thus, the traffic plan μ is such that $\pi_\mu = \pi$. □

5. Monge-Kantorovitch problem

For a sake of completeness, we show that the above formalism is adapted to solve the Monge-Kantorovitch problem. Of course, no result is new here.

Definition 5.1. We call cost of a traffic plan a functional

$$I(\mu) = \int_K c(\gamma(0), \gamma(T(\gamma)))\, d\mu(\gamma),$$

where c is a bounded non-negative lower semicontinuous function which informally represents the cost for transporting a unit of mass from x to y.

Let us notice that $I(\mu) = \int_{X\times X} c(x,y)\, d\pi_\mu(x,y)$ where π_μ is the transference plan associated to the traffic plan μ. Given two measures ν^+ and ν^-, the Monge-Kantorovitch problem consists in minimizing $\int_{X\times X} c(x,y)\, d\pi(x,y)$ under prescribed marginal measures ν^+ and ν^-. By Corollary 4.2, any transference plan can be obtained (in a not unique way) as the transference plan π_μ associated to a traffic plan μ. Thus, the problem of minimizing $I(\mu)$ under prescribed marginal measures ν^+ and ν^- is equivalent to the Monge-Kantorovitch problem. The existence of an optimal transference plan is given by standard lower semicontinuity argument and compactness. The next two propositions uses the same strategy at the level of traffic plans.

Proposition 5.1. *If $(\mu_n)_n$ and μ are traffic plans such that $\mu_n \rightharpoonup \mu$, then*

$$I(\mu) \leq \liminf I(\mu_n).$$

Proof: The application $\gamma \mapsto c(\gamma(0), \gamma(M))$ is lower semicontinuous because of the lower semicontinuity of c. Then Lemma 4.1 asserts that

$$\liminf \int_K c(\gamma(0), \gamma(M))\, d\mu_n(\gamma) \geq \int_K c(\gamma(0), \gamma(M))\, d\mu(\gamma).$$

Set $\epsilon = C/M$. By Markov's inequality, $\mu_n(K \setminus K_\epsilon) \leq \frac{C}{M} = \epsilon$ where

$$K_\epsilon := \{\gamma : T(\gamma) \leq M\}.$$

For such an M, we have

$$\int_K c(\gamma(0), \gamma(M))\, d\mu_n(\gamma) \leq I(\mu_n) + \epsilon ||c||_\infty$$

and

$$\int_K c(\gamma(0), \gamma(M))\, d\mu(\gamma) \geq I(\mu) - \epsilon ||c||_\infty,$$

so that

$$I(\mu_n) + \epsilon ||c||_\infty \geq I(\mu) - \epsilon ||c||_\infty.$$

□

Proposition 5.2. *The problem of minimizing $I(\mu)$, with $\mu \in TP_C(\nu^+, \nu^-)$ admits a solution.*

Proof: Let μ_n be a minimizing sequence. Because of Theorem 4.1, there exists a subsequence such that $\mu_n \rightharpoonup \mu$ and $\pi_{\mu_n} \rightharpoonup \pi_\mu$. In particular, we have $\mu_n^+ \rightharpoonup \mu^+$ and $\mu_n^- \rightharpoonup \mu^-$. Since $\mu_n^+ = \nu^+$ and $\mu_n^- = \nu^-$ for all n, μ is a traffic plan satisfying the constraints and such that $I(\mu) \leq \liminf I(\mu_n)$. Since μ_n is a minimizing sequence, μ is a minimizer of I under the constraints of irrigating and irrigated measures. □

6. Irrigation and traffic models

In this section, the cost functional we consider is taken from two irrigation models proposed in [**19**] and [**15**]. As in these models, we prove that the functional admits a minimizer under the constraint of prescribed irrigating and irrigated measures. In addition, our model permits to handle a prescribed transference plan constraint. We prove the existence of minimizing traffic plans with this new constraint. So we move from an irrigation model to a traffic model. The first three subsections are devoted to the proof of the existence of minimizers of the energy functional under the two different sets of constraints. In the other two subsections, we show that there exists a loop-free minimizer of the energy. A change of variable formula permits us to prove that the energy functional coincides with Q. Xia's one [**19**], [**20**] on loop-free traffic plans.

6.1. Energy of a traffic plan and existence of a minimizer.

We use the convention that $0^{\alpha-1} = \infty$ with $\alpha \in [0, 1)$.

Definition 6.1. Let $\alpha \in [0, 1]$. We call energy of a traffic plan the functional

$$E(\mu) = \int_\Omega \int_{\mathbb{R}^+} |[\chi(\omega,t)]_\chi|^{\alpha-1} |\dot\chi(\omega,t)| \, dt \, d\omega, \tag{5}$$

where χ is a parameterization of μ.

Remark 6.1. This energy will be proved to be a reformulation of the one used in [**19**] (see Proposition 6.5).

Remark 6.2. The application $(\omega,t) \mapsto |[\chi(\omega,t)]_\chi|$ was shown to be measurable in Corollary 4.1. Let us denote $|\dot\chi(\omega,t)|_{\text{sup}} := \limsup_{s\to t} \left| \frac{\chi(\omega,t)-\chi(\omega,s)}{t-s} \right|$ and $|\dot\chi(\omega,t)|_{\text{inf}} := \liminf_{s\to t} \left| \frac{\chi(\omega,t)-\chi(\omega,s)}{t-s} \right|$. Both applications $(\omega,t) \mapsto |\dot\chi(\omega,t)|_{\text{sup}}$ and $(\omega,t) \mapsto |\dot\chi(\omega,t)|_{\text{inf}}$ are measurable since they can be interpreted as a pointwise limit of measurable functions. For almost every ω and for almost every t, $|\dot\chi(\omega,t)|_{\text{inf}} = |\dot\chi(\omega,t)|_{\text{sup}}$ since $\chi(\omega,\cdot)$ is 1-Lipschitz. Thus, the set C where $|\dot\chi(\omega,t)|$ is well defined is measurable. If $|\dot\chi|$ is extended by 0 on $\Omega \times \mathbb{R} \setminus C$ (which is of null measure), the function thus defined is measurable.

Remark 6.3. The energy of a traffic plan could also be written

$$E(\mu) = \int_K \int_{\mathbb{R}^+} |\gamma(t)|_\mu^{\alpha-1} |\dot\gamma(t)| \, dt \, d\mu(\gamma).$$

The traffic problem is the following: given two measures ν^+ and ν^-, and a transference plan π between those measures, we look for minimizers of E with this prescribed transference plan. The irrigation problem is the less constrained case where we specify globally the supply and the demand. This latter case is essentially the same as in [**19**].

Lemma 6.1. *Let μ be a traffic plan. Then, we have*

$$E(\mu) \geq \int_K L(\gamma) \, d\mu(\gamma).$$

Proof: As the multiplicity at a point x is always less than 1, we have $|x|_\mu^{\alpha-1} \geq 1$ and then

$$E(\mu) \geq \int_K \int_{\mathbb{R}^+} |\dot{\gamma}(t)|\, dt\, d\mu(\gamma) = \int_K L(\gamma)\, d\mu(\gamma). \qquad \square$$

6.2. Normalization of a traffic plan.

Lemma 6.2. *Let $\chi\colon [0,1] \to K$ be a parameterization of the traffic plan μ. We define $\tilde{\chi}(\omega)$ the arc-length reparameterization of $\chi(\omega)$ in the usual way. Let*

$$S(\omega,t) = \int_0^t |\dot{\chi}(\omega,r)|\, dr,$$

and let

$$T(\omega,s) = \inf\{t \in [0,\infty) : S(\omega,t) = s\}.$$

Let $\tilde{\chi}(\omega,s) = \chi(\omega, T(\omega,s))$. Then $\tilde{\chi}(\omega) \in K$ is Lebesgue measurable and for all $\omega \in [0,1]$, $\tilde{\chi}(\omega)$ is the arc-length reparameterization of $\chi(\omega)$.

Proof: The map $\tilde{\chi}$ is the composition of the maps $(I,T)\colon [0,1]\times[0,\infty) \to [0,1] \times [0,\infty)$ and $\chi\colon [0,1] \times [0,\infty) \to \mathbb{R}^N$. The measurability of $\tilde{\chi}$ will be a consequence of the measurability of (I,T) and χ, and the fact that $(I,T)^{-1}(N)$ is a null set in $[0,1] \times [0,\infty)$ for any null set N in $[0,1] \times [0,\infty)$.

Let us prove first that (I,T) is measurable. It suffices to prove that the function $T\colon [0,1] \times [0,\infty) \to \mathbb{R}$ is measurable. For that it will be sufficient to prove that $T^{-1}((-\infty,\lambda])$ is measurable for any $\lambda \in \mathbb{R}$. Let $\{t_m\}_m$ be a dense sequence in $[0,\infty)$. Using that T is non decreasing and lower semicontinuous in s we may write

$$T^{-1}((-\infty,\lambda]) = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \{\omega \in [0,1] : T(\omega,t_m) \leq \lambda\} \times \left[0, t_m + \frac{1}{n}\right].$$

Since $\{\omega \in [0,1] : T(\omega,t_m) \leq \lambda\} = \{\omega \in [0,1] : S(\omega,\lambda) \geq t_m\}$ is measurable, we deduce that $T^{-1}((-\infty,\lambda])$ is measurable.

Now, let N be a null set in $[0,1] \times [0,\infty)$ and let B be a Borel set containing N (of total measure less than ϵ). Observe that $F(\omega,s) := \mathbb{1}_B(\omega, T(\omega,s))$ is a measurable map. Now, for a.e. fixed value of each $\omega \in [0,1]$, we have

$$\int_0^\infty F(\omega,s)\, ds = \int_0^\infty \mathbb{1}_B(\omega,t) S_t(\omega,t)\, dt \leq \int_0^\infty \mathbb{1}_B(\omega,t)\, dt,$$

the last inequality being true since $S_t(\omega,t) \leq 1$. Integrating with respect to $\omega \in [0,1]$, and observing that both F and $\mathbb{1}_B$ are measurable in $[0,1]\times$

$[0, \infty)$, we have

$$|(I,T)^{-1}(B)| = \int_0^1 \int_0^\infty \mathbb{1}_B(\omega, T(\omega, s))\, ds\, d\omega \leq \int_0^1 \int_0^\infty \mathbb{1}_B(\omega, t)\, dt\, d\omega \leq \epsilon.$$

We deduce that $(I,T)^{-1}(N)$ is a null set. □

Definition 6.2. We say that $\tilde{\mu}$ is a normalization of a traffic plan μ if for some parameterization χ of μ, $\tilde{\chi}\#\lambda = \tilde{\mu}$, where $\tilde{\chi}(\omega)$ is the arc-length reparameterization of $\chi(\omega)$ defined in Lemma 6.2. Observe that $E(\tilde{\mu}) = E(\mu)$.

Remark 6.4. Due to the fact that $\{\gamma \in K : |\dot{\gamma}| = 1\}$ is not closed under the distance d, it is not true that $\mu_n \rightharpoonup \mu$ implies $\tilde{\mu}_n \rightharpoonup \tilde{\mu}$.

6.3. Existence of a minimizer.

Proposition 6.1. *If $(\mu_n)_n$ is a normalized sequence in TP_C, and μ is a traffic plan such that $\mu_n \rightharpoonup \mu$, then*

$$E(\mu) \leq \liminf E(\mu_n).$$

Proof: Let χ_n, χ' be parameterizations of μ_n and μ, respectively, such that $\chi_n(\omega) \to \chi'(\omega)$ converges in (K, d) for almost every $\omega \in [0,1]$. Because of the upper semicontinuity of multiplicity which was proved in Proposition 4.2 and the lower semicontinuity of $L(\gamma)$, we have

$$\begin{aligned}\liminf_n E(\mu_n) &= \liminf_n \int_\Omega \int_0^{L(\chi_n(\omega))} |[\chi_n(\omega,t)]_{\chi_n}|^{\alpha-1}\, dt\, d\omega \\ &\geq \int_\Omega \int_0^{L(\chi'(\omega))} |[\chi'(\omega,t)]_{\chi'}|^{\alpha-1}\, dt\, d\omega \\ &\geq \int_\Omega \int_0^{L(\chi'(\omega))} |[\chi'(\omega,t)]_{\chi'}|^{\alpha-1} |\dot{\chi}'(\omega,t)|\, dt\, d\omega \\ &= E(\chi') = E(\mu).\end{aligned}$$ □

Proposition 6.2. *The problem of minimizing $E(\mu)$ in $TP(\nu^+, \nu^-)$ admits a solution.*

Proof: In the case $\inf_{TP(\nu^+,\nu^-)} E(\mu) = \infty$, there is nothing to prove. Otherwise, there is some $C < \infty$ such that $\inf_{TP(\nu^+,\nu^-)} E(\mu) \leq C$. Because of Lemma 6.1, $\inf_{TP(\nu^+,\nu^-)} E(\mu) = \inf_{TP_C(\nu^+,\nu^-)} E(\mu)$ so that we can consider a minimizing sequence $(\mu_n)_n$ in $TP_C(\nu^+, \nu^-)$. Since $E(\mu_n) = E(\tilde{\mu}_n)$, without loss of generality, we can take μ_n as being normalized. Because of Theorem 4.1, it is possible to extract a converging subsequence such that $\mu_n \rightharpoonup \mu$, $\nu^+_{\mu_n} \rightharpoonup \nu^+_\mu$, and $\nu^-_{\mu_n} \rightharpoonup \nu^-_\mu$. Since $\nu^+_{\mu_n} = \nu^+$ for all n, and $\nu^-_{\mu_n} = \nu^-$, μ is a traffic plan satisfying the constraints and $E(\mu) \leq \liminf E(\mu_n)$. Since μ_n is a minimizing sequence, μ is a minimizer of E under the constraint of the prescribed irrigating and irrigated measures. □

Proposition 6.3. *The problem of minimizing $E(\mu)$ in $TP(\pi)$ admits a solution.*

Proof: As in the proof of Proposition 6.2, we can consider a minimizing sequence $(\mu_n)_n$ in $TP_C(\pi)$, where C is such that $\inf_{TP(\pi)} E(\mu) \leq C$. Since $E(\mu_n) = E(\tilde{\mu}_n)$, without loss of generality, we can take μ_n as being normalized. Because of Theorem 4.1, it is possible to extract a subsequence, which we denote again by μ_n, such that $\mu_n \rightharpoonup \mu$ and $\pi_{\mu_n} \rightharpoonup \pi_\mu$. Since $\pi_{\mu_n} = \pi$ for all n, μ is a traffic plan satisfying the constraints and such that $E(\mu) \leq \liminf E(\mu_n)$. Since μ_n is a minimizing sequence, μ is a minimizer of E under the constraint of the prescribed transference plan. □

6.4. Loop-free traffic plans.

Definition 6.3. A traffic plan μ is said to be loop-free if there is a parameterization χ of μ so that for almost all $\omega \in [0,1]$, the element $\chi(\omega)$ of K is injective on $[0, T(\chi(\omega))]$.

Definition 6.4. Let μ be a traffic plan. We define the geometric embedding of μ as being the set $G_\mu := \{x : [x]_\mu \neq 0\}$.

Proposition 6.4. *Let μ be a traffic plan such that $E(\mu) < \infty$. There exists a loop-free traffic plan $\tilde{\mu}$ so that $G_{\tilde{\mu}} \subset G_\mu$ and $\pi_{\tilde{\mu}} = \pi_\mu$.*

Proof: Since the geometric embedding and the transference plans are invariant under normalization of a traffic plan μ, we can suppose μ to be normalized. Let χ be a parameterization of μ. Because of Lemma 6.1, $L(\chi(\omega)) < \infty$ for almost all $\omega \in \Omega$. For these ω, we reparameterize the path $\chi(\omega)$, so that we suppress loops. To do so, we introduce the set

$$X_\omega = \{x \in \chi(\omega, \mathbb{R}^+) | \# \chi(\omega, \cdot)^{-1}(x) \cap [0, L(\chi(\omega)] > 1\},$$

which is empty if and only if $\chi(\omega)$ is injective.

Step 1: Existence of a maximal set of injectivity. We shall call a set of injectivity, a set

$$A_\omega = \bigcup_{x \in X_\omega} [t^-_x, t^+_x[$$

such that $\chi(\omega)$ is injective on $[0, L(\chi(\omega))] \setminus A_\omega$, where t_x^- and t_x^+ are elements of $\chi(\omega, \cdot)^{-1}(x)$.

Let us use an iterative process to construct such a set. Let us consider first the set $T_\omega^0 = [0, L(\chi(\omega))]$. If $\chi(\omega)$ is injective on T_ω^0, then the empty set is a set of injectivity. Otherwise, we consider one of the largest interval $[t_1^-, t_1^+[$ where t_1^- and t_1^+ are in $T_\omega^0 \cap \chi(\omega, \cdot)^{-1}(x)$ with x in X_ω. Such an interval exists since $[0, L(\chi(\omega))]$ is bounded. We then set $T_\omega^1 = T_\omega^0 \setminus [t_1^-, t_1^+[$. Continuing this process iteratively, we obtain a decreasing sequence of sets

$$T_\omega^n = T_\omega^{n-1} \setminus [t_n^-, t_n^+[,$$

where $t_n^-, t_n^+ \in T_\omega^{n-1} \cap \chi(\omega, \cdot)^{-1}(x)$ and $x \in X_\omega$. The process stops whenever $\cup_{k=1}^n [t_k^-, t_k^+[$ is a set of injectivity. If the process never ends, the set $\cup_{k=1}^\infty [t_k^-, t_k^+[$ is a set of injectivity. Indeed, let us assume that $s_1, s_2 \in [0, L(\omega)] \setminus \cup_k [t_k^-, t_k^+[$ are such that $\chi(\omega, s_1) = \chi(\omega, s_2)$. Then, by construction,

$$\infty > L(\chi(\omega)) \geq \sum_n |t_n^+ - t_n^-| \geq \sum_n |s_1 - s_2|,$$

thus $s_1 = s_2$. We shall denote by T_ω the set $[0, L(\omega)] \setminus \cup_k [t_k^-, t_k^+[$.

Step 2: Definition of the reparameterization. The set T_ω is a set of time parameters describing an injective subpath of $\chi(\omega)$. Let us consider the non-decreasing continuous function

$$S_\omega(u) = \int_0^u 1\!\!1_{T_\omega}(s)\, ds$$

and let us define $\tau_\omega(t) := \inf\{u \in [0, \infty) : S_\omega(u) = t\}$. Then, $\tau_\omega(t)$ is such that $|T_\omega \cap [0; \tau_\omega(t)]| = t$.

Let us observe that the map $\tau_\omega(t)$ is measurable as a function of (ω, t). Let $\{t_m\}$ be a dense sequence in $[0, \infty)$. Following the proof of Lemma 6.2, since $\tau_\omega(t)$ is non-decreasing, lower semicontinuous, and

$$\{\omega \in [0,1] : \tau_\omega(t_m) \leq \lambda\} = \{\omega \in [0,1] : S_\omega(\lambda) \geq t_m\}$$

it suffices to prove that the sets $\{\omega \in [0,1] : S_\omega(\lambda) \geq t_m\}$ are measurable for any $\lambda \geq 0$. For that, it is sufficient to prove that the sets

$$\begin{aligned} \mathcal{S} = \{\omega \in [0,1] : S_\omega(\lambda) \leq t_m\} &= \{\omega \in [0,1] : |T_\omega \cap [0, \lambda]| \leq t_m\} \\ &= \{\omega \in [0,1] : |T_\omega^c \cap [0, \lambda]| \geq \lambda - t_m\} \end{aligned}$$

are measurable for any $\lambda \geq 0$. Let

$$T_{\omega,p} = [0, L(\omega)] \setminus \cup_{\{k : t_k^+ - t_k^- \geq \frac{1}{p}\}} [t_k^-, t_k^+[$$

and observe that $\cap_p T_{\omega,p} = T_\omega$ Let us prove that for any $p \geq 1$, the set

$$\mathcal{S}_p := \{\omega \in [0,1] : |T^c_{\omega,p} \cap [0,\lambda]| \geq \lambda - t_m\}$$

is measurable. Recall that, since $\chi\colon [0,1] \to K$ is measurable, for each $j \in \mathbb{N}$, there is a compact set $B_j \subseteq [0,1]$ such that $\chi\colon B_j \to K$ is continuous [**10**]. Let us prove that for any $j \in \mathbb{N}$ the set

$$\mathcal{S}_{p,j} := \{\omega \in [0,1] : |T^c_{\omega,p} \cap [0,\lambda]| \geq \lambda - t_m\} \cap B_j$$

is closed, hence, a Borel set. Let $\omega_i \in \mathcal{S}_{p,j}$, $\omega_i \to \omega$. Then, for each of the curves $\chi(\omega_i)$, the sum of the lengths of the loops of length $\geq \frac{1}{p}$ is $\geq \lambda - t_m$. Letting $i \to \infty$, we deduce that the sum of the lengths of the loops of $\chi(\omega)$ of length $\geq \frac{1}{p}$ is also $\geq \lambda - t_m$. In other words, $\omega \in \mathcal{S}_{p,j}$. Since $\mathcal{S}_p = \cup_j \mathcal{S}_{p,j} \cup N$ where N is a null set, we deduce that $\mathcal{S}_p$ is a measurable set. Now, since $\cup_p T^c_{\omega,p} = T^c_\omega$, we have that

$$\begin{aligned}
&\{\omega \in [0,1] : |T^c_\omega \cap [0,\lambda]| \geq \lambda - t_m\} \\
&\qquad = \left\{\omega \in [0,1] : \sup_p |T^c_{\omega,p} \cap [0,\lambda]| \geq \lambda - t_m\right\} \\
&\qquad = \cap_j \cup_k \left\{\omega \in [0,1] : |T^c_{\omega,k} \cap [0,\lambda]| \geq \lambda - t_m - \frac{1}{j}\right\}.
\end{aligned}$$

Hence $\mathcal{S}$ is measurable. We conclude that $\tau_\omega(t)$ is measurable as a function of (ω, t).

We reparameterize the paths $\chi(\omega, s)$ by $\tilde{\chi}(\omega, t) := \chi(\omega, \tau_\omega(t))$. As in Lemma 6.2, to prove that the application $\tilde{\chi}(\omega, t)$ is measurable it suffices to prove that $(I,\tau)^{-1}(N)$ is a null set for any null set $N \subseteq [0,1] \times [0,\infty)$. As in the proof of Lemma 6.2, let B be a Borel set containing N (of total measure less than ϵ). Observe that $G(\omega, s) := 1\!\!1_B(\omega, \tau_\omega(s))$ is a measurable map. Now, for a.e. fixed value of each $\omega \in [0,1]$, we have

$$\int_0^\infty G(\omega, s)\, ds = \int_0^\infty 1\!\!1_B(\omega, u) S'_\omega(u)\, du \leq \int_0^\infty 1\!\!1_B(\omega, u)\, du,$$

the last inequality being true since $S'_\omega(u) \leq 1$. Integrating with respect to $\omega \in [0,1]$, and observing that both G and $1\!\!1_B$ are measurable in $[0,1] \times [0,\infty)$, we have

$$|(I,\tau)^{-1}(B)| = \int_0^1 \int_0^\infty 1\!\!1_B(\omega, \tau_\omega(s))\, ds\, d\omega \leq \int_0^1 \int_0^\infty 1\!\!1_B(\omega, u)\, du\, d\omega \leq \epsilon.$$

We deduce that $(I,\tau)^{-1}(N)$ is a null set. We conclude that $\tilde{\chi}$ is measurable. We can then define $\tilde{\mu} := \tilde{\chi}\#\lambda$.

Step 3: The traffic plan $\tilde{\mu}$ is loop-free. Indeed, if there is an ω such that $\tilde{\chi}(\omega)$ is not injective, there are t_1 and t_2 such that $y = \tilde{\chi}(\omega, t_1) = \tilde{\chi}(\omega, t_2)$ with $t_1 \neq t_2$. Then, since τ_ω is increasing, $\tau_\omega(t_1) \neq \tau_\omega(t_2)$. Thus $\#\chi_\omega^{-1}(y) > 1$ so by definition of A_ω one of these two elements has to be in A_ω. But this is not possible since the image of τ_ω is disjoint from A_ω. Thus, $\tilde{\chi}$ is loop-free. By definition of $\tilde{\chi}$, $\pi_{\tilde{\mu}} = \pi_\mu$ and $G_{\tilde{\mu}} \subset G_\mu$. ☐

6.5. A change of variable formula.

Let μ be a traffic plan and χ a parameterization of μ. It will be called non-trivial if $L(\chi(\omega)) > 0$ on a set of positive measure in $\Omega := [0,1]$. Since we can eliminate the paths whose length is null, without loss of generality we shall assume that for non-trivial traffic plans we have $L(\chi(\omega)) > 0$ a.e. First, we prove that the geometric embedding of a non-trivial traffic plan with finite energy can be covered by a countable set of paths. This permits us to compare our energy with the formulation given by Q. Xia [**19**], [**20**]. For a sake of simplicity, we shall denote in the sequel $[x]$ instead of $[x]_\chi$.

Lemma 6.3. *Let μ be a non-trivial traffic plan with finite energy and χ a parameterization of μ. There exists a sequence $(\omega_j)_j$ such that*

$$|[x]_\chi| = 0 \quad \mathcal{H}^1\text{-a.e., for} \quad x \in \mathbb{R} \setminus \cup_{j=1}^{\infty} \operatorname{Im} \chi(\omega_j). \tag{6}$$

Proof: Let us first prove that we may cover the set

$$D := \{(\omega, t) \in \Omega \times [0, \infty) : 0 < t < L(\chi(\omega))\}$$

with a countable number of sets of the form $D_\omega = \{(\tilde{\omega}, t) \in D : \chi(\tilde{\omega}, t) \in \operatorname{Im} \chi(\omega)\}$. Since $E(\mu)$ is finite and χ is non-trivial, then for almost all $(\omega, t) \in D$, $|[\chi(\omega, t)]| > 0$. For each $\omega \in \Omega$, let

$$D^1_\omega := \{(\tilde{\omega}, t) : \chi(\tilde{\omega}, t) \in \operatorname{Im} \chi(\omega)\}.$$

Observe that

$$\begin{aligned}
\int_\Omega |D^1_\omega| \, d\omega &= \int_\Omega |\{(\tilde{\omega}, t) : \chi(\tilde{\omega}, t) \in \operatorname{Im} \chi(\omega)\}| \, d\omega \\
&= \int_\Omega \int_0^\infty \int_\Omega \mathbb{1}_{\operatorname{Im} \chi(\omega)}(\chi(\tilde{\omega}, t)) \, d\tilde{\omega} \, dt \, d\omega \\
&= \int_\Omega \int_0^\infty \int_\Omega \mathbb{1}_{\operatorname{Im} \chi(\omega)}(\chi(\tilde{\omega}, t)) \, d\omega \, dt \, d\tilde{\omega} \\
&= \int_\Omega \int_0^\infty |[\chi(\tilde{\omega}, t)]| \, dt \, d\tilde{\omega} > 0.
\end{aligned}$$

Hence $d^1 := \sup_\omega |D^1_\omega| > 0$. Let us choose $\omega_1 \in \Omega$ such that

$$|D^1_{\omega_1}| \geq \frac{d^1}{2} > 0.$$

Either D_{ω_1} covers all D, or

$$|D^1_{\omega_1}| < \int_\Omega \int_0^{L(\chi(\omega))} dt\, d\omega.$$

Proceeding iteratively in this way, and assuming that

$$\sum_{j=1}^{k-1} |D^j_{\omega_j}| < \int_\Omega \int_0^{L(\chi(\omega))} dt\, d\omega,$$

we define

$$D^k_\omega := \{(\tilde{\omega}, t) : \chi(\tilde{\omega}, t) \in \operatorname{Im} \chi(\omega) \setminus \cup_{j=1}^{k-1} \operatorname{Im} \chi(\omega_j)\}$$

and we may check that

$$\int_\Omega |D^k_\omega|\, d\omega = \int_{(\cup_{j=1}^{k-1} D^j_{\omega_j})^c} |[\chi(\tilde{\omega}, t)]|\, dt\, d\tilde{\omega} > 0,$$

which implies that $d^k := \max_\omega |D^k_\omega| > 0$. Then we choose $\omega_k \in \Omega$ such that

$$|D^k_{\omega_k}| \geq \frac{d^k}{2} > 0.$$

Either this construction ends in a finite number of steps k and we obtain that

$$\text{a.e. } \omega \in \Omega \quad \operatorname{Im} \chi(\omega) \subseteq \cup_{j=1}^{k} \operatorname{Im} \chi(\omega_j),$$

or we have an infinite number of sets $D^j_{\omega_j}$ and we have

$$\text{a.e. } \omega \in \Omega \quad \operatorname{Im} \chi(\omega) \subseteq \cup_{j=1}^{\infty} \operatorname{Im} \chi(\omega_j). \tag{7}$$

Indeed, if (7) does not hold then

$$\sum_{j=1}^{\infty} |D^j_{\omega_j}| < \int_\Omega \int_0^{L(\chi(\omega))} dt\, d\omega.$$

In particular, we have $d^j \leq 2|D^j_{\omega_j}| \to 0$ as $j \to \infty$, hence

$$\sup_{\omega \in \Omega} |D^j_\omega| \to 0 \quad \text{as } j \to \infty. \tag{8}$$

Since

$$\int_\Omega |D^j_\omega|\, d\omega = \int_{(\cup_{i=1}^{j-1} D^i_{\omega_i})^c} |[\chi(\tilde\omega, t)]|\, dt\, d\tilde\omega$$
$$\geq \int\int_{(\cup_{i=1}^{\infty} D^i_{\omega_i})^c} |[\chi(\tilde\omega, t)]|\, dt\, d\tilde\omega > 0,$$

we obtain a contradiction since the left-hand side tends to 0 as $j \to \infty$ while the right-hand side is a positive constant. We have proved that $\cup_{j=1}^\infty D^j_{\omega_j}$ covers D (modulo a null set), and, therefore (7) holds.

To prove that (6) holds, assume on the contrary that there exists a set C such that $\mathcal{H}^1(C) > 0$,

$$C \cap (\cup_{i=1}^\infty \operatorname{Im} \chi(\omega_i)) = \emptyset, \tag{9}$$

and such that $|[x]| > 0$ for all $x \in C$. Then

$$0 < \int_C |[x]|\, d\mathcal{H}^1(x) = \int_C \int_\Omega \mathbb{1}_{[x]}(\omega)\, d\omega\, d\mathcal{H}^1(x)$$
$$= \int_\Omega \int_C \mathbb{1}_{[x]}(\omega)\, d\mathcal{H}^1(x)\, d\omega = \int_\Omega \mathcal{H}^1(C \cap \operatorname{Im} \chi(\omega))\, d\omega.$$

This implies that there exists a subset Ω_C of Ω such that $\mathcal{H}^1(C \cap \operatorname{Im} \chi(\omega)) > 0$ for any $\omega \in \Omega_C$, hence for any $\omega \in \Omega_C$ the set $I_\omega := \{t \in [0, \infty) : \chi(\omega, t) \in C\}$ is of positive measure. Since

$$\{(\omega, t) : \omega \in \Omega_C,\, t \in I_\omega\} \subseteq \{(\omega, t) : \chi(\omega, t) \in C\},$$

we conclude that $|\{(\omega, t) : \chi(\omega, t) \in C\}| > 0$. This contradicts (9). The lemma follows. □

Definition 6.5. Let μ be a traffic plan and χ a parameterization of μ. For each $\omega \in \Omega$, we define

$$\mathcal{D}^\chi(\omega) = \{x \in \mathbb{R}^N : x \text{ is a double point of } \chi(\omega)\}.$$

We say that χ has simple paths if $\mathcal{H}^1(\mathcal{D}^\chi(\omega)) = 0$ for almost every $\omega \in \Omega$.

Assume that for a given $\omega \in \Omega$, $\chi(\omega)$ is parameterized by arc-length. Let

$$\mathcal{D}_\chi(\omega) = \{t \in [0, \infty) : \exists\, s < t,\, \chi(\omega, t) = \chi(\omega, s)\}.$$

Observe that $\mathcal{H}^1(\mathcal{D}^\chi(\omega)) = 0$ if and only if $|\mathcal{D}_\chi(\omega)| = 0$. Thus, if χ is normalized, χ has simple paths if and only if $|\mathcal{D}_\chi(\omega)| = 0$ for almost every $\omega \in \Omega$.

Our purpose is to prove the following change of variable formula. Notice that, in the case of a loop-free graph, the right-hand side of the identity (11) takes the form (1), so that our framework generalizes [**19**].

Proposition 6.5. *Let χ be a parameterization of a nontrivial traffic plan μ with finite energy. Then, we have*

$$(10)\quad E(\mu) = \int_\Omega \int_0^\infty |[\chi(\omega,t)]|^{\alpha-1} |\dot\chi(\omega,t)|\, dt\, d\omega \geq \int_{\mathbb{R}^N} |[x]_\chi|^\alpha\, d\mathcal{H}^1(x).$$

If we assume, in addition, that χ has simple paths, we have

$$(11)\quad E(\mu) = \int_\Omega \int_0^\infty |[\chi(\omega,t)]|^{\alpha-1} |\dot\chi(\omega,t)|\, dt\, d\omega = \int_{\mathbb{R}^N} |[x]_\chi|^\alpha\, d\mathcal{H}^1(x).$$

Proof: Since the reparameterization $\tilde\chi$ of χ is measurable (Lemma 6.2), and since $[x]_\chi = [x]_{\tilde\chi}$ for all $x \in \mathbb{R}^N$, we may assume that $|\dot\chi(\omega,t)| = 1$ for almost all $\omega \in \Omega$, a.e. $t \in [0, L(\chi(\omega))[$. Let us consider the sequence $(\omega_j)_j$ constructed in Lemma 6.3. We denote by D the set

$$D := \{(\omega,t) \in \Omega \times [0,\infty) : 0 \leq t < L(\chi(\omega))\}.$$

Let us prove first that

$$\int_{D_{\omega_1}} |[\chi(\omega,t)]|^{\alpha-1}\, d\omega\, dt = \int_{\mathrm{Im}\,\chi(\omega_1)} |[x]|^\alpha\, d\mathcal{H}^1(x),$$

where D_{ω_1} is the set

$$D_{\omega_1} = \{(\tilde\omega,t) \in D : \chi(\tilde\omega,t) \in \mathrm{Im}\,\chi(\omega_1)\}.$$

Let us define

$$\Omega_{\omega_1} := \{\omega \in \Omega : \mathrm{Im}\,\chi(\omega) \cap \mathrm{Im}\,\chi(\omega_1) \neq \emptyset\},$$
$$I_\omega = \{t < L(\chi(\omega)) : \chi(\omega,t) \in \mathrm{Im}\,\chi(\omega_1)\},$$

and

$$I'_\omega := \{t \in \mathbb{R}^+ \setminus \mathcal{D}_\chi(\omega) : \chi(\omega,t) \in \mathrm{Im}\,\chi(\omega_1)\}.$$

Notice that

$$D_{\omega_1} = \cup_\omega \{\omega\} \times I_\omega.$$

Let t be in I'_ω. Since $\chi(\omega,t) \in \mathrm{Im}\,\chi(\omega_1)$ and because of the definition of $\mathcal{D}_\chi(\omega_1)$, there is a unique $s = \varphi(t) \in \mathbb{R}^+ \setminus \mathcal{D}_\chi(\omega_1)$ such that $\chi(\omega_1,s) = \chi(\omega,t)$. Let I^*_ω be the set

$$I^*_\omega = \varphi(I'_\omega) = \{s \in \mathbb{R}^+ \setminus \mathcal{D}_\chi(\omega_1) : \chi(\omega_1,s) \in \mathrm{Im}\,\chi(\omega)\}.$$

Then I^*_ω is a Borel set of the same one-dimensional Lebesgue measure as I'_ω. As in the proof of Lemma 6.4, to prove the measurability of the set

$$Q = \{(\omega, s) : \omega \in \Omega_{\omega_1},\ \chi(\omega_1, s) \in \operatorname{Im}\chi(\omega)\},$$

we recall that for each $\epsilon > 0$, there is a compact set $B_\epsilon \subseteq [0,1]$ such that $\chi\colon B_\epsilon \to K$ is continuous [**10**]. Now, one can easily check that $Q \cap B_\epsilon$ is a closed set. We deduce that Q is measurable. Since

$$\begin{aligned}\{(\omega, s) : \omega \in \Omega_{\omega_1},\ \chi(\omega_1, s) \in \operatorname{Im}\chi(\omega) \setminus \mathcal{D}^\chi(\omega_1)\}& \\ = Q \cap \{(\omega, s) : \omega \in \Omega_{\omega_1},\ s \notin \mathcal{D}_\chi(\omega_1)\}&\end{aligned}$$

we deduce that the set

$$\{(\omega, s) : \omega \in \Omega_{\omega_1},\ \chi(\omega_1, s) \in \operatorname{Im}\chi(\omega) \setminus \mathcal{D}^\chi(\omega_1)\}$$

is measurable. Finally observe that $\mathbb{1}_{I^*_\omega}(s) = 1$ if and only if $\omega \in [\chi(\omega_1, s)]$ and $s \notin \mathcal{D}_\chi(\omega_1)$. Thus, we have

$$\int_{\Omega_{\omega_1}} \mathbb{1}_{I^*_\omega}(s)\, d\omega = |[\chi(\omega_1, s)]| \mathbb{1}_{\mathbb{R}^+ \setminus \mathcal{D}_\chi(\omega_1)}.$$

Then, we have

$$\begin{aligned}
\int_{D_{\omega_1}} |[\chi(\omega, t)]|^{\alpha-1}\, d\omega\, dt &= \int_{\Omega_{\omega_1}} \int_{I_\omega} |[\chi(\omega, t)]|^{\alpha-1}\, dt\, d\omega \\
&\geq \int_{\Omega_{\omega_1}} \int_{I'_\omega} |[\chi(\omega, t)]|^{\alpha-1}\, dt\, d\omega \\
&= \int_{\Omega_{\omega_1}} \int_{I^*_\omega} |[\chi(\omega_1, s)]|^{\alpha-1}\, ds\, d\omega \\
&= \int_{\Omega_{\omega_1}} \int_0^\infty \mathbb{1}_{I^*_\omega}(s) |[\chi(\omega_1, s)]|^{\alpha-1}\, ds\, d\omega \\
&= \int_0^\infty \int_{\Omega_{\omega_1}} \mathbb{1}_{I^*_\omega}(s) |[\chi(\omega_1, s)]|^{\alpha-1}\, d\omega\, ds \\
&= \int_0^\infty |[\chi(\omega_1, s)]|^{\alpha-1} \int_{\Omega_{\omega_1}} \mathbb{1}_{I^*_\omega}(s)\, d\omega\, ds \\
&= \int_{[0,\infty) \setminus \mathcal{D}_\chi(\omega_1)} |[\chi(\omega_1, s)]|^{\alpha}\, ds \\
&= \int_{\operatorname{Im}\chi(\omega_1)} |[x]|^\alpha\, d\mathcal{H}^1(x).
\end{aligned}$$

Notice that in the case μ has simple paths, modulo a null set we have the identity

$$I_\omega = I'_\omega.$$

This proves that for a traffic plan with simple paths,

$$\int_{D_{\omega_1}} |[\chi(\omega,t)]|^{\alpha-1}\, d\omega\, dt = \int_{\operatorname{Im}\chi(\omega_1)} |[x]|^\alpha\, d\mathcal{H}^1(x).$$

We may reproduce iteratively the same argument for the arcs forming $\operatorname{Im}\chi(\omega_k) \setminus \cup_{j=1}^{k-1} \operatorname{Im}\chi(\omega_j)$ to obtain

$$\int_{\cup_{j=1}^k D^j_{\omega_j}} |[\chi(\omega,t)]|^{\alpha-1}\, d\omega\, dt \geq \int_{\cup_{j=1}^k \operatorname{Im}\chi(\omega_j)} |[x]|^\alpha\, d\mathcal{H}^1(x).$$

Notice that there is equality in the case μ has simple paths. Letting $k \to \infty$, and using that $\cup_{j=1}^\infty D^j_{\omega_j}$ is a covering (modulo a null set) of

$$D = \{(\omega,t) \in \Omega \times [0,\infty) : 0 \leq t < L(\chi(\omega))\},$$

we obtain

$$\int_\Omega \int_0^{L(\chi(\omega))} |[\chi(\omega,t)]|^{\alpha-1}\, dt\, d\omega \geq \int_{\cup_{j=1}^\infty \operatorname{Im}\chi(\omega_j)} |[x]|^\alpha\, d\mathcal{H}^1(x),$$

and

$$\int_\Omega \int_0^{L(\chi(\omega))} |[\chi(\omega,t)]|^{\alpha-1}\, dt\, d\omega = \int_{\cup_{j=1}^\infty \operatorname{Im}\chi(\omega_j)} |[x]|^\alpha\, d\mathcal{H}^1(x)$$

if μ has simple paths. The proposition follows by using Lemma 6.3. □

Let us denote

$$E_x(\mu) = \int_{\mathbb{R}^N} |[x]_\mu|^\alpha\, d\mathcal{H}^1(x).$$

Proposition 6.6. *The minimum of E on the set of traffic plans is attained at a loop-free traffic plan. Moreover* $\inf E = \inf E_x$ *where both infima can be taken with respect to the set of all traffic plans or the set of loop-free traffic plans.*

Proof: We observe that if μ is a traffic plan and $\tilde\mu$ its associated loop-free traffic plan constructed in Proposition 6.4, we have $E(\tilde\mu) \leq E(\mu)$. To prove it, we observe that when eliminating loops, the multiplicity decreases, hence $E_x(\mu) \geq E_x(\tilde\mu)$. Now, by Proposition 6.5, we have

$$E(\mu) \geq E_x(\mu) \geq E_x(\tilde\mu) = E(\tilde\mu).$$

Our assertions are a simple consequence of Proposition 6.4 and this inequality. □

7. Conclusion, some urban problems

We have shown that a simple Lagrangian formalism generalizing the one in [**15**] could be used for the continuous generalizations of the Gilbert-Steiner irrigation problem. This new formalism seems to be useful to formalize naturally the "who goes where" constraint. Now, this constraint makes essentially sense in a context which is more organizational than physical or biological. Let us point out some shortcomings of the approach, if it were to formalize general traffic problems in a realistic context. We have computed the traffic multiplicity at a point x (in other terms the intensity of the traffic) as the probability measure of the set of paths passing by x. If we were to deal with the classical irrigation problem, opposite paths would cancel while here we added them up. So we have assumed as a basic principle that the cost of two paths whose physical supports coincide but go in opposite directions is the same as if they were going in the same direction. This hypothesis makes actually a lot of sense for urban traffic, since most streets, highways, sidewalks, etc. go both ways and the construction cost seems to depend on the width of an avenue. This width sums up the flows on both directions. Of course, in the energy functional we considered, the multiplicities add when two paths cross (even when they do not coincide on a set of positive length). It is easily checked that the total length and therefore the contribution to the energy of these crossings is zero. The fact that crossing do not matter in traffic planning is not realistic. Crossings should have a special Dirac cost. This, and other more realistic ingredients, like the cost of the stations, and the commutations from one transportation means to another, should be considered in order to match the complexity of the urban transportation problem. We mentioned the Buttazzo-Stepanov [**5**] problem as an attempt in that direction, since the authors consider two transportation media instead of one.

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