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ARITHMETIC BASED FRACTALS ASSOCIATED WITH PASCAL'S TRIANGLE

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Abstract

Our goal is to study Pascal-Sierpinski gaskets, which are certain fractal sets defined in terms of divisibility of entries in Pascal's triangle. The principal tool is a "carry rule" for the addition of the base-q representation of coordinates of points in the unit square. In the case that $q = p$ is prime, we connect the carry rule to the power of p appearing in the prime factorization of binomial coefficients. We use the carry rule to define a family of fractal subsets B_{qr} of the unit square, and we show that when $q = p$ is prime, B_{qr} coincides with the Pascal-Sierpinski gasket corresponding to $N = p^r$. We go on to describe B_{qr} as the limit of an iterated function system of "partial similarities", and we determine its Hausdorff dimension. We consider also the corresponding fractal sets in higher-dimensional Euclidean space.

1. Introduction

Just as the Cantor set is obtained from an interval by excising "middle thirds", so is the Sierpinski gasket (or Sierpinski triangle) obtained from a triangle by excising "middle triangles". Starting with a triangle T, we excise the open triangle U_1 with vertices at the midpoints of the three sides of T. This yields $T_1 = T \backslash U_1$, which is a union of three congruent triangles each similar to T. Performing the same procedure on each of the three triangles in T_1, we excise the union U_2 of the three middle triangles. This yields $T_2 = T_1 \backslash U_2$, which consists now of nine congruent triangles each similar to T. Iterating the procedure, we obtain in the limit the Sierpinski gasket X as a decreasing limit of the T_n's as $n \to \infty$. If ϕ_1, ϕ_2, ϕ_3 are the affine maps of T onto the three triangles in T_1, then $T_{n+1} = \cup \phi_j(T_n)$, and in the limit, $X = \cup \phi_j(X)$. This relation expresses the self-similarity of X.

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It is well known (cf. [**Man**]) that the Sierpinski gasket can also be obtained by operations on Pascal's triangle. We view Pascal's triangle as a quarter-plane suspended from its vertex and tiled by unit squares, so that each square cell of the tiling contains an entry of Pascal's triangle. We color black the cells with odd entries, and we color white those with even entries. If we scale the triangular figure consisting of the top 2^n rows of Pascal's triangle to fit a fixed triangle, and we let E_n be the image of the black squares, we see visually and verify easily that the E_n's decrease to a Sierpinski gasket.

In [**H**], Holter, Lakhtakia, Varadan, Varadan, and Messier extend this idea by performing the following experiment. For a fixed integer $N \geq 2$, they color black the cells of Pascal's triangle whose labels are not divisible by N, the other cells white, they truncate the triangle, and they look for patterns. They observe that when $N = p$ is prime, the truncated and rescaled sets have a limit set that is self-similar and has Hausdorff dimension

$$\beta_p = 1 + \frac{\log((p+1)/2)}{\log p}. \tag{1}$$

They report that when $N = p^r$ is a power of a prime, "visual inspection alone suffices to show that the resulting gaskets are self-similar", and they raise the problem of determining their dimensions. For $N = 6$ they report that "visual inspection extended up to $n = 198$ rows does not reveal any self-similarity in these gaskets". (These sets can be viewed on the interactive web site at `http://www.its.caltech.edu/~mamikon/PasFastC.html`.)

Our goal is to introduce and investigate a family of "fractal" subsets B_{qr} of the plane, defined for integers $q \geq 2$ and $r \geq 1$, which coincide with the Pascal-Sierpinski gaskets of [**H**] in the case that $q = p$ is prime and $N = p^r$. We refer to B_{qr} as the (q,r)-*basket*, since in some sense it is a basket of gaskets. We study the self-similarity properties of B_{qr}, and we determine its Hausdorff dimension.

The paper is organized as follows. In Section 2 we derive the "carry rule", which gives a condition equivalent to a multinomial coefficient being divisible by a fixed prime power p^r. In Section 3 we define for each $q \geq 2$ the *Sierpinski q-gasket* B_q, we show how it is obtained from a triangle by an iterative process of excising subtriangles, and we collect some basic facts for later use. In Section 4 we define for each $q \geq 2$ and $r \geq 1$ the (q,r)-basket B_{qr}. The $(q,1)$-basket B_{q1} coincides with the Sierpinski q-gasket B_q, and $B_{qr} \subset B_{q,r+1}$ for $r \geq 1$.

The carry rule shows that in the case that $q = p$ is prime, B_{qr} coincides with the Pascal-Sierpinski gasket treated in [**H**] for $N = p^r$. In Section 5 we describe in more detail the dynamics of the (q, r)-basket by showing that B_{qr} can be viewed as a limit of an iterated function system of certain "partial self-similarities". If $r > 1$, we show that B_{qr} is obtained from B_q by plugging scaled copies of lower order gaskets B_{qt}, $1 \leq t < r$, into the triangles forming the complementary components of B_q. In Sections 6 and 7 we show that for fixed q, the (q, r)-baskets have Hausdorff dimension $\beta = \beta_q$ determined by the identity $q^\beta = q(q+1)/2$, which is equivalent to the formula (1) with p replaced by q. In Section 8 we indicate how the analysis can be extended to the corresponding fractal sets in higher dimensions and higher order multinomial coefficients.

This collaboration began in connection with a project for professional development of K-12 mathematics teachers. We hope that various of the ideas that appear here can be reformulated to be useful for professional development.

2. The carry rule

We are interested in the prime decompositions of multinomial coefficients. For a given prime number p, we would like to specify the prime power p^r appearing in the prime decomposition of a multinomial coefficient. We begin with the following.

Lemma 2.1. *Let p be a prime number, and let $n \geq 0$. Suppose n has the base-p representation*

$$n = a_k p^k + a_{k-1} p^{k-1} + \cdots + a_1 p + a_0,$$

where $0 \leq a_j \leq p - 1$ for $0 \leq j \leq k$. Then the power p^r of p appearing in the prime decomposition of $n!$ has exponent r given by

$$r = \frac{1}{p-1}\left[n - (a_k + a_{k-1} + \cdots + a_1 + a_0)\right].$$

Proof: We may assume $n \geq 1$. We express $n! = n(n-1)(n-2)\ldots$. The factors that are divisible by p are the multiples $p, 2p, 3p, \ldots, (a_k p^{k-1} + a_{k-1} p^{k-2} + \cdots + a_1)p$ of p. Thus the number of factors divisible by p is $a_k p^{k-1} + a_{k-1} p^{k-2} + \cdots + a_1$. Similarly, the number of factors divisible by p^2 is $a_k p^{k-2} + a_{k-1} p^{k-3} + \cdots + a_2$. We count in this fashion, until we reach the number of factors divisible by p^k, which is a_k. The exponent r

is the sum of these numbers,

$$\begin{aligned} r &= (a_k p^{k-1} + a_{k-1} p^{k-2} + \cdots + a_1) \\ &\quad + (a_k p^{k-2} + a_{k-1} p^{k-3} + \cdots + a_2) + \cdots + a_k \\ &= a_k (p^{k-1} + p^{k-2} + \cdots + 1) \\ &\quad + a_{k-1}(p^{k-2} + p^{k-3} + \cdots + 1) + \cdots + a_2(p+1) + a_1. \end{aligned}$$

Summing geometric series, we obtain

$$r = \frac{a_k p^k + a_{k-1} p^{k-1} + \cdots + a_1 p - (a_k + a_{k-1} + \cdots + a_1)}{p-1}.$$

Substituting $n - a_0$ for $a_k p^k + a_{k-1} p^{k-1} + \cdots + a_1 p$, we obtain the desired identity. □

Lemma 2.2. *Let p be a prime number, and let $m_1, \ldots, m_\ell \geq 0$, $N = m_1 + \cdots + m_\ell$. Suppose that the m_i's and N have base-p representations $m_i = \sum_j a_{ij} p^j$, $1 \leq i \leq \ell$, and $N = \sum_j b_j p^j$. Then the power p^r of p appearing in the prime decomposition of the multinomial coefficient*

$$\frac{N!}{m_1! \ldots m_\ell!} \tag{2}$$

has exponent r given by

$$r = \frac{1}{p-1} \left(\sum_{i,j} a_{ij} - \sum_j b_j \right). \tag{3}$$

Proof: Apply the preceding lemma to each of the factorials, and use $N = m_1 + \cdots + m_\ell$. □

Now we consider the addition algorithm for adding numbers of the form $m_i = \sum_j a_{ij} p^j$ in base-p representation, by adding successively digits in each place and carrying if the sum is $\geq p$. We count the carries according to multiplicity. The number of carries in the jth place is the integer $\kappa_j \geq 0$ defined inductively, starting with $\kappa_{-1} = 0$, by

$$a_{1j} + a_{2j} + \cdots + a_{\ell j} + \kappa_{j-1} = b_j + p\kappa_j, \quad j \geq 0, \tag{4}$$

where $0 \leq b_j \leq p-1$.

In the case that $\ell = 2$, we are adding only two m_i's, and there is never more than one carry, that is, each κ_j is either 0 or 1.

Theorem 2.3 (Carry Rule). *Let p be a prime number, and let $m_1, \dots, m_\ell \geq 0$, $N = m_1 + \cdots + m_\ell$. Suppose that the m_i's and N have base-p representations $m_i = \sum_j a_{ij} p^j$, $1 \leq i \leq \ell$, and $N = \sum_j b_j p^j$. Then the exponent r in the power p^r of p appearing in the prime decomposition of the multinomial coefficient* (2) *is equal to the number of carries in adding the base-p representations of m_i's.*

Proof: From (4) we have

$$\sum_{i,j} a_{ij} - \sum_j b_j = \sum p\kappa_j - \sum \kappa_{j-1} = (p-1) \sum \kappa_j.$$

Thus from (3) we obtain $r = \sum \kappa_j$, as required. □

Corollary 2.4. *Let p be a prime number, and let $m_1, \dots, m_\ell \geq 0$, $N = m_1 + \cdots + m_\ell$. Let the m_i's have base-p representations $m_i = \sum_j a_{ij} p^j$ as above. Then p does not divide the multinomial coefficient* (2) *if and only if there are no carries in adding the base-p representations of the m_i's, that is, if and only if $a_{1j} + a_{2j} + \cdots + a_{\ell j} < p$ for $j \geq 0$.*

3. The Sierpinski q-gasket

We fix $q \geq 2$. For $n \geq 0$, let $\mathcal{G}_n$ be the grid of subsquares of the unit square of sidelength $1/q^n$. We label the squares according to their lower left corners (x, y), and we represent x and y in their base-q expansions

$$x = 0.x_n x_{n-1} \dots x_1 = \frac{1}{q^n}(x_1 + x_2 q + \cdots + x_n q^{n-1}),$$

$$y = 0.y_n y_{n-1} \dots y_1 = \frac{1}{q^n}(y_1 + y_2 q + \cdots + y_n q^{n-1}),$$

where $0 \leq x_j, y_j \leq q - 1$.

Let $\mathcal{E}_n$ be the squares in $\mathcal{G}_n$ whose labels (x, y) satisfy

$$x_j + y_j \leq q - 1, \quad 1 \leq j \leq n. \tag{5}$$

We refer to the squares in $\mathcal{E}_n$ as the "black squares", and to the other squares in $\mathcal{G}_n$ as the "white squares". Let E_n be the union of the (closed) black squares in $\mathcal{G}_n$,

$$E_n = \cup\{S : S \in \mathcal{E}_n\}.$$

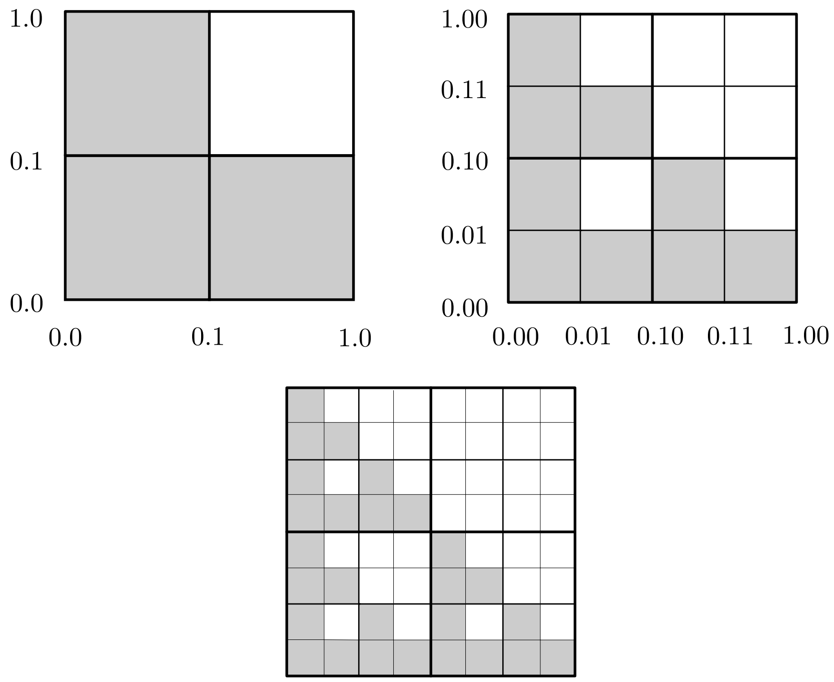

Figure 1. Three stages E_1, E_2, E_3 in construction of Sierpinski gasket.

Now consider the corresponding grid $\mathcal{G}_{n+1}$ and black squares $\mathcal{E}_{n+1}$. The union of the squares in $\mathcal{G}_{n+1}$ with labels $(0.x_{n+1}x_n \dots x_2x_1, 0.y_{n+1}y_n \dots y_2y_1)$, taken over $1 \leq x_1, y_1 \leq q-1$, is the square in $\mathcal{G}_n$ with label $(0.x_{n+1}x_n \dots x_2, 0.y_{n+1}y_n \dots y_2)$. From (5) we see that if a square in $\mathcal{G}_n$ is white, then each of the constituent squares in $\mathcal{G}_{n+1}$ is also white. Thus $\{E_n\}$ is a decreasing sequence of nonempty compact sets.

Definition 3.1. The *Sierpinski q-gasket* B_q is the decreasing limit of the E_n's,

$$B_q = \lim_{n\to\infty} E_n = \cap_{n=0}^{\infty} E_n.$$

Evidently B_q is a nonempty compact set. When $q = 2$, we obtain the usual Sierpinski gasket B_2. Figure 1 indicates the first three stages E_1, E_2, E_3 in the construction of B_2. Figure 2 depicts two stages in the construction of B_3. In this figure, the shaded squares (both light and dark) represent E_1, and the darkly shaded squares represent E_2.

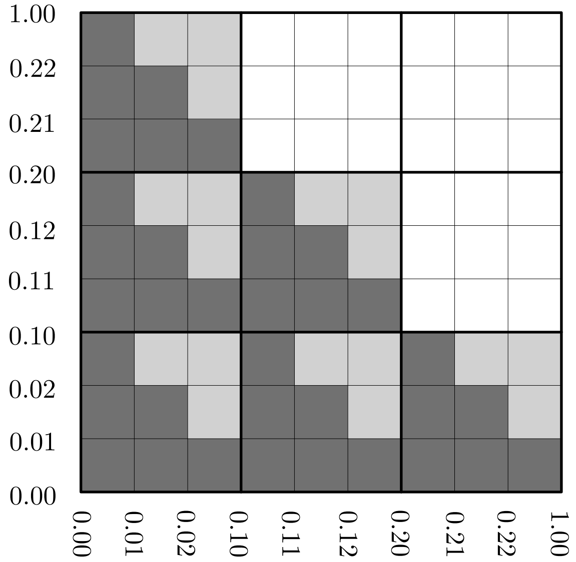

FIGURE 2. Two stages E_1, E_2 in construction of B_3.

We refer to a square in $\mathcal{G}_n$ as *diagonal* if its center lies on the diagonal line $\{x + y = 1\}$. Otherwise the square is *subdiagonal* or *superdiagonal* depending on whether it lies below-left or above-right of the diagonal squares. Note that the diagonal and subdiagonal squares in $\mathcal{G}_1$ are black (the squares in $\mathcal{E}_1$), and the superdiagonal squares in $\mathcal{G}_1$ are white. For each square $S \in \mathcal{E}_1$, we define an affine map ϕ_S of the unit square onto S by scaling by $1/q$ and translating. If the square S has label $(0.a, 0.b)$, where $0 \leq a, b \leq q - 1$, then

$$\phi_S(x, y) = \left(\frac{a + x}{q}, \frac{b + y}{q}\right) = \frac{1}{q}(a + x, b + y).$$

Note that ϕ_S maps the square U in $\mathcal{G}_n$ with label $(0.x_n \ldots x_1, 0.y_n \ldots y_1)$ onto the square V in $\mathcal{G}_{n+1}$ with label $(0.ax_n \ldots x_1, 0.by_n \ldots y_1)$. If U is black, then $x_j + y_j \leq q - 1$ for $1 \leq j \leq n$, so since $a + b \leq q - 1$, the square V is also black. Further, if V is a black square in $\mathcal{G}_{n+1}$, say with label $(0.x_{n+1}x_n \ldots x_1, 0.y_{n+1}y_n \ldots y_1)$, then $V = \phi_S(U)$ for the black square U with label $(0.x_n \ldots x_1, 0.y_n \ldots y_1)$ and S with label $(0.x_{n+1}, 0.y_{n+1})$. Thus

$$E_{n+1} = \cup\{\phi_S(E_n) : S \in \mathcal{E}_1\},$$

and in the limit

$$B_q = \cup\{\phi_S(B_q) : S \in \mathcal{E}_1\}.$$

The maps $\{\phi_S\}_{S\in\mathcal{E}_1}$ form an iterated function system (see [**B**], [**Men**]). The limit B_q is the unique fixed point of the set-mapping

$$E \mapsto \cup\phi_S(E),$$

which is a contraction of the space of nonempty compact subsets of the unit square, endowed with the usual Hausdorff metric.

The Sierpinski q-gasket can also be obtained through an iterative process of excising triangles. We start with the triangle $T = \{(x, y) : x \geq 0, y \geq 0, x + y \leq 1\}$. Let $W = \{(x, y) : x < 1, y < 1, x + y > 1\}$, which is an open triangle disjoint from T.

Lemma 3.2. *The triangle T contains B_q. The boundary ∂T of T is contained in B_q. The triangle W is disjoint from B_q.*

Proof: The squares in $\mathcal{E}_n$ have labels in T, so points of E_n have distance at most $1/2q^n$ from T. Passing to the limit, we obtain $B_q \subset T$, and B_q is disjoint from W. The squares in $\mathcal{G}_n$ bordering on the x-axis or the y-axis have labels with one coordinate equal to 0, hence belong to $\mathcal{E}_n$. Thus the unit intervals on the two coordinate axes are contained in each E_n hence in B_q. Also the diagonal squares in $\mathcal{G}_n$ have labels (x, y) satisfying $x_j + y_j = q - 1$, for all j, so the diagonal squares in $\mathcal{G}_n$ are all black. Thus the diagonal edge of T is contained in each E_n, hence in B_q. □

We consider the $q(q-1)/2$ open triangles $\phi_S(W)$, where S is subdiagonal. We refer to these as the *first-generation triangles*. They play the role of the first middle triangle in the construction of the Sierpinski triangle. Let U_1 be the union of the first-generation triangles, and set $T_1 = T\backslash U_1$. Since W is disjoint from B_q, each $\phi_S(W)$ is disjoint from B_q, U_1 is disjoint from B_q, and $B_q \subset T_1$.

In each black square $S \in \mathcal{E}_1$ we define $q(q+1)/2$ second-generation triangles to be the triangles in $\phi_S(U_1)$. Thus there are $[q(q+1)/2]q(q-1)/2$ second-generation triangles. Let U_2 be the union of the second-generation triangles, and let $T_2 = T_1\backslash U_2$. Again U_2 is disjoint from B_q, and $B_q \subset T_2$. Proceeding by induction, we define the nth-generation triangles to be the images of the $(n-1)$th generation triangles under the maps ϕ_S for $S \in \mathcal{E}_1$. There are $[q(q+1)/2]^{n-1}q(q-1)/2$ triangles in the nth-generation. The union U_n of the nth-generation triangles is disjoint from B_q, so that $T_n = T_{n-1}\backslash U_n$ contains B_q. Passing to the limit, we have $B_q \subset \lim T_n = T\backslash(\cup_{n=1}^{\infty} U_n)$. On the other hand, T_n is contained in the black squares in $\mathcal{G}_{n-1}$, so that $\lim T_n \subset B_q$. We have established the following.

Theorem 3.3. *The Sierpinski q-gasket B_q is obtained from the triangle T by excising the nth-generation triangles, $1 \le n < \infty$,*

$$B_q = \lim T_n = T \setminus (\cup_{n=1}^{\infty} U_n) \,.$$

Now we focus on the case when $q = p$ is prime. We consider the quarter-plane tiled by cells that are unit squares containing the entries of Pascal's triangle, and we rotate the quarter-plane so that it fills the first quadrant. With this representation, the entry in the cell whose lower left corner has coordinates (k, m) is the binomial coefficient $\binom{k+m}{k}$, $0 \le k, m < \infty$. We color the cell white if $\binom{k+m}{k}$ is divisible by p, and we color the cell black if $\binom{k+m}{k}$ is not divisible by p.

Fix $n \ge 0$, and consider the square $[0, p^n] \times [0, p^n]$. The scaling by the factor $1/p^n$ maps this square onto the unit square, and it maps the cells corresponding to the binomial coefficients onto squares in $\mathcal{G}_n$.

Theorem 3.4. *Let p be a prime number. Fix $n \ge 1$, and $0 \le k$, $m < p^n$. The binomial coefficient $\binom{k+m}{k}$ is not divisible by p if and only if the corresponding square in $\mathcal{G}_n$ belongs to $\mathcal{E}_n$. In other words, the cell in Pascal's triangle corresponding to $\binom{k+m}{k}$ is black if and only if the corresponding square in $\mathcal{G}_n$ is black.*

Proof: Let $k = a_{n-1}p^{n-1} + a_{n-2}p^{n-2} + \cdots + a_1 p + a_0$ and $m = b_{n-1}p^{n-1} + b_{n-2}p^{n-2} + \cdots + b_1 p + b_0$ be the base-p representations of k and m. By the carry rule, $\binom{k+m}{m}$ is not divisible by p if and only if there are no carries when we add these representations. This occurs if and only if $a_j + b_j \le q - 1$ for $0 \le j \le n-1$, and this occurs if and only if the square with label

$$(0.a_{n-1} \dots a_0, 0.b_{n-1} \dots b_0) = \frac{1}{p^n}(k, m)$$

belongs to $\mathcal{E}_n$. □

4. The (q, r)-basket

Again we fix $q \ge 2$, and we let $r \ge 1$. Let $\mathcal{E}_{rn}$ be the set of squares in $\mathcal{G}_n$ with labels (x, y) such that when we add the base-q representations of $q^{n-1}x$ and $q^{n-1}y$, there are fewer than r carries. In other words, the squares in $\mathcal{E}_{rn}$ are the squares with labels $(0.x_n x_{n-1} \dots x_1, 0.y_n y_{n-1} \dots y_1)$ such that when we add $x_1 + x_2 q + \cdots + x_n q^{n-1}$ and $y_1 + y_2 q + \cdots + y_n q^{n-1}$, there are fewer than r carries. Occasionally we refer to the squares in $\mathcal{E}_{rn}$ as the "black squares", and we refer to the other squares in $\mathcal{G}_n$ as the "white squares". Again, if a square in $\mathcal{G}_n$ is white, then the squares in $\mathcal{G}_{n+1}$ it contains are white.

Let E_{rn} be the union of the (closed) black squares in $\mathcal{G}_n$,

$$E_{rn} = \cup\{S : S \in \mathcal{E}_{rn}\}.$$

Then $\{E_{rn}\}_{n=1}^{\infty}$ is a decreasing sequence of nonempty compact sets.

Definition 4.1. The (q,r)-*basket* B_{qr} is the decreasing limit of the E_{rn}'s,

$$B_{qr} = \lim_{n\to\infty} E_{rn} = \cap_{n=1}^{\infty} E_{rn}.$$

Thus B_{qr} is a nonempty compact subset of the unit square. When $r = 1$, the definition of $\mathcal{E}_{1n}$ coincides with that of $\mathcal{E}_n$, so that $E_{1n} = E_n$ for each n, and the $(q,1)$-basket coincides with the Sierpinski q-gasket,

$$B_{q1} = B_q.$$

If we allow more carries, we obtain more black squares. Thus if $r \leq s$, then $\mathcal{E}_{rn} \subseteq \mathcal{E}_{sn}$, $E_{rn} \subseteq E_{sn}$, and in the limit,

$$B_{qr} \subseteq B_{qs}, \quad 1 \leq r \leq s.$$

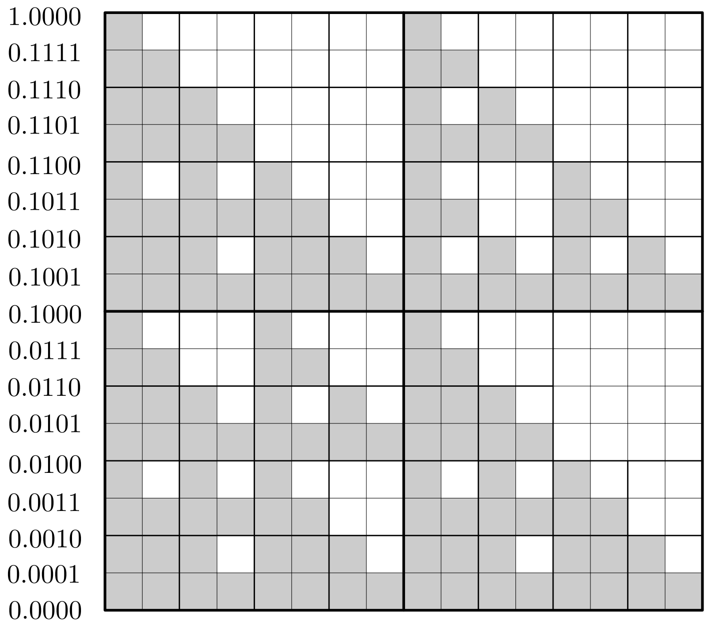

Figure 3. Fourth stage E_{24} in construction of B_{22}.

Figure 3 depicts the fourth stage E_{24} in the construction of B_{22} ($q = 2$, $r = 2$, $n = 4$). In the lower left quarter we see the preceding stage of the construction. In the upper right quarter we see the third stage of the construction of the Sierpinski gasket (see Figure 1).

Now suppose that $q = p$ is prime. We parametrize Pascal's triangle as before, and we consider the cell with coordinates (k, m), which contains the binomial coefficient $\binom{k+m}{k}$. We color the cell white if $\binom{k+m}{k}$ is divisible by p^r, and we color the cell black if $\binom{k+m}{k}$ is not divisible by p^r. Again the scaling by the factor $1/p^n$ maps the square $\{(x, y) : 0 \leq x,\, y \leq p^n\}$ onto the unit square, and it maps the cells corresponding to the binomial coefficients onto squares in $\mathcal{G}_n$. The following theorem is a direct consequence of the carry rule, as in the preceding section.

Theorem 4.2. *Let p be a prime number, and let $r \geq 1$. Fix $n \geq 1$, and $0 \leq k, m < p^n$. The binomial coefficient $\binom{k+m}{k}$ is <u>not</u> divisible by p^r if and only if the corresponding square in $\mathcal{G}_n$ belongs to $\mathcal{E}_{rn}$. In other words, the cell in Pascal's triangle corresponding to $\binom{k+m}{k}$ is black if and only if the corresponding square in $\mathcal{G}_n$ is black.*

This shows that B_{qr} is effectively the Pascal-Sierpinski fractal associated in [**H**] to $N = p^r$.

5. The dynamics of the black squares

We wish to specify which squares are in $\mathcal{E}_{rn}$ in terms of previous stages (smaller values of n or r). We begin with a subdiagonal square T in $\mathcal{G}_1$, with label $(0.a, 0.b)$. Consider the affine map $\phi_T \colon [0,1]^2 \mapsto T$ defined by

$$\phi_T(x, y) = (0.a, 0.b) + \frac{1}{q}(x, y), \quad 0 \leq x, y \leq 1.$$

Under ϕ_T there is a one-to-one correspondence of squares in $\mathcal{G}_{n-1}$ and the squares in $\mathcal{G}_n$ that are contained in T. The square U in $\mathcal{G}_{n-1}$ with label $(0.x_{n-1} \ldots x_1, 0.y_{n-1} \ldots y_1)$ is mapped by ϕ_T to the square V in $\mathcal{G}_n$ with label $(0.ax_{n-1} \ldots x_1, 0.by_{n-1} \ldots y_1)$. Since T is subdiagonal, $a+b \leq q-2$. Consequently there are the same number of carries when we add the x and y coordinates of U and V. We have established the following.

Lemma 5.1. *Let T be a subdiagonal square in $\mathcal{G}_1$, and let ϕ_T be the affine map from the unit square to T defined above. Then for each $n \geq 1$, the squares V in $\mathcal{E}_{rn}$ that are contained in T are exactly those squares of the form $V = \phi_T(U)$ for squares U in $\mathcal{E}_{r,n-1}$.*

Next fix $1 \leq t \leq r$, and let $T_0 \in \mathcal{G}_t$. We say that T_0 is a *superdiagonal square of tier t* if there is a diagonal square $T \in \mathcal{G}_{t-1}$ such that T_0 is a superdiagonal subsquare of T. Suppose the square T_0 has label $(0.a_1 \dots a_t, 0.b_1 \dots b_t)$. Then the label of T is $(0.a_1 \dots a_{t-1}, 0.b_1 \dots b_{t-1})$, and since T is diagonal, $a_j + b_j = q - 1$ for $1 \leq j \leq t-1$. Since T_0 is a superdiagonal subsquare of T, $a_t + b_t \geq q$.

Suppose $n \geq t$, and that $V \in \mathcal{G}_n$ is contained in T_0. Then the label of V has the form $(0.a_1 \dots a_t x_{n-t} \dots x_1, 0.b_1 \dots b_t y_{n-t} \dots y_1)$. The affine map $\phi_{T_0} \colon [0,1]^2 \mapsto T_0$ defined by

$$\phi_{T_0}(x, y) = (0.a_1 \dots a_t, 0.b_1 \dots b_t) + \frac{1}{q^t}(x, y), \quad 0 \leq x, y \leq 1,$$

maps squares in $\mathcal{G}_{n-t}$ to squares in $\mathcal{G}_n$ that are contained in T_0, and it maps the square U with label $(0.x_{n-1} \dots x_1, 0.y_{n-1} \dots y_1)$ onto V. When we add the coordinates of the label of V, we evidently have t more carries than when we add those for U, corresponding to the additional t place values in the coordinates of U. Hence we obtain the following.

Lemma 5.2. *Suppose $1 \leq t \leq r-1$. Let T_0 be a superdiagonal square of tier t in $\mathcal{G}_t$, and let ϕ_{T_0} be the affine map defined above. Then for each $n \geq t$, the squares V in $\mathcal{E}_{rn}$ that are contained in T_0 are exactly those squares of the form $V = \phi_{T_0}(U)$ for squares U in $\mathcal{E}_{r-t,n-t}$.*

Lemma 5.3. *Suppose $t \geq r$. Let T_0 be a superdiagonal square of tier t in $\mathcal{G}_t$. Then for $n \geq t$, T_0 does not contain any squares in $\mathcal{E}_{r,n}$.*

To complete the picture, we focus on the subdiagonal part of a diagonal square in $\mathcal{G}_1$. For this, let S be the sawtooth subset of the unit square that is the union of the diagonal and subdiagonal squares in $\mathcal{G}_{r-1}$. Let T be a diagonal square in $\mathcal{G}_1$, let $\phi_T \colon [0,1]^2 \mapsto T$ be the usual affine map, and let $S_T = \phi_T(S)$. Then S_T is a sawtooth subset of T that is the union of the squares in $\mathcal{G}_r$ that are either diagonal or subdiagonal subsquares of T.

Lemma 5.4. *Let T be a diagonal square in $\mathcal{G}_1$, let S and S_T be the sawtooth sets defined above, and let ϕ_T be the affine map of S onto S_T as above. Then for each $n \geq r$, the squares V in $\mathcal{E}_{rn}$ that are contained in S_T are exactly those squares of the form $V = \phi_T(U)$ for squares U in $\mathcal{E}_{r,n-1}$ that are contained in S.*

Proof: Let V be a square in $\mathcal{G}_n$ that is contained in S_T, and let U be the corresponding square in $\mathcal{G}_{n-1}$ such that $V = \phi_T(U)$. Suppose the label of V is $(0.x_n \dots x_1, 0.y_n \dots y_1)$. Then the label of U is $(0.x_{n-1} \dots x_1, 0.y_{n-1} \dots y_1)$. Let W be the square in $\mathcal{G}_r$ containing V. The label of W is $(0.x_n x_{n-1} \dots x_{n-r+1}, 0.y_n y_{n-1} \dots y_{n-r+1})$. Since W is a diagonal or subdiagonal square,

$$x_j + y_j \leq q - 1, \quad n - r \leq j \leq n.$$

We consider three cases. Suppose first there is an index k such that $n - r + 1 \leq k \leq n - 1$ and $x_k + y_k \leq q - 2$. Then there are no carries in adding the the values in the kth place or beyond for the labels of both U and V. Consequently the number of carries of U is the same as that for V, and $V \in \mathcal{E}_{rn}$ if and only if $U \in \mathcal{E}_{r,n-1}$. Suppose next that $x_j + y_j = q - 1$ for $n - r + 1 \leq j \leq n - 1$, and there is no carry from the $(n - r)$th place in adding the labels of U or V. Then there are no carries in adding the values beyond the $(n - r)$th place for the labels of both U and V. Again the number of carries for U is the same as that for V, and $V \in \mathcal{E}_{rn}$ if and only if $U \in \mathcal{E}_{r,n-1}$. Suppose finally that $x_j + y_j = q - 1$ for $n - r + 1 \leq k \leq n - 1$, and there is a carry from the $(n - r)$th place in adding the labels of U or V. Then there are carries for both U and V in each place from the $(n - r)$th place to the $(n - 1)$th place. Thus there are at least r carries for both U and V, and consequently $U \notin \mathcal{E}_{r,n-1}$, $V \notin \mathcal{E}_{r,n}$. Since these three cases cover all possibilities, we conclude that $V \in \mathcal{E}_{rn}$ if and only if $U \in \mathcal{E}_{r,n-1}$. This proves the lemma. □

These three lemmas allow us to describe the part of the (q, r)-basket B_{qr} lying above the diagonal in terms of the scaled baskets for smaller values of r.

Theorem 5.5. *Let $r \geq 2$ and $1 \leq t \leq r - 1$. Let T_0 be a superdiagonal square of tier t. Then the limit of the squares in $\mathcal{E}_{rn}$ contained in T_0 is the $(q, r - t)$-bucket $B_{q,r-t}$, scaled by $1/q^t$.*

Thus the part of (q, r)-bucket B_{qr} to the upper right of the diagonal $\{x + y = 1\}$ is a union of scaled (q, t)-buckets, $1 \leq t \leq r - 1$, and a row of triangles adjacent to the diagonal. Each of the triangles is the half of a diagonal square in $\mathcal{G}_{r-1}$ to the upper right of the diagonal. In the case $q = 2$ and $r = 4$, Figure 4 indicates the placement of the scaled lower order fractals B_{21}, B_{22}, and B_{23} in the fractal B_{24}.

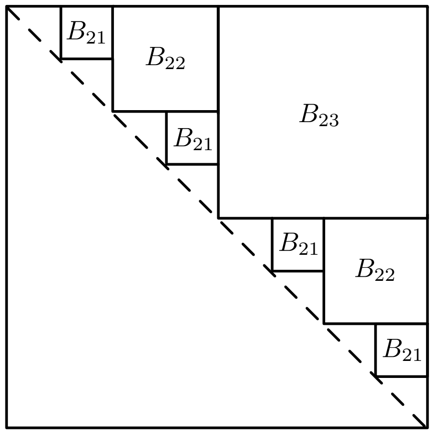

Figure 4. Superdiagonal tiers in construction of B_{24}.

Now recall that B_q is obtained from the triangle T by excising the nth-generation triangles for $n \geq 1$. Let V be a first-generation triangle. Then V is the superdiagonal of a subdiagonal square S. Let ϕ_S be the affine map of the unit square onto S, so that $V = \phi_S(W)$. Then $\phi_S(W \cap B_{qr}) = V \cap B_{qr}$, so that the part of B_{qr} in V can be viewed as obtained by plugging a scaled copy of $W \cap B_{qr}$ into V. Similarly, an nth-generation triangle has the form $V = (\phi_{S_n} \circ \cdots \circ \phi_{S_2} \circ \phi_{S_1})(W)$, where $S_1 \in \mathcal{E}_1$ is subdiagonal, and $S_2, \ldots, S_n \in \mathcal{E}_1$. Again $V \cap B_{qr}$ is the image of $W \cap B_{qr}$ under this composition, so that the part of B_{qr} in V can be viewed as obtained by plugging a scaled copy of $W \cap B_{qr}$ into V. We have established the following.

Theorem 5.6. *Let W be the upper triangle in the unit square, as before. Let $r \geq 2$, and let $X = B_{qr} \cap W$. The (q, r)-basket B_{qr} is obtained from the Sierpinski q-gasket B_q by plugging scaled copies of X into each of the nth-generation triangles in U_n, $n \geq 1$.*

6. Counting the black squares

Let $R_r(n)$ denote the number of squares in $\mathcal{E}_{rn}$, that is, the number of black squares at the nth stage. The black-square count will provide us with one route to determine the Hausdorff dimension of B_{qr}. Since all squares are black when $n < r$, we are only interested in finding $R_r(n)$ for $n \geq r$.

Theorem 6.1. *Fix $q \geq 2$, and let $r \geq 1$. There is a polynomial $P_r(n)$ in n of degree $r-1$ such that*

$$R_r(n) = \left[\frac{q(q+1)}{2}\right]^n P_r(n), \quad n \geq r. \tag{6}$$

Further,

$$P_r(n) = \frac{1}{(r-1)!}\left(\frac{q-1}{q+1}\right)^{2(r-1)} n^{r-1} + \mathcal{O}(n^{r-2}), \quad n \geq r. \tag{7}$$

Proof: We have already observed in Section 4 that $R_1(n) = [q(q+1)/2]^n$. Hence the theorem holds for $r = 1$, and in this case $P_1 = 1$. Thus we assume that $r \geq 2$, and we make an induction hypothesis that

$$\left[\frac{q(q+1)}{2}\right]^{-n} R_k(n) = a_k n^{k-1} + \mathcal{O}(n^{k-2}), \quad n \geq k,\ 1 \leq k < r, \tag{8}$$

where the big-oh term in (8) is a polynomial in n of degree $\leq k-2$.

We assume $n \geq r$, and we count black squares. There are $q(q-1)/2$ subdiagonal squares in $\mathcal{G}_1$. In view of Lemma 5.1, we see that each of these contains $R_r(n-1)$ squares in $\mathcal{E}_{rn}$. Thus there are a total of

$$\frac{q(q-1)}{2} R_r(n-1) \tag{9}$$

squares of $\mathcal{E}_{rn}$ contained in the subdiagonal squares of $\mathcal{G}_1$.

Let $S_r(n)$ denote the number of superdiagonal squares in $\mathcal{E}_{rn}$. Each such square is in tier t for some $1 \leq t \leq r-1$. There are q^{t-1} diagonal squares in $\mathcal{G}_{t-1}$, and each of these contains $q(q-1)/2$ superdiagonal subsquares in $\mathcal{G}_t$. Thus there are a total of $q^t(q-1)/2$ superdiagonal squares in tier t. In view of Lemma 5.2, we see that each of these contains $R_{r-t}(n-t)$ squares in $\mathcal{E}_{rn}$. Thus

$$S_r(n) = \sum_{t=1}^{r-1} \frac{q^t(q-1)}{2} R_{r-t}(n-t). \tag{10}$$

There are $S_r(n-1)$ superdiagonal squares in $\mathcal{E}_{r,n-1}$, so there are $R_r(n-1) - S_r(n-1)$ diagonal and subdiagonal squares in $\mathcal{E}_{r,n-1}$. There are q diagonal squares in $\mathcal{G}_1$, so that in view of Lemma 5.4 we see that there are

$$qR_r(n-1) - qS_r(n-1) \tag{11}$$

squares of $\mathcal{E}_{rn}$ contained in the diagonal squares of $\mathcal{G}_1$.

If we add (9), (10), and (11) and combine like terms, we obtain

$$R_r(n) = \frac{q(q+1)}{2} R_r(n-1) - qS_r(n-1) + S_r(n), \quad n \geq r. \tag{12}$$

Now consider

$$\left[\frac{q(q+1)}{2}\right]^{-n} S_r(n) = \sum_{t=1}^{r-1} \frac{q^t(q-1)}{2} \left[\frac{q(q+1)}{2}\right]^{-n+t} R_{r-t}(n-t).$$

By our induction hypothesis, the right-hand side is a polynomial in n of order $r-2$. Only the summand $t=1$ contributes to the n^{r-2} term, and we find using (8) with $k=r-1$ that

$$\left[\frac{q(q+1)}{2}\right]^{-n} S_r(n) = \frac{q(q-1)}{2} \left[\frac{q(q+1)}{2}\right]^{-1} a_{r-1}n^{r-2} + \mathcal{O}(n^{r-3}). \tag{13}$$

Substituting $n-1$ for n in (13), and dividing by $q(q+1)/2$, we also have

$$\left[\frac{q(q+1)}{2}\right]^{-n} S_r(n-1) = \frac{q(q-1)}{2} \left[\frac{q(q+1)}{2}\right]^{-2} a_{r-1}n^{r-2} + \mathcal{O}(n^{r-3}). \tag{14}$$

Multiplying (12) by $[q(q+1)/2]^{-n}$, using (6), (8), (13), and (14), and doing some algebra, we obtain the recursion relation

$$P_r(n) = P_r(n-1) + \frac{(q-1)^2}{(q+1)^2} a_{r-1}n^{r-2} + \mathcal{O}(n^{r-3}).$$

Thus $P_r(n)$ is a polynomial in n of degree $r-1$, unique up to an additive constant, and in fact we obtain (essentially by integrating) that

$$P_r(n) = \frac{1}{r-1} \frac{(q-1)^2}{(q+1)^2} a_{r-1}\, n^{r-1} + \mathcal{O}(n^{r-2}).$$

Thus

$$a_r = \frac{1}{r-1} \frac{(q-1)^2}{(q+1)^2} a_{r-1}.$$

Since $a_1 = 1$, this recursion relation has a unique solution, which is given by the leading coefficient in (7). □

7. Hausdorff dimension

Let E be a subset of $\mathbb{R}^d$, and let $s > 0$. For each $\delta > 0$, let $\Lambda_s^{(\delta)}(E)$ denote the infimum of the sums $\sum r_j^s$, taken over all covers of E by balls with radii r_j satisfying $r_j \le \delta$. As δ decreases, the infimum is taken over fewer covers, and $\Lambda_s^{(\delta)}(E)$ increases. Its limit $\Lambda_s(E) = \lim_{\delta\to 0} \Lambda_s^{(\delta)}(E)$ is the *s-dimensional Hausdorff measure* of E. The *Hausdorff dimension of E* is the infimum of s such that $\Lambda_s(E) = 0$. For background information on Hausdorff measures, see [**R**].

Theorem 7.1. *Fix $q \ge 2$ and $r \ge 1$. If $q^s > q(q+1)/2$, then $\Lambda_s(B_{qr}) = 0$.*

Proof: Let $\delta = 1/q^n$. Each square in $\mathcal{E}_{rn}$ is contained in a disk of radius δ. Using Theorem 6.1, we obtain

$$\Lambda_s^{(\delta)}(B_{qr}) \le R_r(n)\left(\frac{1}{q^n}\right)^s \sim \left[\frac{q(q+1)}{2q^s}\right]^n n^{r-1}.$$

Since this tends to 0 as $n \to \infty$, $\Lambda_s^{(\delta)}(B_{qr}) \to 0$ as $\delta \to 0$, and $\Lambda_s(B_{qr}) = 0$. □

Fix $q \ge 2$ and associated grids $\mathcal{G}_n$. For E a subset of the unit square $[0,1]^2$ in $\mathbb{R}^2$, define

$$\lambda_s^{(\delta)}(E) = \inf \sum (\text{sidelength } G_j)^s,$$

where the infimum is taken over all finite covers of E by sets $G_j \in \cup\mathcal{G}_n$. It is easy to see that there are constants $c, C > 0$ such that

$$c\lambda_s^{(\delta)}(E) \le \Lambda_s^{(\delta)}(E) \le C\lambda_s^{(\delta)}(E),$$

for all compact subsets E of the unit square (though not for arbitrary subsets). We aim to compute $\lambda_s^{(\delta)}(B_{qr})$ explicitly. First we prove two lemmas.

Lemma 7.2. *If U is a (closed) square in $\mathcal{E}_{rn}$, then B_{qr} contains interior points of U.*

Proof: If $U \in \mathcal{E}_{rn}$ has label $(0.a_n \dots a_2a_1, 0.b_n \dots b_2b_1)$, then $(0.a_n \dots a_2a_110, 0.b_n \dots b_2b_101)$ is an interior point of U. From the definition of $\mathcal{E}_{rn}$, we see that it labels a square in $\mathcal{E}_{rk}$ for all $k \ge n+2$, hence it belongs to B_{qr}. □

Lemma 7.3. *Let $U \in \mathcal{E}_{r,n-1}$. Then either all q^2 subsquares of U in $\mathcal{G}_n$ are contained in $\mathcal{E}_{rn}$, or exactly $q(q+1)/2$ of the subsquares of U in $\mathcal{G}_n$ are contained in $\mathcal{E}_{rn}$.*

Proof: Let U have label $(0.a_n \dots a_2, 0.b_n \dots b_2)$. Subsquares U_0 of U then have labels of the form $(0.a_n \dots a_2a_1, 0.b_n \dots b_2b_1)$. Since $U \in \mathcal{E}_{r,n-1}$, there are at most $r-1$ carries when we add the coordinates of U. Define k, $0 \le k \le q-1$, so that $k = 1$ if $a_2 + b_2 \ne q-1$, and otherwise k is the largest integer such that $a_j + b_j = q-1$ for $2 \le j \le k$. For a subsquare U_0 with label as above, there are no new carries if $a_1 + b_1 \le q-1$, while there are k more carries for U_0 if $a_1 + b_1 \ge q$. If now the number of carries for U is $< r-k$, then all subsquares U_0 have $< r$ carries, and all q^2 subsquares belong to $\mathcal{E}_{rn}$. If on the other hand the number of carries for U is $\ge r-k$, then the subsquares U_0 with $< r$ carries are precisely the subsquares for which $a_1 + b_1 \le q-1$, and there are exactly $q(q+1)/2$ such subsquares, namely the diagonal and subdiagonal subsquares. □

Theorem 7.4. *Fix $q \geq 2$ and $r \geq 1$. Define β by $q^\beta = q(q+1)/2$. Let $\delta = 1/q^n$. Then the infimum defining $\lambda_s^{(\delta)}(E)$ is attained for the cover $\mathcal{E}_{rn}$ of B_{qr} by black squares in $\mathcal{G}_n$. Thus*

$$\lambda_s^{(\delta)}(B_{qr}) = P_r(n), \quad \delta = 1/q^n,\ n \geq r,$$

where $P_r(n)$ is the polynomial of Theorem 6.1. In particular,

$$\lambda_\beta^{(\delta)}(B_q) = 1, \quad 0 < \delta < 1.$$

If $r \geq 2$, then as $\delta \to 0$,

$$\lambda_\beta^{(\delta)}(B_{qr}) \sim \left(\log \frac{1}{\delta}\right)^{r-1}.$$

Proof: Let $\{G_1, \ldots, G_\ell\}$ be a finite cover of B_{qr} by (closed) squares in $\cup_{k=n}^\infty \mathcal{G}_k$, so the sidelength of each G_j is $\leq 1/q^n$. By discarding G_j's that are contained in a larger G_k, we can assume that no G_j is a subsquare of another. Then the interior of each G_i is disjoint from the other G_j's.

Suppose that G_i is a white square, that is, G_i does not belong to one of the $\mathcal{E}_{rk}$'s. If $p \in G_i \cap B_{qr}$, then p lies on a boundary segment of G_i. Let $1/q^m$ be the minimal sidelength of the G_j's, and let U be a black square of sidelength $1/q^m$ that contains p. By Lemma 7.2, U contains points of B_{qr} in its interior, and any such point belongs to one of the G_j's, hence U must be contained in one of the G_j's, and consequently p is in one of the G_j's other than G_i. Thus the G_j's, for $j \neq i$ already cover B_{qr}. By discarding one by one the G_i's that are white, we can then assume that each G_i is black, that is, each G_i belongs to $\mathcal{E}_{rk}$ for some k, $n \leq k \leq m$.

Let G_i be a square in the cover of minimal sidelength $1/q^m$. Suppose G_i is a subsquare of $V \in \mathcal{G}_{m-1}$. Then V is black, that is, $V \in \mathcal{E}_{r,m-1}$, and by Lemma 7.3, at least $q(q+1)/2$ subsquares of V are black. Since G_i has minimal sidelength, and since the G_j's are disjoint, each of these black subsquares of V must be among the G_j's. If we replace these squares by the parent square V, and we note that

$$\sum_{G_j \subset U} (\text{sidelength } G_j)^\beta \geq \frac{q(q+1)}{2}\left(\frac{1}{q^m}\right)^\beta = \left(\frac{1}{q^{m-1}}\right)^\beta = (\text{sidelength } U)^\beta,$$

we obtain a cover by fewer squares for which the sum defining the minimum is at least as small. By successively replacing smaller squares by their parent squares, we eventually arrive at the situation where all the squares among the G_j's are in $\mathcal{E}_{rn}$. Since no subset of $\mathcal{E}_{rn}$ covers

B_{rq}, we conclude that the infimum defining $\lambda_\beta^{(\delta)}(B_{qr})$ is attained for the cover $\mathcal{E}_{rn}$.

The remaining assertions of the theorem follow from the definitions of $R_r(n)$ and $P_r(n)$ and Theorem 6.1. □

Theorem 7.5. *Fix $q \geq 2$ and $r \geq 1$. Define β by $q^\beta = q(q+1)/2$. The Sierpinski q-gasket B_q has finite positive β-dimensional Hausdorff measure. For $r \geq 2$, the (q,r)-basket B_{qr} has infinite though σ-finite β-dimensional Hausdorff measure.*

Proof: The first statement follows from Theorem 7.4 and the comparability of $\lambda_s^{(\delta)}$ and $\Lambda_s^{(\delta)}$. For the second statement, suppose first that $r = 2$. By Theorem 5.6, B_{q2} is obtained from B_q by plugging scaled copies of B_q into the complementary triangles of B_q. In this construction, there are $q(q-1)/2$ triangles with scaling factor $1/q$, and at the nth stage there are $[q(q+1)/2]^{n-1}q(q-1)/2$ triangles with scaling factor $1/q^n$. Thus the β-dimensional Hausdorff measure of the copy of B_q plugged in to the triangles at the nth stage is $\sim 1/q^{n\beta}$, and

$$\Lambda_\beta(B_{q2}) \geq c\sum_{n=1}^{\infty}\left(\frac{q(q+1)}{2}\right)^n \frac{q(q-1)}{2}\frac{1}{q^{n\beta}} = c\frac{q(q-1)}{2}\sum 1 = +\infty.$$

Thus B_{q2} has infinite though σ-finite β-dimensional Hausdorff measure. If now $r > 2$, B_{qr} is the union of B_q and a countable number of scaled copies of B_{qt} for $t < r$. By induction on r, we see that B_{qr} has infinite though σ-finite β-dimensional Hausdorff measure. □

Note that the fact that B_{qr} has σ-finite β-dimensional Hausdorff measure already implies that $\Lambda_s(B_{qr}) = 0$ for $s > \beta$. This provides another route to Theorem 7.1, which depends only on the (easy) case $r = 1$ of Theorem 6.1. In any event, we have the following corollary to Theorem 7.5.

Corollary 7.6. *Fix $q \geq 2$ and $r \geq 1$, and define β by $q^\beta = q(q+1)/2$. Then the (q,r)-basket B_{qr} has Hausdorff dimension β.*

8. Baskets in higher dimensions

We can define baskets in d-dimensional Euclidean space for any $d \geq 2$ in the same way as in the two-dimensional case. Carrying over the same notation, we let $\mathcal{G}_n$ be the grid of subcubes of the unit cube $[0,1]^d$ of sidelength $1/q^n$, and we label a cube in $\mathcal{G}_n$ by the d-tuple of coordinates of its lower left corner. We express the coordinates in base-q, and we define $\mathcal{E}_{rn}$ to be the subset of $\mathcal{G}_n$ of cubes with the property that when we add the coordinates of the label, there are fewer than r carries. Again we define E_{rn} be the union of the (closed) cubes in $\mathcal{E}_{rn}$. The E_{rn}'s form a decreasing sequence of nonempty compact sets. We define their limit to be the (q,r)-*basket* in $\mathbb{R}^d$, and we denote it by B_{qr}.

If $r = 1$, we obtain the analogue $B_q = B_{q1}$ in $\mathbb{R}^d$ of the Sierpinski q-gasket. It is the limit of the iterated function system consisting of functions of the form

$$\phi_{\mathbf{k}}(\mathbf{x}) = \frac{1}{q}(\mathbf{k}+\mathbf{x}), \quad \mathbf{x} \in [0,1]^d,$$

where $\mathbf{k} = (k_1, \ldots, k_d)$, and the k_j's are nonnegative integers satisfying $k_1 + \cdots + k_d \le q-1$. Each $\mathbf{k}/q$ is the label of a diagonal or subdiagonal cube in $\mathcal{G}_1$. In this case there are $q^{d-1}(q+1)/2$ cubes in $\mathcal{E}_1$, hence $[q^{d-1}(q+1)/2]^n$ cubes in $\mathcal{E}_n$. Consequently

$$\begin{aligned}\sum\{(\text{sidelength } U)^\beta : U \in \mathcal{E}_n\} &= \left[\frac{q^{d-1}(q+1)}{2}\right]^n \cdot \frac{1}{q^{n\beta}} \\ &= \left[\frac{q^{d-1}(q+1)}{2}q^{-\beta}\right]^n.\end{aligned}$$

This has a finite nonzero limit as $n \to \infty$ when

$$q^\beta = \frac{q^{d-1}(q+1)}{2},$$

that is, for $\beta = \beta_q$ defined by

$$\beta_q = d - 1 + \frac{\log((q+1)/2)}{\log q}.$$

It is straightforward to verify that β_q is the Hausdorff dimension of B_q. Again B_q has finite positive β_q-dimensional Hausdorff measure, and for $r \ge 2$, B_{qr} has infinite though σ-finite β_q-dimensional Hausdorff measure.

In the case that $q = p^r$ is a prime power, there is a connection between the (q,r)-basket and the d-dimensional analogue of Pascal's triangle consisting of multinomial coefficients. The connection can be made through the following theorem, which is a direct consequence of the carry rule.

Theorem 8.1. *Let p be a prime number, and let $r \ge 1$. For $n \ge 1$ and $0 \le k_1, \ldots, k_d < p^n$, the multinomial coefficient*

$$\frac{(k_1 + \cdots k_d)!}{k_1! \ldots k_d!}$$

is not divisible by p^r if and only if the cube in $\mathcal{G}_n$ with label $q^{-n}(k_1, \ldots, k_d)$ belongs to $\mathcal{E}_{rn}$.

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