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A REPRESENTATION FORMULA FOR RADIALLY WEIGHTED BIHARMONIC FUNCTIONS IN THE UNIT DISC

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Abstract

Let $w\colon \mathbb{D} \to (0,\infty)$ be a radial continuous weight function. We consider the weighted biharmonic equation

$$\Delta w^{-1}\Delta u = 0 \quad \text{in } \mathbb{D}$$

with Dirichlet boundary conditions $u=f_0$ and $\partial_n u=f_1$ on $\mathbb{T}=\partial\mathbb{D}$. Under some extra conditions on the weight function w, we establish existence and uniqueness of a distributional solution u of this biharmonic Dirichlet problem. Furthermore, we give a representation formula for the solution u. The key to our analysis is a series representation of Almansi type.

0. Introduction

Denote by $\mathbb{D}$ the unit disc and let $\Delta = \partial^2/\partial x^2+\partial^2/\partial y^2$ be the Laplacian in the complex plane. Let $w\colon \mathbb{D} \to (0,\infty)$ be a radial continuous weight function. In this paper we study the weighted biharmonic equation

$$\Delta w^{-1}\Delta u = 0 \quad \text{in } \mathbb{D}. \tag{0.1}$$

We interpret equation (0.1) in the distributional sense. Let $u \in \mathcal{D}'(\mathbb{D})$; here $\mathcal{D}'(\Omega)$ denotes the space of distributions in Ω. We call u a solution of (0.1) if $\Delta u = wh$ in $\mathcal{D}'(\mathbb{D})$ for some harmonic function h in $\mathbb{D}$. We shall see that every solution u of (0.1) is of type $C^2(\mathbb{D})$ (see Remark 1.2). A solution u of (0.1) is sometimes called a w-biharmonic function.

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Of particular interest is the weighted biharmonic Dirichlet problem:

$$
\begin{cases} \Delta w^{-1}\Delta u = 0 & \text{in } \mathbb{D}, \\ u = f_0 & \text{on } \mathbb{T}, \\ \partial_n u = f_1 & \text{on } \mathbb{T}. \end{cases} \tag{0.2}
$$

Here $\mathbb{T} = \partial\mathbb{D}$ is the unit circle, ∂_n denotes differentiation in the inward normal direction and the boundary datas f_j $(j = 0, 1)$ are distributions on $\mathbb{T}$.

Let us make a few comments on the interpretation of (0.2). The first equation in (0.2) is interpreted in the above distributional sense of (0.1). For $0 \le r < 1$ we consider the function u_r defined by

$$
u_r(e^{i\theta}) = u(re^{i\theta}), \quad e^{i\theta} \in \mathbb{T}. \tag{0.3}
$$

The middle boundary condition in (0.2) is interpreted that

$$
\lim_{r\to 1} u_r = f_0 \quad \text{in } \mathcal{D}'(\mathbb{T});
$$

here $\mathcal{D}'(\mathbb{T})$ denotes the space of distributions on $\mathbb{T}$. Similarly, we interpret the last boundary condition in (0.2) to mean that

$$
\partial_n u := \lim_{r\to 1} (u_r - f_0)/(1-r) = f_1 \quad \text{in } \mathcal{D}'(\mathbb{T}),
$$

where f_0 is as above.

The purpose of this paper is to study the questions of existence, uniqueness, and representation of distributional solutions of the Dirichlet problem (0.2). Let us now describe our results.

Assume that the radial weight function $w\colon \mathbb{D} \to (0,\infty)$ is area integrable, that is, $\int_0^1 w(t)\,dt < \infty$. We can then show uniqueness of a distributional solution of the Dirichlet problem (0.2) (see Theorem 2.1). Assume also that the moment condition

$$
A_{|k|} = \int_0^1 t^{2|k|+1} w(t)\,dt \ge c(1+|k|)^{-N}, \quad k \in \mathbb{Z}, \tag{0.4}
$$

$(c, N > 0)$ is satisfied. We can then show existence of a distributional solution u of the Dirichlet problem (0.2) for arbitrary distributional boundary data $f_j \in \mathcal{D}'(\mathbb{T})$ $(j = 0, 1)$. Furthermore, we show that the solution u of the Dirichlet problem (0.2) admits the representation

$$
u(z) = (F_{w,r} * f_0)(e^{i\theta}) + (H_{w,r} * f_1)(e^{i\theta}), \quad z = re^{i\theta} \in \mathbb{D}, \tag{0.5}
$$

in terms of two functions F_w and H_w in $\mathbb{D}$; here $F_{w,r} = (F_w)_r$ and $H_{w,r} = (H_w)_r$ as in (0.3), and by $*$ we denote convolution (see Theorem 3.2).

The kernel functions F_w and H_w appearing in (0.5) are given by the formulas

$$F_w(z) = \frac{1-|z|^2}{|1-z|^2} + \int_r^1 \int_0^s \left(\sum_{k=-\infty}^{\infty} \frac{|k|}{A_{|k|}} \left(\frac{t}{s}\right)^{2|k|+1} r^{|k|} e^{ik\theta} \right) w(t)\, dt\, ds \quad \text{and}$$

$$H_w(z) = \int_r^1 \int_0^s \left(\sum_{k=-\infty}^{\infty} \frac{1}{A_{|k|}} \left(\frac{t}{s}\right)^{2|k|+1} r^{|k|} e^{ik\theta} \right) w(t)\, dt\, ds$$

for $z = re^{i\theta} \in \mathbb{D}$ (see Proposition 4.1). We also mention that the two functions F_w and H_w solve the Dirichlet boundary value problems

$$\begin{cases} \Delta w^{-1} \Delta F_w = 0 & \text{in } \mathbb{D}, \\ F_w = \delta_1 & \text{on } \mathbb{T}, \\ \partial_n F_w = 0 & \text{on } \mathbb{T}, \end{cases} \quad \text{and} \quad \begin{cases} \Delta w^{-1} \Delta H_w = 0 & \text{in } \mathbb{D}, \\ H_w = 0 & \text{on } \mathbb{T}, \\ \partial_n H_w = \delta_1 & \text{on } \mathbb{T}, \end{cases}$$

in the above distributional sense; here $\delta_{e^{i\theta}}$ denotes the unit Dirac mass at $e^{i\theta} \in \mathbb{T}$.

The moment condition (0.4) is satisfied provided the weight function w has enough mass near the boundary $\mathbb{T}$ (see the end of Section 3). In particular, this moment condition is satisfied by all the so-called standard weights $w = w_\alpha$ defined by

$$w_\alpha(z) = (1-|z|^2)^\alpha, \quad z \in \mathbb{D},$$

where $\alpha > -1$. In the context of these weights the case $\alpha = 0$ is often referred to as the (classical) unweighted case. The standard weight $w = w_1$ has attracted special attention in recent papers (see [**7**], [**8**], [**15**]).

The above representation formula (0.5) generalizes earlier work by Abkar and Hedenmalm [**1**] in the unweighted case $w = w_0$ to our weighted context.

We consider in some more detail the above standard weight $w = w_1$. In this case we show that the corresponding kernels $F_1 = F_{w_1}$ and $H_1 = H_{w_1}$ are given by the formulas

$$F_1(z) = \frac{1}{2} \frac{(1-|z|^2)^3}{|1-z|^2} - |z|^2 \frac{(1-|z|^2)^3}{|1-z|^4} + \frac{1}{2} \frac{(1-|z|^2)^5}{|1-z|^6} \quad \text{and}$$

$$H_1(z) = \frac{1}{2} \frac{(1-|z|^2)^3}{|1-z|^2} + \frac{1}{4} \frac{(1-|z|^2)^4}{|1-z|^4} \qquad \text{for } z \in \mathbb{D}$$

(see Proposition 4.2). Using the first of these formulas we verify that the function F_1 is not positive in $\mathbb{D}$.

The motivation for our study of (weighted) biharmonic boundary value problems of the above type comes from Bergman space theory where such techniques have shown to be an important tool (see [**2**], [**3**], [**6**]). A recent achievement in this direction of research is the proof of positivity of certain weighted biharmonic Green functions (see [**5**], [**14**]).

The key to our analysis of the Dirichlet problem (0.2) is a series expansion of w-biharmonic functions which generalizes the classical Almansi representation of biharmonic functions. This generalized Almansi type representation is then used to prove existence and uniqueness of solutions of the Dirichlet problem (0.2).

This paper is organized as follows. In Section 1 we study an Almansi type representation of w-biharmonic functions. In Section 2 we use this Almansi representation to show uniqueness of solutions of the Dirichlet problem (0.2). In Section 3 we consider the problem of representation of a w-biharmonic function in terms of the two associated kernels F_w and H_w. In Section 4 we give some formulas for the functions F_w and H_w, and some related work is mentioned.

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1. Almansi representation

Our first task is to generalize the well-known Almansi representation of biharmonic functions. For this we need some preparation. Let $w\colon \mathbb{D} \to (0,\infty)$ be a radial continuous weight function. We consider the associated functions w_k defined by

$$w_0(r) = r^2 \int_0^1 \log(1/t) t w(rt)\, dt, \qquad 0 \le r < 1, \quad \text{and}$$

$$w_k(r) = r^2 \int_0^1 (1 - t^{2k}) t w(rt)\, dt/(2k), \quad 0 \le r < 1, \quad \text{for } k \ge 1.$$

We shall also need the functions v_k defined by $v_k(r) = r^k w_k(r)$.

In the following lemmas we establish some basic properties of these functions.

Lemma 1.1. *Let the w_k's be as above. Then the function w_k is of type $C^2[0,1)$ and satisfies the ordinary differential equation initial value problem*

$$\begin{cases} w_k''(r) + (2k+1) w_k'(r)/r = w(r), & 0 < r < 1, \\ w_k(0) = w_k'(0) = 0. \end{cases} \tag{1.1}$$

Furthermore, we have the estimates

$$0 \le w_k(r) \le r^2 \sup_{[0,r]} w/(4k) \quad \text{and} \quad 0 \le w_k'(r) \le r \sup_{[0,r]} w/(2k+2)$$

for $0 < r < 1$ and $k \ge 1$. In particular, the function w_k radially extends to a function in $C^2(\mathbb{D})$.

Proof: We assume that $k \ge 1$. The case $k = 0$ can be handled similarly. The first estimate of w_k is immediate from the definition. By a change of variables we have that

$$2kw_k(r) = \int_0^r tw(t)\,dt - \frac{1}{r^{2k}} \int_0^r t^{2k+1} w(t)\,dt.$$

By differentiation we see that

$$w_k'(r) = \frac{1}{r^{2k+1}} \int_0^r t^{2k+1} w(t)\,dt. \tag{1.2}$$

Similarly we check that (1.2) holds also for $k = 0$. An easy estimation now yields the estimate for w_k'. By another differentiation we see that w_k satisfies (1.1). Since the compatibility condition $w_k'(0) = 0$ is satisfied, the function w_k radially extends to a function in $C^2(\mathbb{D})$. □

For easy reference later we shall derive one more estimate of the w_k's. Note that by (1.2) the function w_k is increasing. We have that

$$w_k(r) \le w_k(1) = \frac{1}{2k} \int_0^1 (1 - t^{2k}) tw(t)\,dt \le \int_0^1 (1-t)tw(t)\,dt.$$

Thus the w_k's are uniformly bounded if $\int_0^1 (1-t)w(t)\,dt < \infty$.

We now consider the v_k's.

Lemma 1.2. *Let $v_k(r) = w_k(r) r^k$ for $0 \le r < 1$ and $k \ge 0$. Then the function v_k satisfies the ordinary differential equation*

$$v_k''(r) + \frac{1}{r} v_k'(r) - \frac{k^2}{r^2} v_k(r) = r^k w(r), \quad 0 < r < 1. \tag{1.3}$$

Proof: This is a straightforward verification using Lemma 1.1. □

We shall consider series expansions involving the functions w_k.

Lemma 1.3. *Let $\{c_k\}_{k=-\infty}^{\infty}$ be a sequence of complex numbers such that*

$$\limsup_{|k|\to\infty} |c_k|^{1/|k|} \le 1.$$

Then the series

$$u(z) = \sum_{k=-\infty}^{\infty} c_k w_{|k|}(r) r^{|k|} e^{ik\theta} = \sum_{k=-\infty}^{\infty} c_k v_{|k|}(r) e^{ik\theta}, \quad z = re^{i\theta} \in \mathbb{D},$$

converges in $C^2(\mathbb{D})$. We also have that

$$\Delta u = wh \quad \text{in } \mathbb{D}, \quad \text{where} \quad h(z) = \sum_{k=-\infty}^{\infty} c_k r^{|k|} e^{ik\theta}, \quad z = re^{i\theta} \in \mathbb{D}.$$

Proof: Recall that by Lemma 1.1 the radial function $w_{|k|}$ is in $C^2(\mathbb{D})$ and thus the same is true for the function $z \mapsto w_{|k|}(r) r^{|k|} e^{ik\theta}$. Using the estimates in Lemma 1.1 it is straightforward to check that

$$\sum_{k=-\infty}^{\infty} |c_k| \sup_{0\le r\le t} \big|\partial^\alpha \big(w_{|k|}(r) r^{|k|} e^{ik\theta}\big)\big| < \infty$$

for $|\alpha| \le 2$; here $\partial = (\partial/\partial x, \partial/\partial y)$ and standard multi-index notation is used. The convergence of the series expansion of u now follows.

We now compute Δu. Recall that in polar coordinates (r,θ) the Laplacian takes the form

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

A computation using Lemma 1.2 shows that

$$\Delta u = \sum_{k=-\infty}^{\infty} c_k \left(v''_{|k|}(r) + \frac{1}{r} v'_{|k|}(r) - \frac{k^2}{r^2} v_{|k|}(r) \right) e^{ik\theta} = wh \quad \text{in } \mathbb{D},$$

which concludes the proof of the lemma. □

Remark 1.1. We remark that in the context of the above lemma the term by term differentiated series

$$\sum_{k=-\infty}^{\infty} c_k \partial^\alpha \big(v_{|k|}(r) e^{ik\theta}\big), \quad |\alpha| \le 2,$$

converges absolutely and uniformly in every smaller disc $t\mathbb{D}$ $(0 < t < 1)$.

The following theorem generalizes the well-known Almansi representation of biharmonic functions.

Theorem 1.1. *Let $w\colon \mathbb{D} \to (0,\infty)$ be a radial continuous weight function, and let the w_k's be as above. Then a function u in $\mathbb{D}$ satisfies the weighted biharmonic equation*

$$\Delta w^{-1}\Delta u = 0 \quad \text{in } \mathbb{D} \tag{1.4}$$

if and only if u has a convergent series expansion of the form

$$u(z) = \sum_{k=-\infty}^{\infty} \big(a_k + b_k w_{|k|}(r)\big) r^{|k|} e^{ik\theta}, \quad z = re^{i\theta} \in \mathbb{D}, \tag{1.5}$$

for some complex numbers a_k, b_k, $k \in \mathbb{Z}$, such that

$$\limsup_{|k|\to\infty} |a_k|^{1/|k|} \le 1 \quad \text{and} \quad \limsup_{|k|\to\infty} |b_k|^{1/|k|} \le 1.$$

Furthermore, the series (1.5) *is convergent in the topology of* $C^2(\mathbb{D})$.

Proof: By Lemma 1.3, we know that the series (1.5) converges in $C^2(\mathbb{D})$ and that u so defined satisfies (1.4).

Let now u be a solution of (1.4). We proceed to show that u has a series expansion of the form (1.5). We know that $\Delta u = wh$ in $\mathcal{D}'(\mathbb{D})$, where $h = \sum b_k r^{|k|} e^{ik\theta}$ is harmonic in $\mathbb{D}$. Consider the function $u_1 = \sum b_k w_{|k|}(r) r^{|k|} e^{ik\theta}$. By Lemma 1.3 we know that $\Delta u_1 = wh$ in $\mathbb{D}$. Thus by Weil's lemma (see [**9**, Theorem 4.4.1]) we conclude that $h_1 = u - u_1$ is harmonic in $\mathbb{D}$. An expansion of h_1 now yields (1.5). $\square$

Remark 1.2. Let the weight w be as above. Note that by Theorem 1.1 every w-biharmonic function is in $C^2(\mathbb{D})$.

We inform the reader that a different generalization of the Almansi representation to weights of the form $w(z)=|z|^{2\alpha}$ for $z\in\mathbb{D}$, where $\alpha>-1$, has previously been given by Hedenmalm in [**4**, Lemma 3.1].

2. Uniqueness of the Dirichlet problem

The purpose of this section is to show uniqueness of solutions of the Dirichlet problem (0.2). We shall need some asymptotic properties of the functions w_k.

Proposition 2.1. *Let $w\colon \mathbb{D} \to (0,\infty)$ be a radial continuous weight function which is area integrable in the sense that $\int_0^1 w(t)\,dt < \infty$. Let the functions w_k $(k \geq 0)$ be as above and consider the sequence of moments $\{A_k\}_{k=-1}^{\infty}$ defined by*

$$A_{-1} = \int_0^1 \log(t) t w(t)\,dt \quad \textit{and} \quad A_k = \int_0^1 t^{2k+1} w(t)\,dt \quad \textit{for } k \geq 0.$$

Then as $r \to 1$ we have the asymptotic expansions:

$$\begin{aligned} w_0(r) &= -A_{-1} - A_0(1-r) + o(1-r) && \textit{and} \\ w_k(r) &= (A_0 - A_k)/(2k) - A_k(1-r) + o(1-r) && \textit{for } k \geq 1. \end{aligned}$$

Proof: We first compute the expansion of w_0. By a change of variables we see that

$$w_0(r) = \log(r) \int_0^r t w(t)\,dt - \int_0^r \log(t) t w(t)\,dt. \tag{2.1}$$

We first consider the second integral in (2.1). A computation shows that

$$\begin{aligned} \int_0^r \log(t) t w(t)\,dt &= A_{-1} - \int_r^1 \log(t) t w(t)\,dt \\ &= A_{-1} - \left[\int_0^t s w(s)\,ds \log(t) \right]_{t=r}^1 + \int_r^1 \int_0^t s w(s)\,ds \frac{dt}{t} \\ &= A_{-1} + \log(r) \int_0^r t w(t)\,dt + (A_0 + o(1)) \big[\log t \big]_{t=r}^1 \\ &= A_{-1} + \log(r) \int_0^r t w(t)\,dt - A_0 \log r + o(1-r). \end{aligned}$$

Substituting this expansion into (2.1) we see that

$$w_0(r) = -A_{-1} + A_0 \log(r) + o(1-r) = -A_{-1} - A_0(1-r) + o(1-r),$$

which is the above expansion of w_0.

We now turn to the expansion of w_k. Similarly as above we see by a change of variables that

$$2k w_k(r) = \int_0^r t w(t)\,dt - \frac{1}{r^{2k}} \int_0^r t^{2k+1} w(t)\,dt. \tag{2.2}$$

We consider the second integral in (2.2). By computation we have that

$$\begin{aligned}\int_0^r t^{2k+1}w(t)\,dt &= A_k - \int_r^1 t^{2k+1}w(t)\,dt\\ &= A_k - \left[\int_0^t sw(s)\,ds t^{2k}\right]_{t=r}^1 + \int_r^1\int_0^t sw(s)\,ds 2kt^{2k-1}\,dt\\ &= A_k - A_0 + r^{2k}\int_0^r tw(t)\,dt + 2kA_0(1-r) + o(1-r).\end{aligned}$$

Substituting this expansion into (2.2) we see that

$$\begin{aligned}2kw_k(r) &= \frac{1}{r^{2k}}\Big((A_0-A_k) - 2kA_0(1-r) + o(1-r)\Big)\\ &= (A_0-A_k) - 2kA_k(1-r) + o(1-r),\end{aligned}$$

which is the expansion of w_k. □

Using the result of Proposition 2.1 we can compute asymptotic formulas for the coefficients in (1.5). We obtain that

$$a_0 + b_0w_0(r) = (a_0 - A_{-1}b_0) - A_0b_0(1-r) + o(1-r) \quad \text{and}$$

$$\begin{aligned}\big(a_k + b_kw_{|k|}(r)\big)r^{|k|} &= a_k + \frac{A_0 - A_{|k|}}{2|k|}b_k\\ &\quad - \left(|k|a_k + \frac{A_0 + A_{|k|}}{2}b_k\right)(1-r) + o(1-r) \quad \text{for } k\neq 0.\end{aligned}$$

We can now give a general uniqueness result for equation (0.1).

Theorem 2.1. *Let $w\colon \mathbb{D}\to(0,\infty)$ be a radial continuous weight function which is area integrable in the sense that $\int_0^1 w(t)\,dt<\infty$. Let u be a solution of* (0.1) *and assume that*

$$\lim_{r\to 1} u_r/(1-r) = 0 \quad \textit{in } \mathcal{D}'(\mathbb{T}),$$

where $u_r(e^{i\theta}) = u(re^{i\theta})$ for $0\le r<1$ and $e^{i\theta}\in\mathbb{T}$. Then $u(z)=0$ for all $z\in\mathbb{D}$.

Proof: By assumption we have that

$$\hat{u}_r(k) = \frac{1}{2\pi}\int_{\mathbb{T}} u_r(e^{i\theta})e^{-ik\theta}\,d\theta = o(1-r)$$

as $r\to 1$. By Theorem 1.1 we have that

$$\hat{u}_r(k) = \big(a_k + b_kw_{|k|}(r)\big)r^{|k|},$$

where $a_k, b_k \in \mathbb{C}$. By the above asymptotic formulas for these quantities we conclude that

$$\begin{cases} a_0 - A_{-1}b_0 = 0 \\ -A_0 b_0 = 0 \end{cases} \qquad \text{and} \qquad \begin{cases} a_k + \dfrac{A_0 - A_{|k|}}{2|k|} b_k = 0 \\ |k|a_k + \dfrac{A_0 + A_{|k|}}{2} b_k = 0 \end{cases}$$

for $k \neq 0$. Since $A_k > 0$ for $k \geq 0$, we conclude that $a_k = b_k = 0$ for all $k \in \mathbb{Z}$. Thus $u(z) = 0$ for $z \in \mathbb{D}$. $\square$

We remark that Theorem 2.1 gives uniqueness of solutions of the Dirichlet problem (0.2). In fact, the assumption on u in the theorem can be phrased as $u = \partial_n u = 0$ on $\mathbb{T}$ in the distributional sense.

In the unweighted case $w = w_0$ the result of Theorem 2.1 is known (see [**11**, Proposition 1.1]).

3. A solution of the Dirichlet problem

We now turn to the problem of representation of w-biharmonic functions in terms of boundary values. First we discuss existence of distributional boundary values.

Proposition 3.1. *Let $w\colon \mathbb{D} \to (0,\infty)$ be a radial continuous weight function such that $\int_0^1 (1-t)w(t)\,dt < \infty$. Let u be given by* (1.5)*. Then u satisfies for some positive constants C and N the growth conditions*

$$(3.1)\quad |u(z)| \leq C(1-|z|)^{-N} \quad \textit{and} \quad |\Delta u(z)| \leq Cw(z)(1-|z|)^{-N}, \quad z\in\mathbb{D},$$

if and only if the coefficients a_k and b_k are of polynomial growth, that is,

$$|a_k| + |b_k| \leq C(1+|k|)^N, \quad k \in \mathbb{Z},$$

for some (different) constants C and N.

Proof: We first recall that the w_k's are uniformly bounded (see the paragraph after Lemma 1.1).

Assume now that u satisfies (3.1). Recall that $\Delta u = wh$, where $h = \sum b_k r^{|k|} e^{ik\theta}$ (see Lemma 1.3). By the second estimate in (3.1) we see that the harmonic function h is of tempered growth in $\mathbb{D}$ which means that the b_k's are of polynomial growth. Since the w_k's are uniformly bounded, the sum $u_1 = \sum b_k w_{|k|}(r) r^{|k|} e^{ik\theta}$ is easily seen to be of tempered growth in $\mathbb{D}$. By the first estimate in (3.1) we see that the harmonic function $h_1 = \sum a_k r^{|k|} e^{ik\theta}$ is of tempered growth in $\mathbb{D}$ which means that the a_k's also are of polynomial growth.

Assume now that the coefficients a_k and b_k are of polynomial growth. Using the uniform boundedness of the w_k's it is easy to see that u is of tempered growth in $\mathbb{D}$. Since $\Delta u = wh$, where $h = \sum b_k r^{|k|} e^{ik\theta}$, the second growth condition in (3.1) is also satisfied. □

We now compute the boundary values.

Proposition 3.2. *Let $w\colon \mathbb{D} \to (0,\infty)$ be a radial continuous weight function which is area integrable in the sense that $\int_0^1 w(t)\,dt < \infty$. Let u be a w-biharmonic function satisfying* (3.1). *Then there exists the limits*

$$f_0 = \lim_{r\to 1} u_r \quad \textit{and} \quad f_1 = \lim_{r\to 1}(u_r - f_0)/(1-r) \quad \textit{in } \mathcal{D}'(\mathbb{T}).$$

Furthermore, the distributions f_0 and f_1 are given by the Fourier series:

$$f_0 = (a_0 - A_{-1}b_0) + \sum_{k\neq 0}\left(a_k + \frac{A_0 - A_{|k|}}{2|k|}b_k\right)e^{ik\theta} \quad \textit{and}$$

$$f_1 = -A_0 b_0 - \sum_{k\neq 0}\left(|k|a_k + \frac{A_0 + A_{|k|}}{2}b_k\right)e^{ik\theta} \qquad \textit{in } \mathcal{D}'(\mathbb{T}),$$

respectively; here a_k and b_k are the coefficients in (1.5) *and the A_k's are as in Proposition 2.1.*

Proof: Consider the series expansion (1.5). By Proposition 3.1 we know that the coefficients a_k and b_k are of polynomial growth.

We compute the boundary limit of u. Let $\varphi \in C^\infty(\mathbb{T})$. We have that

$$\langle u_r, \varphi\rangle = \sum_{k=-\infty}^{\infty}\big(a_k + b_k w_{|k|}(r)\big)r^{|k|}\hat{\varphi}(-k);$$

here $\langle\cdot,\cdot\rangle$ denotes the usual distributional pairing and $\hat{\varphi}(k) = \int_{\mathbb{T}} \varphi(e^{i\theta})e^{-ik\theta}\,d\theta/2\pi$ is the k-th Fourier coefficient of φ. By the Lebesgue dominated convergence theorem and Proposition 2.1 we have that

$$\langle u_r, \varphi\rangle \to (a_0 - A_{-1}b_0)\hat{\varphi}(0) + \sum_{k\neq 0}\left(a_k + \frac{A_0 - A_{|k|}}{2|k|}b_k\right)\hat{\varphi}(-k) = \langle f_0, \varphi\rangle$$

as $r \to 1$.

We now compute the normal derivative of u. Note that there is an estimate

$$0 \le w_k(1) - w_k(r) \le C(1-r), \quad 0 < r < 1, \tag{3.2}$$

where C is an absolute constant; here $w_k(1) = (A_0 - A_k)/(2k)$. Indeed, by (1.2) we have that $0 \le w_k'(r) \le \int_0^1 w(t)\,dt = C$ and an integration yields (3.2). The estimate (3.2) allows us to use Lebesgue's theorem to compute the limit $\lim_{r\to 1}\langle (u_r - f_0)/(1-r), \varphi\rangle$. By Proposition 2.1 we have that

$$\begin{aligned}\lim_{r\to 1}\big\langle (u_r - f_0)/(1-r), \varphi\big\rangle &= -A_0 b_0 \hat{\varphi}(0) - \sum_{k\neq 0}\bigg(|k| a_k + \frac{A_0 + A_{|k|}}{2} b_k\bigg)\hat{\varphi}(-k)\\ &= \langle f_1, \varphi\rangle,\end{aligned}$$

which concludes the proof. □

Let $w\colon \mathbb{D} \to (0,\infty)$ be a radial continuous area integrable weight function. We shall consider the functions F_w and H_w defined by the formulas

$$F_w(z) = \sum_{k=-\infty}^{\infty}\left(\frac{A_0 + A_{|k|}}{2A_{|k|}} - \frac{|k|}{A_{|k|}} w_{|k|}(r)\right) r^{|k|} e^{ik\theta} \quad \text{and}$$

$$\begin{aligned}H_w(z) &= -\frac{A_{-1}}{A_0} - \frac{1}{A_0} w_0(r)\\ &\quad + \sum_{k\neq 0}\left(\frac{A_0 - A_{|k|}}{2|k| A_{|k|}} - \frac{1}{A_{|k|}} w_{|k|}(r)\right) r^{|k|} e^{ik\theta}\end{aligned}$$

for $z = re^{i\theta} \in \mathbb{D}$. Note that since $\lim_{k\to\infty} A_k^{1/k} = 1$ the functions F_w and H_w are well-defined w-biharmonic functions in $\mathbb{D}$ (see Theorem 1.1). We also write $F_{w,r}(e^{i\theta}) = F_w(re^{i\theta})$ for $e^{i\theta} \in \mathbb{T}$ and $0 \leq r < 1$, and similarly for H_w.

We can represent a fairly general w-biharmonic function in terms of the above kernels F_w and H_w.

Theorem 3.1. *Let $w\colon \mathbb{D} \to (0,\infty)$ be a radial continuous weight function which is area integrable. Let u be a w-biharmonic function satisfying* (3.1). *Then the function u admits the representation*

$$u(z) = (F_{w,r} * f_0)(e^{i\theta}) + (H_{w,r} * f_1)(e^{i\theta}), \quad z = re^{i\theta} \in \mathbb{D}, \tag{3.3}$$

where the f_j's are as in Proposition 3.2.

Proof: We compute the Fourier coefficients. For $k \neq 0$ we have that

$$
\begin{aligned}
&(F_{w,r} * f_0 + H_{w,r} * f_1)\hat{}(k) \\
&\qquad = \left(\frac{A_0 + A_{|k|}}{2A_{|k|}} - \frac{|k|}{A_{|k|}} w_{|k|}(r)\right) r^{|k|} \left(a_k + \frac{A_0 - A_{|k|}}{2|k|} b_k\right) \\
&\qquad\quad + \left(\frac{A_0 - A_{|k|}}{2|k|A_{|k|}} - \frac{1}{A_{|k|}} w_{|k|}(r)\right) r^{|k|} \left(-|k|a_k - \frac{A_0 + A_{|k|}}{2} b_k\right) \\
&\qquad = \big(a_k + b_k w_{|k|}(r)\big) r^{|k|} = \hat{u}_r(k).
\end{aligned}
$$

Similarly we see that

$$
\begin{aligned}
(F_{w,r} * f_0 + H_{w,r} * f_1)\hat{}(0) &= (a_0 - A_{-1} b_0) + \left(-\frac{A_{-1}}{A_0} - \frac{1}{A_0} w_0(r)\right)(-A_0 b_0) \\
&= \big(a_0 + b_0 w_0(r)\big) = \hat{u}_r(0).
\end{aligned}
$$

By uniqueness of Fourier coefficients we now conclude that $u_r = F_{w,r} * f_0 + H_{w,r} * f_1$, which yields (3.3). □

Remark 3.1. Let $f_0, f_1 \in \mathcal{D}'(\mathbb{T})$ and let u be defined by (3.3). Then by Theorem 1.1 the function u is w-biharmonic in $\mathbb{D}$.

Let $w \colon \mathbb{D} \to (0, \infty)$ be a radial continuous weight function which is area integrable. Assume also that the moments $\{A_k\}$ of w are such that

$$
A_k \geq c(1+k)^{-N}, \quad k \geq 0,
$$

for some positive constants c and N. Then the coefficients of F_w and H_w are of polynomial growth. Propositions 3.1 and 3.2 now apply to show that the functions F_w and H_w solve the boundary value problems:

$$
\begin{cases}
\Delta w^{-1} \Delta F_w = 0 & \text{in } \mathbb{D}, \\
F_w = \delta_1 & \text{on } \mathbb{T}, \\
\partial_n F_w = 0 & \text{on } \mathbb{T},
\end{cases}
\quad \text{and} \quad
\begin{cases}
\Delta w^{-1} \Delta H_w = 0 & \text{in } \mathbb{D}, \\
H_w = 0 & \text{on } \mathbb{T}, \\
\partial_n H_w = \delta_1 & \text{on } \mathbb{T},
\end{cases}
$$

in the distributional sense. Here $\delta_{e^{i\theta}}$ denotes the unit Dirac mass at $e^{i\theta} \in \mathbb{T}$. The function H_w is sometimes called the harmonic compensator (see [**5**]).

We can now solve the Dirichlet problem (0.2) for arbitrary distributional boundary datas.

Theorem 3.2. *Let $w\colon \mathbb{D} \to (0,\infty)$ be a radial continuous weight function which is area integrable and such that*

$$A_k = \int_0^1 t^{2k+1} w(t)\,dt \geq c(1+k)^{-N}, \quad k \geq 0, \tag{3.4}$$

for some positive constants c and N. Let $f_0, f_1 \in \mathcal{D}'(\mathbb{T})$ and let u be defined by (3.3)*. Then the function u solves the Dirichlet boundary value problem*

$$\begin{cases} \Delta w^{-1} \Delta u = 0 & \textit{in } \mathbb{D}, \\ u = f_0 & \textit{on } \mathbb{T}, \\ \partial_n u = f_1 & \textit{on } \mathbb{T}, \end{cases}$$

in the distributional sense.

Proof: The function u defined by (3.3) is always w-biharmonic in $\mathbb{D}$ (see Remark 3.1).

We now turn to the boundary values of u. Let us first consider the function u_1 defined by $u_1(z) = (H_{w,r} * f_1)(e^{i\theta})$ for $z = re^{i\theta} \in \mathbb{D}$. By Propositions 3.1 and 3.2 we know that

$$H_{w,r}/(1-r) \to \delta_1 \quad \text{in } \mathcal{D}'(\mathbb{T}).$$

Convolving with $f_1 \in \mathcal{D}'(\mathbb{T})$ we see that

$$u_{1,r}/(1-r) = (H_{w,r} * f_1)/(1-r) \to f_1 \quad \text{in } \mathcal{D}'(\mathbb{T}).$$

Thus $u_1 = 0$ and $\partial_n u_1 = f_1$ on $\mathbb{T}$ in the distributional sense.

Similarly as above we see that the function u_0 defined by $u_0(z) = (F_{w,r} * f_0)(e^{i\theta})$ for $z = re^{i\theta} \in \mathbb{D}$ is such that $u_0 = f_0$ and $\partial_n u_0 = 0$ on $\mathbb{T}$ in the distributional sense. We now conclude that the function $u = u_0 + u_1$ is such that $u = f_0$ and $\partial_n u = f_1$ on $\mathbb{T}$ in the distributional sense. $\square$

The moment condition (3.4) is satisfied provided the weight function $w\colon \mathbb{D} \to (0,\infty)$ has enough mass near the boundary $\mathbb{T}$. Let us first consider the so-called standard weights $w = w_\alpha$ defined by

$$w_\alpha(z) = (1-|z|^2)^\alpha, \quad z \in \mathbb{D},$$

where $\alpha > -1$. In this case it is known that

$$\int_0^1 t^{2k+1}(1-t^2)^\alpha\,dt = \frac{1}{2}\frac{\Gamma(k+1)\Gamma(\alpha+1)}{\Gamma(k+\alpha+2)},$$

where $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt$ for $x > 0$ is the gamma function (see [**13**, Theorem 8.20]; see also [**6**, Section 1.1]). A computation using Stirling's formula (see [**13**, Section 8.22]) shows that an estimate of the form (3.4) holds in this case.

Let us now consider a general area integrable weight function $w\colon \mathbb{D} \to (0,\infty)$. By partial integration we see that

$$\int_0^1 t^{2k+1} w(t)\,dt = (2k+1)\int_0^1 \left(\int_t^1 w(s)\,ds\right) t^{2k}\,dt.$$

A comparison argument now shows that (3.4) holds also for every weight w such that

$$\inf_{0<r<1} \left(\int_r^1 w(t)\,dt\right) / (1-r^2)^\alpha > 0$$

for some $\alpha > 0$.

4. Formulas for the functions F_w and H_w

We now give some formulas for the functions F_w and H_w.

Proposition 4.1. *Let $w\colon \mathbb{D} \to (0,\infty)$ be a radial continuous weight function which is area integrable. Then the functions F_w and H_w admit the representations*

$$F_w(z) = \frac{1-|z|^2}{|1-z|^2} + \int_r^1\int_0^s \left(\sum_{k=-\infty}^{\infty} \frac{|k|}{A_{|k|}} \left(\frac{t}{s}\right)^{2|k|+1} r^{|k|} e^{ik\theta}\right) w(t)\,dt\,ds \quad \textit{and}$$

$$H_w(z) = \int_r^1\int_0^s \left(\sum_{k=-\infty}^{\infty} \frac{1}{A_{|k|}} \left(\frac{t}{s}\right)^{2|k|+1} r^{|k|} e^{ik\theta}\right) w(t)\,dt\,ds$$

for $z = re^{i\theta} \in \mathbb{D}$, respectively, where $A_k = \int_0^1 t^{2k+1} w(t)\,dt$ for $k \geq 0$.

Proof: We first derive the formula for H_w. Let

$$W_0(r) = -\frac{A_{-1}}{A_0} - \frac{1}{A_0} w_0(r) \quad \text{and} \quad W_{|k|}(r) = \frac{A_0 - A_{|k|}}{2|k| A_{|k|}} - \frac{1}{A_{|k|}} w_{|k|}(r)$$

for $k \neq 0$. By Lemma 1.1 the function $W_{|k|}$ satisfies the differential equation

$$W''_{|k|}(r) + (2|k|+1)\frac{1}{r} W'_{|k|}(r) = -\frac{1}{A_{|k|}} w(r)$$

and $W'_{|k|}(0) = 0$. By Proposition 2.1 we also have that $W_{|k|}(1) = 0$. A standard argument shows that $W_{|k|}$ has the representation

$$W_{|k|}(r) = \frac{1}{A_{|k|}} \int_r^1 \int_0^s \left(\frac{t}{s}\right)^{2|k|+1} w(t)\,dt\,ds.$$

In fact, an integration of (1.2) yields the above formula. A computation now shows that

$$H_w(z) = \sum_{k=-\infty}^{\infty} W_{|k|}(r) r^{|k|} e^{ik\theta}$$
$$= \int_r^1 \int_0^s \left(\sum_{k=-\infty}^{\infty} \frac{1}{A_{|k|}} \left(\frac{t}{s}\right)^{2|k|+1} r^{|k|} e^{ik\theta} \right) w(t)\, dt\, ds,$$

which is the formula for H_w.

We now derive the formula for F_w. Arguing as above we first see that

$$\frac{A_0 + A_{|k|}}{2A_{|k|}} - \frac{|k|}{A_{|k|}} w_{|k|}(r) = 1 + \frac{|k|}{A_{|k|}} \int_r^1 \int_0^s \left(\frac{t}{s}\right)^{2|k|+1} w(t)\, dt\, ds.$$

A computation now shows that

$$F_w(z) = \sum_{k=-\infty}^{\infty} \left(1 + \frac{|k|}{A_{|k|}} \int_r^1 \int_0^s \left(\frac{t}{s}\right)^{2|k|+1} w(t)\, dt\, ds \right) r^{|k|} e^{ik\theta}$$
$$= P(z) + \int_r^1 \int_0^s \left(\sum_{k=-\infty}^{\infty} \frac{|k|}{A_{|k|}} \left(\frac{t}{s}\right)^{2|k|+1} r^{|k|} e^{ik\theta} \right) w(t)\, dt\, ds,$$

where $P(z) = (1 - |z|^2)/|1 - z|^2$ is the usual Poisson kernel. □

Remark 4.1. Let $\kappa = \sum_{k=-\infty}^{\infty} |k| e^{ik\theta}$ in $\mathcal{D}'(\mathbb{T})$. The formulas in Proposition 4.1 make evident that

$$F_w(z) = P(z) + (\kappa * H_{w,r})(e^{i\theta}), \quad z = re^{i\theta} \in \mathbb{D}, \tag{4.1}$$

where $P(z) = (1 - |z|^2)/|1 - z|^2$ is the usual Poisson kernel for the unit disc $\mathbb{D}$. Let us include here a direct proof of (4.1). A computation shows that

$$\partial_n P = -\sum_{k=-\infty}^{\infty} |k| e^{ik\theta} = -\kappa \quad \text{in } \mathcal{D}'(\mathbb{T}).$$

Let us denote by u the right-hand side in (4.1). Using properties of H_w we can easily verify that $u = \delta_1$ and $\partial_n u = 0$ on $\mathbb{T}$ in the distributional sense. A uniqueness argument (see Theorem 2.1) then gives that $u = F_w$ which proves (4.1).

For easy reference we also record the following series expansions of the functions F_w and H_w:

$$F_w(z) = \sum_{k=-\infty}^{\infty} \left(1 + \frac{|k|}{A_{|k|}} \int_r^1 \int_0^s \left(\frac{t}{s}\right)^{2|k|+1} w(t)\,dt\,ds\right) r^{|k|} e^{ik\theta} \quad \text{and}$$

$$H_w(z) = \sum_{k=-\infty}^{\infty} \left(\frac{1}{A_{|k|}} \int_r^1 \int_0^s \left(\frac{t}{s}\right)^{2|k|+1} w(t)\,dt\,ds\right) r^{|k|} e^{ik\theta}$$

for $z = re^{i\theta} \in \mathbb{D}$. The validity of these expansions is clear by Proposition 4.1.

The formula for H_w in Proposition 4.1 has appeared in a different context in [**14**, Formula (9)] (see also [**5**, Formula (8.5)]). Here we also want to mention that the function

$$K_w(z,\zeta) = \frac{1}{2}\sum_{k\geq 0} \frac{1}{A_k}(z\bar{\zeta})^k + \frac{1}{2}\sum_{k<0} \frac{1}{A_{|k|}}(\bar{z}\zeta)^{|k|}, \quad (z,\zeta) \in \mathbb{D}\times\mathbb{D},$$

has the interpretation as the kernel function for the Bergman space of square area integrable with respect to the weight w harmonic functions in $\mathbb{D}$; here $\|f\|_w^2 = \int_{\mathbb{D}} |f(z)|^2 w(z)\,dA(z)$, where dA is the normalized Lebesgue area measure in $\mathbb{D}$.

Let us mention two cases where explicit formulas for the kernels F_w and H_w are available. We consider first the unweighted situation. In this case the functions $F_0 = F_{w_0}$ and $H_0 = H_{w_0}$ have the explicit expressions

$$F_0(z) = \frac{1-|z|^2}{|1-z|^2} + (1-|z|^2)\Re\left(\frac{z}{(1-z)^2}\right) \quad \text{and}$$

$$H_0(z) = \frac{1}{2}\frac{(1-|z|^2)^2}{|1-z|^2} \qquad \text{for } z \in \mathbb{D}.$$

Here $\Re(z)$ denotes the real part of the complex number z. The function F_0 can also be written

$$F_0(z) = \frac{1}{2}\frac{(1-|z|^2)^2}{|1-z|^2} + \frac{1}{2}\frac{(1-|z|^2)^3}{|1-z|^4}, \quad z \in \mathbb{D}$$

(see [**1**, Formula (0-5)]). Hedenmalm has pointed out to us that the equality of these two expressions for the function F_0 follows by a straightforward computation using the identity $2\Re(z) = 1+|z|^2-|1-z|^2$ (see formula (4.4) below). We mention in passing that for the harmonic compensator function H_0 a sharp monotonicity estimate is known (see [**10**]).

We next consider the weight $w = w_1$.

Proposition 4.2. *The functions $F_1 = F_{w_1}$ and $H_1 = H_{w_1}$ have the explicit expressions*

$$F_1(z) = \frac{1}{2}\frac{(1-|z|^2)^3}{|1-z|^2} - |z|^2\frac{(1-|z|^2)^3}{|1-z|^4} + \frac{1}{2}\frac{(1-|z|^2)^5}{|1-z|^6} \quad \textit{and}$$

$$H_1(z) = \frac{1}{2}\frac{(1-|z|^2)^3}{|1-z|^2} + \frac{1}{4}\frac{(1-|z|^2)^4}{|1-z|^4} \qquad \textit{for } z \in \mathbb{D}.$$

Furthermore, the function F_1 assumes negative as well as positive values in every neighborhood of $z = 1$.

Proof: A straightforward computation shows that

$$A_k = \int_0^1 t^{2k+1}(1-t^2)\,dt = \frac{2}{(2k+2)(2k+4)}, \quad k \geq 0,$$

and that

$$\int_r^1 \int_0^s \left(\frac{t}{s}\right)^{2|k|+1} (1-t^2)\,dt\,ds = \frac{1}{2|k|+2}\frac{1-r^2}{2} - \frac{1}{2|k|+4}\frac{1-r^4}{4}.$$

By Proposition 4.1 we now have that

$$\begin{aligned}
F_1(z) = \sum_{k=-\infty}^{\infty} \Bigg(1 + &\frac{|k|(2|k|+2)(2|k|+4)}{2} \\
&\times \left(\frac{1}{2|k|+2}\frac{1-r^2}{2} - \frac{1}{2|k|+4}\frac{1-r^4}{4}\right)\Bigg) r^{|k|}e^{ik\theta} \\
= \sum_{k=-\infty}^{\infty} r^{|k|}e^{ik\theta} &+ \frac{1-r^2}{2}\sum_{k=-\infty}^{\infty} |k|(|k|+2)r^{|k|}e^{ik\theta} \\
&- \frac{1-r^4}{4}\sum_{k=-\infty}^{\infty} |k|(|k|+1)r^{|k|}e^{ik\theta} \\
= \frac{1-|z|^2}{|1-z|^2} &+ (1-|z|^2)\Re\left(\sum_{k=1}^{\infty} k(k+2)z^k\right) \\
&- \frac{1-|z|^4}{2}\Re\left(\sum_{k=1}^{\infty} k(k+1)z^k\right),
\end{aligned}$$

where $\Re(z)$ denotes the real part of the complex number z. Now using

$$\sum_{k=1}^{\infty} k(k+1)z^k = \frac{2z}{(1-z)^3} \quad \text{and} \quad \sum_{k=1}^{\infty} k(k+2)z^k = \frac{2z}{(1-z)^3} + \frac{z}{(1-z)^2},$$

we obtain that

$$F_1(z) = \frac{1-|z|^2}{|1-z|^2} + (1-|z|^2)\Re\left(\frac{2z}{(1-z)^3} + \frac{z}{(1-z)^2}\right) - \frac{1-|z|^4}{2}\Re\left(\frac{2z}{(1-z)^3}\right),$$

which simplifies to

$$F_1(z) = \frac{1-|z|^2}{|1-z|^2} + (1-|z|^2)\Re\left(\frac{z}{(1-z)^2}\right) + (1-|z|^2)^2\Re\left(\frac{z}{(1-z)^3}\right), \quad z \in \mathbb{D}. \tag{4.2}$$

We shall next derive a similar formula for the function H_1. By Proposition 4.1 we again have

$$\begin{aligned} H_1(z) &= \sum_{k=-\infty}^{\infty} \frac{(2|k|+2)(2|k|+4)}{2}\left(\frac{1}{2|k|+2}\frac{1-r^2}{2} - \frac{1}{2|k|+4}\frac{1-r^4}{4}\right) r^{|k|}e^{ik\theta} \\ &= \frac{1-r^2}{2}\sum_{k=-\infty}^{\infty}(|k|+2)r^{|k|}e^{ik\theta} - \frac{1-r^4}{4}\sum_{k=-\infty}^{\infty}(|k|+1)r^{|k|}e^{ik\theta} \\ &= \left(1-r^2-\frac{1-r^4}{4}\right)\sum_{k=-\infty}^{\infty} r^{|k|}e^{ik\theta} + \left(\frac{1-r^2}{2}-\frac{1-r^4}{4}\right)\sum_{k=-\infty}^{\infty}|k|r^{|k|}e^{ik\theta} \\ &= \frac{3-4|z|^2+|z|^4}{4}\frac{1-|z|^2}{|1-z|^2} + \frac{(1-|z|^2)^2}{2}\Re\left(\frac{z}{(1-z)^2}\right), \end{aligned}$$

which simplifies to

$$H_1(z) = \frac{3-|z|^2}{4}\frac{(1-|z|^2)^2}{|1-z|^2} + \frac{(1-|z|^2)^2}{2}\Re\left(\frac{z}{(1-z)^2}\right), \quad z \in \mathbb{D}. \tag{4.3}$$

We shall next rewrite the real part expressions in (4.2) and (4.3). In fact, we have that

$$(4.4)\quad 2\Re\left(\frac{z}{(1-z)^2}\right)=\frac{(1-|z|^2)^2}{|1-z|^4}-\frac{1+|z|^2}{|1-z|^2} \qquad \text{and}$$

$$(4.5)\quad 2\Re\left(\frac{z}{(1-z)^3}\right)=\frac{(1-|z|^2)^3}{|1-z|^6}-\frac{(1-|z|^2)(2|z|^2+1)}{|1-z|^4}-\frac{|z|^2}{|1-z|^2}.$$

To prove (4.4) we compute, using the formula $2\Re(z)=1+|z|^2-|1-z|^2$, that

$$\begin{aligned}
2\Re\left(\frac{z}{(1-z)^2}\right) &= \frac{1}{|1-z|^4}2\Re\big(z(1-\bar{z})^2\big) \\
&= \frac{1}{|1-z|^4}\big((1+|z|^2)2\Re(z)-4|z|^2\big) \\
&= \frac{1}{|1-z|^4}\big((1+|z|^2)(1+|z|^2-|1-z|^2)-4|z|^2\big) \\
&= \frac{(1-|z|^2)^2}{|1-z|^4}-\frac{1+|z|^2}{|1-z|^2}.
\end{aligned}$$

Let us now turn to the proof of (4.5). In analogy with the formula $2\Re(z)=1+|z|^2-|1-z|^2$ used above we have that

$$2\Re(z^2)=1+|z|^4+|1-z|^4-2(1+|z|^2)|1-z|^2.$$

A computation now shows that

$$\begin{aligned}
2\Re\left(\frac{z}{(1-z)^3}\right) &= \frac{1}{|1-z|^6}2\Re\big(z(1-\bar{z})^3\big) \\
&= \frac{1}{|1-z|^6}\big(-6|z|^2+(1+3|z|^2)2\Re(z)-|z|^2 2\Re(z^2)\big) \\
&= \frac{1}{|1-z|^6}\Big(-6|z|^2+(1+3|z|^2)(1+|z|^2-|1-z|^2) \\
&\qquad -|z|^2\big(1+|z|^4+|1-z|^4-2(1+|z|^2)|1-z|^2\big)\Big) \\
&= \frac{1}{|1-z|^6}\Big(1-3|z|^2+3|z|^4-|z|^6+(2|z|^4-|z|^2-1)|1-z|^2 \\
&\qquad -|z|^2|1-z|^4\Big) \\
&= \frac{(1-|z|^2)^3}{|1-z|^6}+\frac{2|z|^4-|z|^2-1}{|1-z|^4}-\frac{|z|^2}{|1-z|^2},
\end{aligned}$$

which gives formula (4.5).

We now substitute (4.4) and (4.5) into (4.2) to obtain

$$\begin{aligned} F_1(z) &= \frac{1-|z|^2}{|1-z|^2} + \frac{1-|z|^2}{2}\left(\frac{(1-|z|^2)^2}{|1-z|^4} - \frac{1+|z|^2}{|1-z|^2}\right) \\ &\quad + \frac{(1-|z|^2)^2}{2}\left(\frac{(1-|z|^2)^3}{|1-z|^6} - \frac{(1-|z|^2)(2|z|^2+1)}{|1-z|^4} - \frac{|z|^2}{|1-z|^2}\right) \\ &= \frac{1}{2}\frac{(1-|z|^2)^3}{|1-z|^2} - |z|^2\frac{(1-|z|^2)^3}{|1-z|^4} + \frac{1}{2}\frac{(1-|z|^2)^5}{|1-z|^6}, \end{aligned}$$

which gives the formula for F_1 in the proposition. The formula for H_1 follows similarly.

We now consider the function F_1 in some more detail. First, for $z = x \in \mathbb{R}$, we have that

$$F_1(x) = (1+x)^3/(1-x) > 0$$

for $-1 < x < 1$. We now demonstrate that F_1 also assumes negative values. For z on the level set $(1-|z|^2)/|1-z|^2 = c > 0$ we have that

$$F_1(z) = (1-|z|^2)\big(-2c^2 + c(1+c)^2(1-|z|^2)\big)/2$$

which is clearly negative for z close to 1. □

We remark that the formula for H_1 in (4.3) is known (see [**8**, Proof of Lemma 2.2]).

The last assertion of Proposition 4.2 can be interpreted saying that the full biharmonic maximum principle suggested in [**5**] does not hold true for the particular weight $w = w_1$. Indeed, the above function F_1 is such that $F_1 = \delta_1 \geq 0$ and $\partial_n F_1 = 0$ on $\mathbb{T}$ while F_1 is not of constant sign in $\mathbb{D}$. This fact has been pointed out earlier by Hedenmalm (private discussion); we have merely filled in the details.

As a direction for future research it is of interest to know detailed regularizing properties of the Dirichlet problem (0.2) discussed in the introduction. We plan to return to this topic in a forthcoming paper [**12**]. In this context we want to mention the papers by Shimorin [**14**] and Hedenmalm, Jakobsson and Shimorin [**5**] where it is shown that the harmonic compensator function H_w is nonnegative in $\mathbb{D}$, that is, $H_w(z) \geq 0$ for $z \in \mathbb{D}$, if the weight function $w\colon \mathbb{D} \to (0,\infty)$ is logarithmically subharmonic in $\mathbb{D}$. Here the function w is said to be logarithmically subharmonic in $\mathbb{D}$ if the function $z \mapsto \log w(z)$ is subharmonic there.

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