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ON THE HEIGHT OF FOLIATED SURFACES WITH VANISHING KODAIRA DIMENSION

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Abstract

We prove that the height of a foliated surface of Kodaira dimension zero belongs to $\{1, 2, 3, 4, 5, 6, 8, 10, 12\}$. We also construct an explicit projective model for Brunella's very special foliation.

1. Introduction

A *foliated surface* consists of a pair $(S, \mathcal{F})$, where S is a complex surface and $\mathcal{F}$ is a saturated singular holomorphic foliation. By a *singular holomorphic foliation* we mean an element of $\mathbb{P}H^0(S, TS \otimes \mathcal{L})$, for some line bundle $\mathcal{L}$. The line bundle $\mathcal{L}$ is the *cotangent bundle* of $\mathcal{F}$ and will be denoted by $T^*\mathcal{F}$. We say that a singular holomorphic foliation is *saturated* if any represent of $\mathcal{F}$ in $H^0(S, TS \otimes T^*\mathcal{F})$ has a finite singular set. On this paper all the foliated surfaces will be projective.

The sections of $T^*\mathcal{F}$ can be interpreted as the 1-forms along the leaves of $\mathcal{F}$. Keeping in mind the analogous case of projective manifolds, it is natural to ask whether the integers $h^0(S, TS \otimes T^*\mathcal{F}^{\otimes k})$ for $k \in \mathbb{Z}_{>0}$, are birational invariants of $(S, \mathcal{F})$. The answer turns out to be no as in the case of projective surfaces with arbitrary singularities. One needs to restrict to a class with mild singularities. A nice surprise, pointed out in [**6**], is that the reduced foliated surfaces singularities (in the sense of Seidenberg) form such class. Moreover Seidenberg proved that every foliated surface is birationally equivalent to a reduced foliated surface.

A birational classification of reduced foliated surfaces according to theirs Kodaira dimension have been carried out recently, cf. [**2**], [**5**], [**6**]. Recall that the Kodaira dimension of a reduced foliated surface $(S, \mathcal{F})$, $\operatorname{kod}(S, \mathcal{F})$ for short, is defined as

$$\operatorname{kod}(S, \mathcal{F}) = \limsup_{k \to \infty} \frac{\log h^0(S, T^*\mathcal{F}^{\otimes k})}{\log k}.$$

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For a foliated surface $(S, \mathcal{F})$ of non-negative Kodaira dimension the *height of* $(S, \mathcal{F})$, $h(S, \mathcal{F})$ for short, is defined in [**7**] as the smallest integer k such that $T^*\mathcal{F}^{\otimes k}$ has $\mathrm{kod}(S, \mathcal{F}) + 1$ algebraically independent sections. In particular, if $\mathrm{kod}(S, \mathcal{F}) = 0$ then

$$h(S, \mathcal{F}) = \min\{k \in \mathbb{Z}_{>0} \mid h^0(S, T^*\mathcal{F}^{\otimes k}) \neq 0\}.$$

The purpose of this note is to prove the following

Theorem 1. *If* $(S, \mathcal{F})$ *is a reduced foliated surface of Kodaira dimension zero then*

$$h(S, \mathcal{F}) \in \{1, 2, 3, 4, 5, 6, 8, 10, 12\}.$$

Moreover, if S *is not rational or* $\mathcal{F}$ *admits a rational first integral then*

$$h(S, \mathcal{F}) \in \{1, 2, 3, 4, 6\}.$$

It is interesting to note that for projective surfaces S with Kodaira dimension zero it is well known that (cf. [**1**])

$$\min\{k \in \mathbb{Z}_{>0} \mid h^0(S, KS^{\otimes k}) \neq 0\} \in \{1, 2, 3, 4, 6\}.$$

For foliated surfaces the exceptional cases where $h(S, \mathcal{F}) \in \{5, 8, 10, 12\}$ correspond to foliated rational surfaces obtained as quotients of linear foliations on Abelian surfaces with complex multiplication.

2. Height versus transformation groups

2.1. Minimal and relatively minimal foliated surfaces. A foliated surface $(S, \mathcal{F})$ is a *relatively minimal foliated surface* if, and only, it satisfies the following universal property: any bimeromorphic morphism $(S, \mathcal{F}) \to (S', \mathcal{F}')$ onto a reduced foliated surface is in fact a biholomorphism. It turns out that every foliated surface admits a *relatively minimal model*, i.e., is birationally equivalent to a relatively minimal foliated surface. To verify this it is sufficient to apply Seidenberg's Theorem to obtain a reduced foliation and then do successive contractions of the so called $\mathcal{F}$*-exceptional curves*, i.e., the smooth rational curves of self-intersection -1 whose contraction still yields a reduced foliation, cf. [**2**, Proposition 1, p. 73].

In general we don't have the uniqueness of the relatively minimal model and when we do have the uniqueness we say that the foliated surface admits a *minimal model*, or equivalently, it is birationally equivalent to a *minimal foliated surface*. The minimal foliated surfaces can also be characterized by an universal property, namely: any bimeromorphic map $(S', \mathcal{F}') \dashrightarrow (S, \mathcal{F})$ onto a reduced foliated surface is in fact a morphism.

It has to be noted that the definition of minimal model above is not the only one in the literature. In [**5**] a more *functorial* definition, build-up on the concepts of $\mathbb{Q}$-gorenstein and canonical singularities, is made and from the point of view of Mori-Theory it is much more natural. We have adopted the definitions above (the same that are presented in [**2**]) since they are build-up on the concept of reduced singularities which is widely known in the theory of foliations of surfaces.

2.2. Foliated surfaces with vanishing Kodaira dimension. Concerning the classification of foliated surfaces of Kodaira dimension zero the key fact is the following

Theorem 2.1. *Let $(S,\mathcal{F})$ be a relatively minimal foliated surface. If the Kodaira dimension of $(S,\mathcal{F})$ is zero then there exists a ramified covering $\pi\colon S'\to S$ and a birational morphism $\rho\colon S'\to S''$ such that the foliation $\rho_*\pi^*\mathcal{F}$ is generated by a global holomorphic vector field v. Moreover v is in at least one of the following classes:*

(a) *v is tangent to a smooth elliptic fibration;*

(b) *v generates a Kroenecker foliation on an Abelian Surface;*

(c) *v is tangent to the suspension of a representation $\pi_1(E)\to\mathrm{Aut}(\mathbb{P}^1)$, where E is an elliptic curve;*

(d) *v is birationally equivalent to a linear vector field on $\mathbb{P}^2$.*

The above result first appeared, in a slightly different form, as Theorem 5 in McQuillan's paper [**5**]. The statement above, as it is, can be found in [**2**, pp. 88, 110 and 119].

On the proof of Theorem 2.1 presented in [**2**, pp. 110–112, 119–127] the ramified covering π is constructed using the *ramified covering trick*: if σ is a non-zero section of $T^*\mathcal{F}^{\otimes h(\mathcal{F})}$ then S' is the minimal resolution of $\widetilde{S}$ the preimage of the graph of σ under the map

$$E(T^*\mathcal{F}) \xrightarrow{\otimes h(\mathcal{F})} E(T^*\mathcal{F}^{\otimes h(\mathcal{F})}),$$

where $E(T^*\mathcal{F})$ is the total space of $T^*\mathcal{F}$. It follows that π is ramified along the zero set of σ. Note also that by construction $[k(S') : \pi^*k(S)] = h(\mathcal{F})$ and that ξ is a cyclic covering, i.e., $k(S')$ is a cyclic extension of $\pi^*k(S)$. In particular we have that the ramified covering π is Galois. More explicitly, observe that $\mathbb{C}^*$ acts naturally on $E(T^*\mathcal{F})$ and the action of the subgroup generated by ξ, where ξ is a primitive root of unity of order $h(\mathcal{F})$, leaves $\widetilde{S}$ invariant. Thus, after resolving the singularities of $\widetilde{S}$, ξ induces an automorphism g_ξ of order $h(\mathcal{F})$ of the foliated surface $(S',\pi^*\mathcal{F})$.

Note that in general $\pi^*\mathcal{F}$ is not tangent to a holomorphic flow. This will be the case for a relatively minimal reduction $\rho_*\pi^*\mathcal{F}$ of $\mathcal{F}$, where $\rho\colon S' \to S''$ is a birational morphism obtained by successive contractions of $\pi^*\mathcal{F}$-exceptional curves. In general the birational map $\rho\circ g_\xi\circ\rho^{-1}$ is not an automorphism. Although we can guarantee in cases (a), (b) and (c) of Theorem 2.1 that it will be an automorphism since $\rho_*\pi^*\mathcal{F}$ is a minimal foliation.

We close this section remarking that every reduced foliated surface of Kodaira dimension zero and tangent to a holomorphic flow has trivial cotangent bundle. In fact if $(S,\mathcal{F})$ is foliated surface tangent to a global holomorphic vector field v then $T\mathcal{F} = \mathcal{O}_S((v)_0)$. If the divisor $(v)_0$ is non trivial then $h^0(S, T^*\mathcal{F}^{\otimes k}) = 0$ for every $k \in \mathbb{Z}_{>0}$ and consequently $\operatorname{kod}(S,\mathcal{F}) = -\infty$. In particular the fixed points of the holomorphic flow (equivalently the zeros of v) are isolated.

2.3. The height of quotients of holomorphic actions. Let $(S,\mathcal{F})$ be a reduced foliated surface tangent to a holomorphic vector field v. If $\phi\colon S \to S$ is an automorphism of $(S,\mathcal{F})$ then there exists a rational function $g \in k(S)$ such that $\phi_*(v) = g\cdot v$. When $\operatorname{kod}(S,\mathcal{F}) = 0$ then from the triviality of $T\mathcal{F}$ it follows that g is in fact a constant. Thus, in the case of Kodaira dimension zero, the automorphism group of $(S,\mathcal{F})$ admits a natural character $\lambda\colon \operatorname{Aut}(S,\mathcal{F}) \to \mathbb{C}^*$ defined by the relation: if $\phi\colon S \to S$ is an automorphism of $(S,\mathcal{F})$ then $\phi_*(v) = \lambda(\phi)\cdot v$.

Proposition 2.1. *Let $(S',\mathcal{F}')$ be a reduced foliated surface of Kodaira dimension zero tangent to a holomorphic flow and G be a finite subgroup of $\operatorname{Aut}(S',\mathcal{F}')$. If $(S,\mathcal{F})$ is the minimal resolution of $(S',\mathcal{F}')/G$ then*

(i) $\operatorname{kod}(S,\mathcal{F}) = -\infty$ *and $(S,\mathcal{F})$ is a rational fibration, or*

(ii) $\operatorname{kod}(S,\mathcal{F}) = 0$ *and* $h(S,\mathcal{F}) = [G : \ker\lambda]$.

Proof: Let $(S'',\mathcal{F}'') = (S',\mathcal{F}')/G$ and $\rho\colon S' \to S''$ be the natural quotient map. Note that in general S'' will be singular, but with mild singularities: all its singularities are cyclic quotient singularities. Thus the sheaf $T^*\mathcal{F}''$ might fail to be a line bundle: it may be not locally free around the singularities of S'/G, cf. [**5**], [**2**]. Anyway, some power of $T^*\mathcal{F}''$ is a line-bundle and, for most purposes, we can deal with $T^*\mathcal{F}''$ as if it were a line-bundle. In particular, if we denote by $\operatorname{Fix}(G)$ the set

$$\{p \in S' \mid \text{there exists } g \neq \mathrm{Id} \in G \text{ such that } g(p) = p\}$$

then we can apply the formulas of [**2**, p. 29] to assure that

$$T^*\mathcal{F}' = \rho^*T^*\mathcal{F}'' \otimes \mathcal{O}_{S'}(R) \qquad (1)$$

where R is an effective $\mathbb{Q}$-divisor with support equal to the union of irreducible components of $\mathrm{Fix}(G)$ which are not $\mathcal{F}'$-invariant.

Suppose first that R is a nontrivial divisor. Since $T^*\mathcal{F}'$ is the trivial bundle then it follows from (1) that $\rho^*T^*\mathcal{F}'' = \mathcal{O}_{S'}(-R)$. Therefore $\rho^*T^*\mathcal{F}''$ and (consequently) $T^*\mathcal{F}''$ are not pseudo-effective. From Miyaoka's Theorem, see [**2**, p. 89], we deduce that $(S,\mathcal{F})$ is a (maybe singular) rational fibration, i.e., we are in case (i).

Suppose now that R is a trivial divisor. Let $v \in H^0(S', T\mathcal{F}')$ be a nontrivial vector field and $k = [G : \ker\lambda]$. From the definition of $\lambda\colon G \to \mathbb{C}^*$ it follows that $v^{\otimes k} \in H^0(S', T\mathcal{F}'^{\otimes k})$ is invariant under the action of G. Thus $\rho_* v^{\otimes k}$ is meaningful and it follows from (1) that it defines a trivialization of $T\mathcal{F}''^{\otimes k}$: seing $\rho_* v^{\otimes k}$ as a section of $T\mathcal{F}''^{\otimes k}$ it is a nowhere vanishing section. In particular $h^0(S'', T^*\mathcal{F}''^{\otimes k}) = 1$.

If $\pi\colon (S,\mathcal{F}) \to (S'',\mathcal{F}'')$ denotes the minimal resolution of $(S'',\mathcal{F}'')$ then using the fact that $(S', T^*\mathcal{F}')$ is reduced it can be easily verified that

$$T^*\mathcal{F} = \pi^* T^*\mathcal{F}'' \otimes \mathcal{O}_{S''}(E)$$

for some effective $\mathbb{Q}$-divisor E supported on the exceptional locus of π. Thus, $T^*\mathcal{F}^{\otimes k}$ is trivial,

$$h^0(S, T^*\mathcal{F}^{\otimes k}) = h^0(S'', T^*\mathcal{F}''^{\otimes k}) = 1$$

and, consequently, $h(S,\mathcal{F}) \leq [G : \ker\lambda]$.

From the definition of λ it follows that $h(S,\mathcal{F}) \geq [G : \ker\lambda]$, establishing the proposition. ☐

3. Quotients of holomorphic actions

Having at hand Theorem 2.1 and Proposition 2.1 we will deduce Theorem 1 from a case-by-case analysis.

Let $(S,\mathcal{F})$ be a foliated surface tangent to vector field v and $G \subset \mathrm{Aut}(S,\mathcal{F})$ be a finite subgroup. Let us study the different possibilities:

3.1. Case (a): Elliptic fibrations. In this case there exists a holomorphic map $\pi\colon S \to B$ from S to an algebraic curve B with connected fibers. If B^* is the set of regular values of π and $S^* = \pi^{-1}(B^*)$ then the restriction of π to S^* is a locally trivial fibration.

If $u\colon \widetilde{B} \to B^*$ is the universal covering of B^* then $\widetilde{S}$, the fibered product of u and π, is a trivial fibration over $\widetilde{B}$, i.e., $\widetilde{S} = \widetilde{B} \times E$ for some elliptic curve E.

The action of G on S lifts to an action $\widetilde{S} = \widetilde{B} \times E$. Moreover if $g \in G$ then φ_g, the automorphism induced by g in $\widetilde{B} \times E$, is of the form

$$\varphi_g(x, y) = (\alpha_g(x), \beta_g \cdot y + \gamma_g(x))$$

for coordinates (x, y) where $x \in \widetilde{B}$ and $y \in \mathbb{C}/\Gamma = E$. It follows that the morphism $\lambda\colon G \to \mathbb{C}^*$ is given by $g \mapsto \beta_g$. This is sufficient to establish that

$$[G : \ker \lambda] \in \{1, 2, 3, 4, 6\}.$$

3.2. Case (b): Kroenecker foliations. Here $S = A$ is an abelian surface. First recall that the automorphism group of A fits into the exact sequence

$$0 \longrightarrow A \longrightarrow \mathrm{Aut}(A) \longrightarrow \mathrm{Hol}(A) \longrightarrow 0$$

where $\mathrm{Hol}(A)$ is the *holonomy part* of $\mathrm{Aut}(A)$ which can be identified with a subgroup of $GL_2(\mathbb{C})$ and A acts on itself by translations.

If $G \subset \mathrm{Aut}(S, \mathcal{F}) \subset \mathrm{Aut}(A)$ is a finite subgroup then the character $\lambda\colon G \to \mathbb{C}^*$ factors through the natural projection $G \to G/(G \cap A)$ since the translations act trivially on vector fields.

In general the holonomy part of $\mathrm{Aut}(A)$ is a finite group of order 1, 5, 10 or $2^m \cdot 3^n$, where $m \leq 5$ and $n \leq 2$, see [**8**]. Although since finite subgroups of $\mathbb{C}^*$ are cyclic we have just to bound the order of the cyclic subgroups of $GL_2(\mathbb{C})$ which preserves a lattice Γ on $\mathbb{C}^2$. Forgetting the complex structure of A we are lead to bound the order of elements of $GL_4(\mathbb{Z})$.

Let $g \in GL_4(\mathbb{Z})$ be an element of finite order. Thus all the eigenvalues are roots of the unity with minimal polynomial of degree at most 4 and if k denotes the order of g then $\phi(k) \in \{1, 2, 3, 4\}$, where ϕ is Euler's function. We have the following possibilities: $\phi(k) = 1$ and $k = 1$; $\phi(k) = 2$ and $k \in \{2, 3, 4, 6\}$; or $\phi(k) = 4$ and $k \in \{5, 8, 10, 12\}$.

We conclude that in case (b)

$$[G : \ker \lambda] \in \{1, 2, 3, 4, 5, 6, 8, 10, 12\}.$$

3.3. Case (c): Suspension over an elliptic curve. Suppose now that S is a $\mathbb{P}^1$-bundle over an elliptic curve E. The vector field v induces a Riccati foliations without invariant fibers. Here we are in a dual situation to the case (a). If $u\colon \mathbb{C} \times \mathbb{P}^1 \to S$ is the universal covering of S then we can choose coordinates $(x, y) \in \mathbb{C} \times \mathbb{P}^1$ in such a way that v lifts to the vector field ∂_x. We can therefore conclude that the character $\lambda\colon G \to \mathbb{C}^*$ factors through the morphism $G \subset \mathrm{Aut}(S) \to \mathrm{Aut}(E)$. In particular, as in case (a), it follows that

$$[G : \ker \lambda] \in \{1, 2, 3, 4, 6\}.$$

3.4. Case (d): Rational surfaces. Suppose now that S is a rational surface and v is a holomorphic vector field. on this case we do not have in general that $(S, \mathcal{F})$ admits a minimal model. Although we can suppose without loss of generality that $S = \mathbb{P}^2$ and that v is a vector field on $\mathbb{P}^2$ we have to consider G as a finite subgroup of $\mathrm{Bir}(\mathbb{P}^2)$ instead of $\mathrm{Aut}(\mathbb{P}^2)$.

Up to the end of this section (x, y) will stand for the coordinates of a $\mathbb{C}^2 \subset \mathbb{P}^2$. We will distinguish two cases:

(d.1) $v = x\partial_x + \lambda y \partial_y$ for some $\lambda \in \mathbb{C} \setminus \mathbb{Q}$;

(d.2) $v = x\partial_x + \partial_y$.

3.4.1. Case (d.1): Quotients of $\mathbb{C}^* \times \mathbb{C}^*$. If the vector field v is of the form (d.1) then the only algebraic curves invariant by v are the lines $x = 0$, $y = 0$ and the line at infinity. Thus if $\varphi \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is a birational map preserving the foliation induced v then φ belongs to $\mathrm{Aut}(\mathbb{C}^* \times \mathbb{C}^*)$, the group of algebraic automorphisms of $\mathbb{C}^2 \setminus \{x \cdot y = 0\} \cong \mathbb{C}^* \times \mathbb{C}^*$. We now find ourselves on a situation completely similar to the case (b); the group $\mathrm{Aut}(\mathbb{C}^* \times \mathbb{C}^*)$ fits on the splitting exact sequence

$$0 \longrightarrow \mathbb{C}^* \times \mathbb{C}^* \longrightarrow \mathrm{Aut}(\mathbb{C}^* \times \mathbb{C}^*) \longrightarrow GL_2(\mathbb{Z}) \longrightarrow 0,$$

where the homomorphism $\mathrm{Aut}(\mathbb{C}^* \times \mathbb{C}^*) \to GL_2(\mathbb{Z})$ is given by the action on fundamental group of $\mathbb{C}^* \times \mathbb{C}^*$. We remark, for the sake of clearness, that this homomorphism admits a right inverse given by

$$\begin{aligned} GL_2(\mathbb{Z}) &\longrightarrow \mathrm{Aut}(\mathbb{C}^* \times \mathbb{C}^*) \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\longmapsto \left((x, y) \longmapsto (x^a \cdot y^b, x^c \cdot y^d) \right). \end{aligned}$$

We can check that $\lambda \colon G \to \mathbb{C}^*$ factors through the morphism $G \subset \mathrm{Aut}(\mathbb{C}^* \times \mathbb{C}^*) \to GL(2, \mathbb{Z})$ and, in complete analogy with case (b), we have reduced our problem to bound the order of cyclic elements of $GL_2(\mathbb{Z})$. Therefore

$$[G : \ker \lambda] \in \{1, 2, 3, 4, 6\}$$

in case (d.1).

3.4.2. Case (d.2): Quotients of $\mathbb{C}^* \times \mathbb{C}$. It remains to treat the case $v = x\partial_x + \partial_y$. Note that the only algebraic curves invariant by v are $\{x = 0\}$ and the line at infinity. Note also that v is tangent to rational 1-form

$$\omega = \frac{dx}{x} + dy.$$

As in case (d.1) if φ leaves $\mathcal{F}$, the foliation induced by v, invariant then ϕ must be biregular when restricted to $\mathbb{C}^2 \setminus \{x = 0\} \cong \mathbb{C}^* \times \mathbb{C}$. A simple argument shows that every $\varphi \in \mathrm{Aut}(\mathbb{C}^* \times \mathbb{C})$ is of the form

$$\varphi(x, y) = (\alpha_1 \cdot x^{\epsilon}, \alpha_2 x^{\beta} \cdot y + f(x)),$$

where $\alpha_1, \alpha_2 \in \mathbb{C}^*$, $\epsilon \in \{-1, 1\}$, $\beta \in \mathbb{Z}$ and $f \in \mathbb{C}\left[x, \frac{1}{x}\right]$. Moreover, since $\mathcal{F}$ does not admit a rational first integral, if φ preserves $\mathcal{F}$ then $\varphi^*\omega = \kappa \cdot \omega$ for some $\kappa \in \mathbb{C}^*$.

Comparing the dy component of ω and $\varphi^*\omega$ we see that $\beta = 0$ and comparing the dx component we deduce that f must be constant. Thus, as a simple computation shows, if φ is a birational map of $\mathbb{P}^2$ which preserves $\mathcal{F}$ then it must be of the form

$$\varphi(x, y) = (\alpha \cdot x^{\epsilon}, \epsilon \cdot y + \beta)$$

where $\alpha \in \mathbb{C}^*$, $\beta \in \mathbb{C}$ and $\epsilon \in \{-1, 1\}$. In particular

$$[G : \ker \lambda] \in \{1, 2\}$$

in case (d.2).

3.5. Proof of the Main Theorem. Let $(S, \mathcal{F})$ be a foliated surface of Kodaira dimension zero. From Theorem 2.1, Proposition 2.1 and the analysis of quotient of foliated surfaces generated by vector fields just made it follows at once that

$$h(S, \mathcal{F}) \in \{1, 2, 3, 4, 5, 6, 8, 10, 12\}.$$

If $\mathcal{F}$ admits a rational first integral then $\mathcal{F}$ is an elliptic fibration and the arguments in §3.1 allow us to conclude that $h(S, \mathcal{F}) \in \{1, 2, 3, 4, 5, 6\}$.

Suppose now that $h(S, \mathcal{F}) \in \{5, 8, 10, 12\}$. Again from the analysis above it follows that $(S, \mathcal{F})$ is birationally equivalent to the quotient of a Kroenecker foliation. We can apply [**8**, Theorem 2.1] to conclude that S is rational. □

4. Some examples

4.1. Examples with $h(S, \mathcal{F}) = 2$. If $\mathcal{F}$ is a Kroenecker foliation of an abelian surface A then quotient of $\mathcal{F}$ by the canonical involution of A (multiplication by -1) is a foliation of Kodaira dimension zero on a $K3$ surface.

Another example already appeared in case (e.2) of §3.

4.2. Examples with $h(S, \mathcal{F}) \in \{3, 4, 6\}$. Examples of rational foliated surfaces with $h(S, \mathcal{F}) \in \{3, 4, 6\}$ have already appeared in the literature. For $h(S, \mathcal{F}) = 3$ we have the *very special foliation* of Brunella [**2**, pp. 57–59]. For $h(S, \mathcal{F}) \in \{3, 4, 6\}$ we have the pencils of foliations studied by Lins Neto in [**4**], see also the appendix of [**3**]. They are obtained through quotients of $E \times E$ by a diagonal automorphism of order 3, 4 or 6. In §5 we will determine an explicit model for Brunella's very special foliation on the projective plane.

To construct examples with $h(S, \mathcal{F}) \in \{3, 4, 6\}$ with S not rational we fix $k \in \{3, 4, 6\}$ and E an elliptic curve with an automorphism g of order k. Let C be an algebraic curve with a cyclic automorphism h of order k such that $C/\langle h \rangle$ is not rational. It is a trivial matter to verify that quotient of the natural elliptic fibration on $C \times E$ by the cyclic group generated by $h \times g$ is a foliated surface with the wanted property.

4.3. Examples with $h(S, \mathcal{F}) \in \{5, 8, 10, 12\}$. Let ξ_n be a primitive root of the unity of order n, $n \in \{5, 8, 10, 12\}$. Settle

$$\Gamma_n = \begin{pmatrix} 1 & \xi_n & \xi_n^2 & \xi_n^3 \\ 1 & \xi_n^2 & \xi_n^{2k} & \xi_n^{3k} \end{pmatrix}$$

with $k = 2, 3, 3, 5$ corresponding to each value n respectively. Then $A_n = \mathbb{C}^2/\Gamma_k$ is an abelian surface (in fact it is an abelian surface of CM-type, cf. [**8**]) and

$$\begin{aligned} \varphi \colon \quad \mathbb{C}^2 &\longrightarrow \mathbb{C}^2 \\ (x, y) &\longmapsto (\xi \cdot x, \xi^k \cdot y) \end{aligned}$$

induces an automorphism of A_n of order n which we still denote by φ.

The foliations $\mathcal{F}_x$ and $\mathcal{F}_y$ induced, respectively, by the vector fields ∂_x and ∂_y are invariant under the action of φ. Moreover $\varphi_* \partial_x = \xi_n \partial x$ and $\varphi_* \partial_y = \xi_n^k \partial y$. Since the foliations $\mathcal{F}_x$ and $\mathcal{F}_y$ do not admit rational first integrals then from Proposition 2.1 we have that

$$h\left(\frac{(A_n, \mathcal{F}_x)}{\langle \varphi \rangle}\right) = h\left(\frac{(A_n, \mathcal{F}_y)}{\langle \varphi \rangle}\right) = n.$$

An interesting fact is that the property $h(S,\mathcal{F}) \in \{5,10\}$ characterizes the birational equivalence class of $(S,\mathcal{F})$, cf. [**3**, Corrigendum].

5. A projective model for Brunella's very special foliation

Let T be an automorphism of $\mathbb{P}^2$ which cyclically permutes three noncolinear points p_1, p_2 and p_3. In a suitable system of projective coordinates the automorphism T is of the form $T(x:y:z) = (y:z:x)$. If we look for foliations of $\mathbb{P}^2$ of degree 1, i.e., elements of $\mathbb{P}H^0(\mathbb{P}^2, T\mathbb{P}^2)$, then among these there only two foliations which have p_1, p_2 and p_3 as singular set and are invariant by T. They are induced by the vector fields $v = x\partial_x + \xi y \partial_y + \xi^2 z \partial_z$, where ξ is one of the two primitive roots of the unity of order 3. Note that these two foliations are indistinguishable: they are conjugate under the involution $(x:y:z) \mapsto (x:z:y)$. Denote these foliations by $\mathcal{F}_\xi$.

Let S be the minimal desingularization of $\mathbb{P}^2/\langle T\rangle$ and $\mathcal{H}$ be the foliation of S induced by the quotient of $\mathcal{F}_\xi$. Then $\mathcal{H}$ is the very special foliation as defined by Brunella, cf. [**2**]. Note that S is birationally equivalent to $\mathbb{P}^2$. In [**2**, pp. 63–64] the problem of determining simple explicit equations for a foliation of $\mathbb{P}^2$ birationally equivalent to $\mathcal{H}$ and of minimal degree is proposed.

Of course the degree of any model for $\mathcal{H}$ cannot be smaller than 2. Foliations of degree 0 and 1 are tangent to holomorphic vector fields and either are birationally equivalent to rational fibrations or to reduced foliations of Kodaira dimension zero and height one.

We claim that the homogeneous one-form

$$(2)\quad \Omega = (-y^2z - xz^2 + 2xyz)\,dx + (3xyz - 3x^2z)\,dy + (x^2z - 2xy^2 + x^2y)\,dz$$

induces a foliation of $\mathbb{P}^2$ which is birationally equivalent to $\mathcal{H}$. Moreover since $\Omega \in \mathbb{P}H^0(\mathbb{P}^2, {\Omega^1}_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(4))$ then the foliation induced by Ω has degree 2. In particular it is of minimal degree in the birational equivalence class of $\mathcal{H}$.

In order to determine Ω we have followed a very simple minded strategy. First note that the rational map $\phi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ which sends $(x:y:z)$ to

$$((x+y+z)^3 : (x+y+z)(x+\xi y+\xi^2 z)(x+\xi^2 y+\xi z) : (x+\xi^2 y+\xi z)^3)$$

is such that $\phi \circ T = \phi$ and has generic degree 3. Therefore ϕ defines a birational equivalence between $\mathbb{P}^2$ and its quotient by the group generated by T. So we have just to determine Ω from the system of equations

$$\phi^*\Omega(v) = 0$$

$$\phi^*\Omega(x\partial_x + y\partial_y + z\partial_z) = 0.$$

With some patience (or a computer algebra system) one arrives at equation (2).

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